IV. ERGODIC THEORY OF REGULAR LINEAR SFDE's

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IV. ERGODIC THEORY OF LINEAR SFDE's

1. Plan

Use state space M_2 . For regular linear sfde's (VIII), (IX), consider the following themes:

- I) Existence of a "perfect" cocycle on M_2 that is a modification of the trajectory field $(x(t), x_t) \in M_2$.
- II) Existence of almost sure Lyapunov exponents

$$\lim_{t \to \infty} \frac{1}{t} \log \| (x(t), x_t) \|_{M_2}$$

The multiplicative ergodic theorem and *hyperbolicity* of the cocycle.

III) The Stable Manifold Theorem, (viz. "random saddles") for hyperbolic systems.

2. Regular Linear Systems. White Noise

Linear sfde's on \mathbf{R}^d driven by *m*-dimensional Brownian motion $W := (W_1, \dots, W_m)$, with smooth coefficients.

$$dx(t) = H(x(t - d_1), \cdots, x(t - d_N), x(t), x_t)dt + \sum_{i=1}^{m} g_i x(t) dW_i(t), \quad t > 0$$

$$(VIII)$$

$$(x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$$

(VIII) is defined on

 $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}}, P)$ = canonical complete filtered Wiener space.

 Ω := space of all continuous paths ω : $\mathbf{R} \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space \mathbf{R}^m , with compact open topology;

 $\mathcal{F} := \text{completed Borel } \sigma\text{-field of } \Omega;$

 $\mathcal{F}_t := \text{completed sub-}\sigma\text{-field of }\mathcal{F} \text{ generated by the evaluations}$ $\omega \to \omega(u), \ u \leq t, \quad t \in \mathbf{R}.$

P := Wiener measure on Ω .

 $dW_i(t) =$ Itô stochastic differentials.

Several finite delays $0 < d_1 < d_2 < \cdots < d_N \leq r$ in drift term; no delays in diffusion coefficient.

 $H: (\mathbf{R}^d)^{N+1} \times L^2([-r, 0], \mathbf{R}^d) \to \mathbf{R}^d$ is a fixed continuous linear map, $g_i, i = 1, 2, \dots, m$, fixed (deterministic) $d \times d$ -matrices.

Recall regularity theorem:

Theorem III.4.([Mo], Stochastics, 1990])

(VIII) is regular with respect to the state space $M_2 = \mathbf{R}^d \times \mathbf{L}^2([-r, 0], \mathbf{R}^d)$. There is a measurable version $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ of the trajectory field $\{(x(t), x_t) : t \in \mathbf{R}^+, (x(0), x_0) = (v, \eta) \in M_2\}$ of (VIII) with the following properties:

- (i) For each $(v,\eta) \in M_2$ and $t \in \mathbf{R}^+$, $X(t,(v,\eta),\cdot) = (x(t), x_t)$ a.s., is \mathcal{F}_t -measurable and belongs to $L^2(\Omega, M_2; P)$.
- (ii) There exists $\Omega_0 \in \mathcal{F}$ of full measure such that, for all $\omega \in \Omega_0$, the map $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times M_2 \to M_2$ is continuous.
- (iii) For each t ∈ R⁺ and every ω ∈ Ω₀, the map X(t, ·, ω) : M₂ → M₂ is continuous linear; for each ω ∈ Ω₀, the map R⁺ ∋ t ↦ X(t, ·, ω) ∈ L(M₂) is measurable and locally bounded in the uniform operator norm on L(M₂). The map [r, ∞) ∋ t ↦ X(t, ·, ω) ∈ L(M₂) is continuous for all ω ∈ Ω₀.
- (iv) For each $t \ge r$ and all $\omega \in \Omega_0$, the map

$$X(t,\cdot,\omega):M_2\to M_2$$

is compact.

Compactness of semi-flow for $t \ge r$ will be used below to define hyperbolicity for (VIII) and the associated exponential dichotomies.

Lyapunov Exponents. Hyperbolicity

Version X of the flow constructed in Theorem III.4 is a multiplicative $L(M_2)$ -valued linear cocycle over the canonical Brownian shift $\theta : \mathbf{R} \times \Omega \to \Omega$ on Wiener space:

$$\theta(t,\omega)(u) := \omega(t+u) - \omega(t), \quad u, t \in \mathbf{R}, \quad \omega \in \Omega.$$

Indeed we have

Theorem IV.1([M], 1990)

There is an \mathcal{F} -measurable set $\hat{\Omega}$ of full P-measure such that $\theta(t, \cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$ for all $t \ge 0$ and

$$X(t_2, \cdot, \theta(t_1, \omega)) \circ X(t_1, \cdot, \omega) = X(t_1 + t_2, \cdot, \omega)$$

for all $\omega \in \hat{\Omega}$ and $t_1, t_2 \ge 0$.

The Cocycle Property

Proof of Theorem IV.1. (Sketch)

For simplicity consider the case of a single delay d_1 ; i.e. N = 1. First step.

Approximate the Brownian motion W in (VIII) by smooth adapted processes $\{W^k\}_{k=1}^{\infty}$:

$$W^{k}(t) := k \int_{t-(1/k)}^{t} W(u) \, du - k \int_{-(1/k)}^{0} W(u) \, du, \quad t \ge 0, \ k \ge 1.$$
(1)

Exercise: Check that each W^k is a *helix* (i.e. has stationary increments):

$$W^{k}(t_{1}+t_{2},\omega) - W^{k}(t_{1},\omega) = W^{k}(t_{2},\theta(t_{1},\omega)), \quad t_{1},t_{2} \in \mathbf{R}, \ \omega \in \Omega.$$
(2)

Let $X^k : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ be the stochastic (semi)flow of the random fde's:

$$dx^{k}(t) = H(x^{k}(t - d_{1}), x^{k}(t), x_{t}^{k})dt + \sum_{i=1}^{m} g_{i}x(t)(W_{i}^{k})'(t) dt - \frac{1}{2}\sum_{i=1}^{m} g_{i}^{2}x^{k}(t) dt \quad t > 0$$

$$(VIII - k)$$

$$(x^{k}(0), x_{0}^{k}) = (v, \eta) \in M_{2} := \mathbf{R}^{d} \times L^{2}([-r, 0], \mathbf{R}^{d})$$

If $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ is the flow of (VIII) constructed in Theorem III.4, then

$$\lim_{k \to \infty} \sup_{0 \le t \le T} \|X^k(t, \cdot, \omega) - X(t, \cdot, \omega)\|_{L(M_2)} = 0$$
(3)

for every $0 < T < \infty$ and all ω in a Borel set $\hat{\Omega}$ of full Wiener measure which is invariant under $\theta(t, \cdot)$ for all $t \ge 0$ ([Mo], Stochastics,

1990). This convergence may be proved using the following stochastic variational method:

Let $\phi : \mathbf{R}^+ \times \Omega \to \mathbf{R}^{d \times d}$ be the $d \times d$ -matrix-valued solution of the linear Itô sode (without delay):

$$d\phi(t) = \sum_{i=1}^{m} g_i \phi(t) \ dW_i(t) \qquad t > 0 \\ \phi(0,\omega) = I \in \mathbf{R}^{d \times d} \qquad \text{a.a. } \omega$$

$$(4)$$

Denote by $\phi^k : \mathbf{R}^+ \times \Omega \to \mathbf{R}^{d \times d}, \ k \ge 1$, the $d \times d$ -matrix solution of the random family of linear ode's:

$$d\phi^{k}(t) = \sum_{i=1}^{m} g_{i}\phi^{k}(t)(W_{i}^{k})'(t) - \frac{1}{2}\sum_{i=1}^{m} g_{i}^{2}\phi^{k}(t) dt \qquad t > 0 \\ \phi^{k}(0, \cdot) = I \in \mathbf{R}^{d \times d}.$$

$$(4')$$

Let $\hat{\Omega}$ be the sure event of all $\omega \in \Omega$ such that

$$\phi(t,\omega) := \lim_{k \to \infty} \phi^k(t,\omega) \tag{5}$$

exists uniformly for t in compact subsets of \mathbf{R}^+ . Each ϕ^k is an $\mathbf{R}^{d \times d}$ -valued *cocycle over* θ , viz.

$$\phi^k(t_1 + t_2, \omega) = \phi^k(t_2, \theta(t_1, \omega))\phi^k(t_1, \omega)$$
(6)

for all $t_1, t_2 \in \mathbf{R}^+$ and $\omega \in \Omega$. From the definition of $\hat{\Omega}$ and passing to the limit in (6) as $k \to \infty$, conclude that $\{\phi(t, \omega) : t > 0, \omega \in \Omega\}$, is an $\mathbf{R}^{d \times d}$ -valued *perfect* cocycle over θ , viz.

(i)
$$P(\hat{\Omega}) = 1;$$

- (ii) $\theta(t, \cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$ for all $t \ge 0$;
- (iii) $\phi(t_1 + t_2, \omega) = \phi(t_2, \theta(t_1, \omega))\phi(t_1, \omega)$ for all $t_1, t_2 \in \mathbf{R}^+$ and every $\omega \in \hat{\Omega}$;

(iv) $\phi(\cdot, \omega)$ is continuous for every $\omega \in \hat{\Omega}$.

Alternatively use the perfection theorem in ([M-S], AIHP, 1996, Theorem 3.1, p. 79-82) for crude cocycles with values in a metrizable second countable topological group. Observe that $\phi(t, \omega) \in GL(\mathbf{R}^d)$.

Define $\hat{H}: \mathbf{R}^+ \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ by

$$\hat{H}(t,v_1,v,\eta,\omega)$$

$$:=\phi(t,\omega)^{-1}[H(\phi_t(\cdot,\omega)(-d_1,v_1),\phi(t,\omega)(v),\phi_t(\cdot,\omega)\circ(id_J,\eta))]$$
(7)

for $\omega \in \Omega, t \ge 0, v, v_1 \in \mathbf{R}^d, \eta \in L^2([-r, 0], \mathbf{R}^d)$, where

$$\phi_t(\cdot,\omega)(s,v) = \begin{cases} \phi(t+s,\omega)(v) & t+s \ge 0 \\ \\ v & -r \le t+s < 0 \end{cases}$$

and

$$(id_J,\eta)(s) = (s,\eta(s)), \quad s \in J.$$

Define $\hat{H}^k : \mathbf{R}^+ \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ by a relation similar to (7) with ϕ replaced by ϕ^k . Then the random fde's

$$y'(t) = \hat{H}(t, y(t - d_1), y(t), y_t, \omega) \qquad t > 0
 (y(0), y_0) = (v, \eta) \in M_2$$
(8)

$$y^{k'}(t) = \hat{H}^{k}(t, y^{k}(t - d_{1}), y^{k}(t), y^{k}_{t}, \omega) \qquad t > 0$$

$$(y^{k}(0), y^{k}_{0}) = (v, \eta) \in M_{2}$$

$$(9)$$

have unique *non-explosive* solutions

$$y, y^k : [-r, \infty) \times \Omega \to \mathbf{R}^d$$

([Mo], Stochastics, 1990, pp. 93-98). Itô's formula implies that

$$X(t, v, \eta, \omega) = (\phi(t, \omega)(y(t, \omega)), \phi_t(\cdot, \omega) \circ (id_J, y_t))$$
(10)

The chain rule gives a similar relation for X^k with ϕ replaced by ϕ^k (*Exercise*; [Mo], Stochastics, 1990, pp. 96-97).

Get the convergence

$$\lim_{k \to \infty} |\hat{H}^k(t, v_1, v, \eta, \omega) - \hat{H}(t, v_1, v, \eta, \omega)| = 0$$
(11)

uniformly for (t, v_1, v, η) in bounded sets of $\mathbf{R}^+ \times \mathbf{R}^d \times M_2$. Use Gronwall's lemma and (11) to deduce (3).

Second step.

Fix $\omega \in \hat{\Omega}$ and use uniqueness of solutions to the approximating equation (VIII-k) and the helix property (2) of W^k to obtain the cocycle property for (X^k, θ) :

$$X^{k}(t_{2},\cdot,\theta(t_{1},\omega)) \circ X^{k}(t_{1},\cdot,\omega) = X^{k}(t_{1}+t_{2},\cdot,\omega)$$

for all $\omega \in \hat{\Omega}$ and $t_1, t_2 \ge 0, k \ge 1$.

Third step.

Pass to limit as $k \to \infty$ in the above identity and use the convergence (3) in operator norm to get the perfect cocycle property for X.

The a.s. Lyapunov exponents

$$\lim_{t \to \infty} \frac{1}{t} \log \| X(t, (v(\omega), \eta(\omega)), \omega) \|_{M_2},$$

(for a.a. $\omega \in \Omega$, $(v, \eta) \in L^2(\Omega, M_2)$) of the system (VIII) are characterized by the following "spectral theorem". Each $\theta(t, \cdot)$ is ergodic and preserves Wiener measure *P*. The proof of Theorem IV.2 below uses compactness of $X(t, \cdot, \omega) : M_2 \to M_2, t \ge r$, together with an infinitedimensional version of Oseledec's multiplicative ergodic theorem due to Ruelle (1982).

Theorem IV.2. ([Mo], Stochastics, 1990)

Let $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ be the flow of (VIII) given in Theorem III.4. Then there exist

- (a) an \mathcal{F} -measurable set $\Omega^* \subseteq \Omega$ such that $P(\Omega^*) = 1$ and $\theta(t, \cdot)(\Omega^*) \subseteq \Omega^*$ for all $t \ge 0$,
- (b) a fixed (non-random) sequence of real numbers $\{\lambda_i\}_{i=1}^{\infty}$, and
- (c) a random family $\{E_i(\omega) : i \ge 1, \omega \in \Omega^*\}$ of (closed) finite-codimensional subspaces of M_2 , with the following properties:

(i) If the Lyapunov spectrum $\{\lambda_i\}_{i=1}^{\infty}$ is infinite, then $\lambda_{i+1} < \lambda_i$ for all $i \ge 1$ and $\lim_{i \to \infty} \lambda_i = -\infty$; otherwise there is a fixed (non-random) integer $N \ge 1$ such that $\lambda_N = -\infty < \lambda_{N-1} < \cdots < \lambda_2 < \lambda_1$; (ii) each map $\omega \mapsto E_i(\omega), i \ge 1$, is \mathcal{F} -measurable into the Grassmannian of M_2 ;

- (iii) $E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2, i \ge 1, \omega \in \Omega^*;$
- (iv) for each $i \ge 1$, codim $E_i(\omega)$ is fixed independently of $\omega \in \Omega^*$;
- (v) for each $\omega \in \Omega^*$ and $(v, \eta) \in E_i(\omega) \setminus E_{i+1}(\omega)$,

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} = \lambda_i, \ i \ge 1;$$

(vi) Top Exponent:

$$\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \log \|X(t, \cdot, \omega)\|_{L(M_2)} \quad \text{for all } \omega \in \Omega^*;$$

(vii) Invariance:

$$X(t,\cdot,\omega)(E_i(\omega)) \subseteq E_i(\theta(t,\omega))$$

for all $\omega \in \Omega^*$, $t \ge 0$, $i \ge 1$.

Spectral Theorem

Proof of Theorem IV.2 is based on Ruelle's discrete version of Oseledec's multiplicative ergodic theorem in Hilbert space ([Ru], Ann. of Math. 1982, Theorem (1.1), p. 248 and Corollary (2.2), p. 253):

Theorem IV.3 ([Ru], 1982)

Let (Ω, \mathcal{F}, P) be a probability space and $\tau : \Omega \to \Omega$ a *P*-preserving transformation. Assume that *H* is a separable Hilbert space and $T : \Omega \to L(H)$ a measurable map (w.r.t. the Borel field on the space of all bounded linear operators L(H)). Suppose that $T(\omega)$ is compact for almost all $\omega \in \Omega$, and $E \log^+ ||T(\cdot)|| < \infty$. Define the family of linear operators $\{T^n(\omega) : \omega \in \Omega, n \ge 1\}$ by

$$T^{n}(\omega) := T(\tau^{n-1}(\omega)) \circ \cdots T(\tau(\omega)) \circ T(\omega)$$

for $\omega \in \Omega$, $n \ge 1$.

Then there is a set $\Omega_0 \in \mathcal{F}$ of full *P*-measure such that $\tau(\Omega_0) \subseteq \Omega_0$, and for each $\omega \in \Omega_0$, the limit

$$\lim_{n \to \infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm and is a positive compact self-adjoint operator on H. Furthermore each $\Lambda(\omega)$ has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

where the μ_i 's are distinct. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at $\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigen-space $F_i(\omega)$, with $m_i(\omega) := \dim F_i(\omega)$. Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\bigoplus_{j=1}^{i-1} F_j(\omega) \right]^{\perp}, \quad E_{\infty}(\omega) := \ker \Lambda(\omega).$$

Then

$$E_{\infty}(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = H$$

and

$$\lim_{n \to \infty} \frac{1}{n} \log \|T^n(\omega)x\|_H = \begin{cases} \mu_i(\omega), & \text{if } x \in E_i(\omega) \setminus E_{i+1}(\omega) \\ -\infty & \text{if } x \in \ker \Lambda(\omega). \end{cases}$$

Proof.

[Ru], Ann. of Math., 1982, pp. 248-254.

The following "perfect" version of Kingman's subadditive ergodic theorem is also used to construct the shift invariant set Ω^* appearing in Theorem IV.2 above.

Theorem IV.4([M, 1990])("Perfect" Subadditive Ergodic Theorem)

Let $f : \mathbf{R}^+ \times \Omega \to \mathbf{R} \cup \{-\infty\}$ be a measurable process on the complete probability space (Ω, \mathcal{F}, P) such that

(i) $E \sup_{0 \le u \le 1} f^+(u, \cdot) < \infty$, $E \sup_{0 \le u \le 1} f^+(1 - u, \theta(u, \cdot)) < \infty$; (ii) $f(t_1 + t_2, \omega) \le f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ for all $t_1, t_2 \ge 0$ and every $\omega \in \Omega$.

Then there exist a set $\hat{\hat{\Omega}} \in \mathcal{F}$ and a measurable $\tilde{f} : \Omega \to \mathbf{R} \cup \{-\infty\}$ with the properties:

(a)
$$P(\hat{\Omega}) = 1, \ \theta(t, \cdot)(\hat{\Omega}) \subseteq \hat{\Omega} \text{ for all } t \ge 0;$$

(b) $\tilde{f}(\omega) = \tilde{f}(\theta(t, \omega)) \text{ for all } \omega \in \hat{\Omega} \text{ and all } t \ge 0;$
(c) $\tilde{f}^+ \in \mathbf{L}^1(\Omega, \mathbf{R}; P);$
(d) $\lim_{t \to \infty} (1/t) f(t, \omega) = \tilde{f}(\omega) \text{ for every } \omega \in \hat{\Omega}.$

If θ is ergodic, then there exist $f^* \in \mathbf{R} \cup \{-\infty\}$ and $\tilde{\tilde{\Omega}} \in \mathcal{F}$ such that

(a)'
$$P(\tilde{\Omega}) = 1, \theta(t, \cdot)(\tilde{\Omega}) \subseteq \tilde{\Omega}, t \ge 0;$$

(b)' $\tilde{f}(\omega) = f^* = \lim_{t \to \infty} (1/t) f(t, \omega)$ for every $\omega \in \tilde{\tilde{\Omega}}.$

Proof.

[Mo], Stochastics, 1990, Lemma 7, pp. 115–117.

Proof of Theorem IV.2 is an application of Theorem IV.3. Requires Theorem IV.4 and the following sequence of lemmas.

Lemma 1

For each integer $k \geq 1$ and any $0 < a < \infty$,

$$E \sup_{0 \le t \le a} \|\phi(t,\omega)^{-1}\|^{2k} < \infty;$$
$$E \sup_{0 \le t_1, t_2 \le a} \|\phi(t_2,\theta(t_1,\cdot))\|^{2k} < \infty.$$

Proof.

Follows by standard sode estimates, the cocycle property for ϕ and Hölder's inequality. ([M], pp. 106-108).

The next lemma is a crucial estimate needed to apply Ruelle-Oseledec theorem (Theorem IV.3).

Lemma 2

 $E \sup_{0 \le t_1, t_2 \le r} \log^+ \|X(t_2, \cdot, \theta(t_1, \cdot))\|_{L(M_2)} < \infty.$

Proof.

If $y(t, (v, \eta), \omega)$ is the solution of the fde (8), then using Gronwall's inequality, taking $E \sup_{0 \le t_1, t_2 \le r} \log^+ \sup_{\|(v, \eta)\| \le 1}$ and applying Lemma 1, gives

$$E \sup_{0 \le t_1, t_2 \le r} \log^+ \sup_{\|(v,\eta)\| \le 1} \|(y(t_2, (v,\eta), \theta(t_1, \cdot)), y_{t_2}(\cdot, (v,\eta), \theta(t_1, \cdot)))\|_{M_2} < \infty.$$

Conclusion of lemma now follows by replacing ω' with $\theta(t_1, \omega)$ in the formula

$$X(t_2, (v, \eta), \omega')$$

= $(\phi(t_2, \omega')(y(t_2, (v, \eta), \omega')), \phi_{t_2}(\cdot, \omega') \circ (id_J, y_{t_2}(\cdot, (v, \eta), \omega')))$

and Lemma 1.

The existence of the Lyapunov exponents is obtained by interpolating the discrete limit

$$\frac{1}{r} \lim_{k \to \infty} \frac{1}{k} \log \|X(kr, (v(\omega), \eta(\omega)), \omega)\|_{M_2},$$
(12)

a.a. $\omega \in \Omega$, $(v, \eta) \in L^2(\Omega, M_2)$, between delay periods of length r. This requires the next two lemmas.

Lemma 3

Let $h: \Omega \to \mathbf{R}^+$ be \mathcal{F} -measurable and suppose $E \sup_{0 \le u \le r} h(\theta(u, \cdot))$ is finite. Then

$$\Omega_1 := \Bigl(\lim_{t \to \infty} \frac{1}{t} h(\theta(t, \cdot) = 0 \Bigr)$$

is a sure event and $\theta(t, \cdot)(\Omega_1) \subseteq \Omega_1$ for all $t \ge 0$.

Proof.

Use interpolation between delay periods and the discrete ergodic theorem applied to the L^1 function

$$\hat{h} := \sup_{0 \le u \le r} h(\theta(u, \cdot).$$

([Mo], Stochastics, 1990, Lemma 5, pp. 111-113.)

Lemma 4

Suppose there is a sure event Ω_2 such that $\theta(t, \cdot)(\Omega_2) \subseteq \Omega_2$ for all $t \ge 0$, and the limit (12) exists (or equal to $-\infty$) for all $\omega \in \Omega_2$ and all $(v, \eta) \in M_2$. Then there is a sure event Ω_3 such that $\theta(t, \cdot)(\Omega_3) \subseteq \Omega_3$ and

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} = \frac{1}{r} \lim_{k \to \infty} \frac{1}{k} \log \|X(kr, (v, \eta), \omega)\|_{M_2},$$
(13)

for all $\omega \in \Omega_3$ and all $(v, \eta) \in M_2$.

Proof:

Take $\Omega_3 := \hat{\Omega} \cap \Omega_1 \cap \Omega_2$. Use cocycle property for X, Lemma 2 and Lemma 3 to interpolate. ([Mo], Stochastics 1990, Lemma 6, pp. 113-114.)

Proof of Theorem IV.2. (Sketch)

Apply Ruelle-Oseledec Theorem (Theorem IV.3) with

$$T(\omega) := X(r, \omega) \in L(M_2)$$
, compact linear for $\omega \in \hat{\Omega}$;

$$\tau: \Omega \to \Omega; \quad \tau := \theta(r, \cdot).$$

Then cocycle property for X implies

$$X(kr,\omega,\cdot) = T(\tau^{k-1}(\omega)) \circ T(\tau^{k-2}(\omega)) \circ \cdots \circ T(\tau(\omega)) \circ T(\omega)$$
$$:= T^k(\omega)$$

for all $\omega \in \hat{\Omega}$.

Lemma 2 implies

$$E\log^+ \|T(\cdot)\|_{L(M_2)} < \infty.$$

Theorem IV.3 gives a random family of compact self-adjoint positive linear operators $\{\Lambda(\omega) : \omega \in \Omega_4\}$ such that

$$\lim_{n \to \infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm and is a positive compact operator on M_2 for $\omega \in \Omega_4$, a (continuous) shift-invariant set of full measure. Furthermore each $\Lambda(\omega)$ has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

where the μ'_i s are distinct, with no accumulation points except possibly $-\infty$. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at

 $\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigen-space $F_i(\omega)$, with $m_i(\omega) := \dim F_i(\omega)$. Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\bigoplus_{j=1}^{i-1} F_j(\omega) \right]^{\perp}, \quad E_{\infty}(\omega) := \ker \Lambda(\omega).$$

Then

$$E_{\infty}(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2$$

Note that $\operatorname{codim} E_i(\omega) = \sum_{j=1}^{i-1} m_j(\omega) < \infty$. Also

$$\lim_{k \to \infty} \frac{1}{k} \log \|X(kr, (v, \eta), \omega)\|_{M_2} = \begin{cases} \mu_i(\omega), \text{ if } (v, \eta) \in E_i(\omega) \setminus E_{i+1}(\omega) \\ -\infty & \text{ if } (v, \eta) \in \ker \Lambda(\omega). \end{cases}$$

The functions

$$\omega \mapsto \mu_i(\omega), \quad \omega \mapsto m_i(\omega), \quad \omega \mapsto N(\omega)$$

are invariant under the ergodic shift $\theta(r, \cdot)$. Hence they take the fixed values μ_i , m_i , N almost surely, respectively.

Lemma 4 gives a continuous-shift-invariant sure event $\Omega^* \subseteq \Omega_4$ such that

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} = \frac{1}{r} \lim_{k \to \infty} \frac{1}{k} \log \|X(kr, (v, \eta), \omega)\|_{M_2}$$
$$= \frac{\mu_i}{r} =: \lambda_i,$$

for $(v,\eta) \in E_i(\omega) \setminus E_{i+1}(\omega), \ \omega \in \Omega^*, i \ge 1.$

 $\{\lambda_i := \frac{\mu_i}{r} : i \ge 1\}$ is the Lyapunov spectrum of (VIII).

Since Lyapunov spectrum is discrete with no finite accumulation points, then $\{\lambda_i : \lambda_i > \lambda\}$ is finite for all $\lambda \in \mathbf{R}$.

To prove invariance of the Oseledec space $E_i(\omega)$ under the cocycle (X, θ) use the random field

$$\lambda((v,\eta),\omega) := \lim_{t \to \infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2} \qquad (v,\eta) \in M_2, \quad \omega \in \Omega^*$$

and the relations

$$E_i(\omega) := \{ (v,\eta) \in M_2 : \lambda((v,\eta),\omega) \le \lambda_i \},$$
$$\lambda(X(t,(v,\eta),\omega), \theta(t,\omega)) = \lambda((v,\eta),\omega), \quad \omega \in \Omega^*, \ t \ge 0$$

([Mo], Stochastics 1990, p. 122).

The non-random nature of the Lyapunov exponents $\{\lambda_i\}_{i=1}^{\infty}$ of (VIII) is a consequence of the fact the θ is ergodic. (VIII) is said to be *hyperbolic* if $\lambda_i \neq 0$ for all $i \geq 1$. When (VIII) is hyperbolic the flow satisfies a *stochastic saddle-point property* (or exponential dichotomy) (cf. the deterministic case with $E = C([-r, 0], \mathbf{R}^d), g_i \equiv 0, i = 1, ..., m$, in Hale [H], Theorem 4.1, p. 181).

Theorem IV.5 (Random Saddles)([Mo], Stochastics, 1990)

Suppose the sfde (VIII) is hyperbolic. Then there exist

- (a) a set $\tilde{\Omega}^* \in \mathcal{F}$ such that $P(\tilde{\Omega}^*) = 1$, and $\theta(t, \cdot)(\tilde{\Omega}^*) = \tilde{\Omega}^*$ for all $t \in \mathbf{R}$, and
- (b) a measurable splitting

$$M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \tilde{\Omega}^*,$$

with the following properties:

- (i) $\mathcal{U}(\omega), \mathcal{S}(\omega), \omega \in \tilde{\Omega}^*$, are closed linear subspaces of M_2 , dim $\mathcal{U}(\omega)$ is finite and fixed independently of $\omega \in \tilde{\Omega}^*$.
- (ii) The maps $\omega \mapsto \mathcal{U}(\omega), \ \omega \mapsto \mathcal{S}(\omega)$ are \mathcal{F} -measurable into the Grassmannian of M_2 .
- (iii) For each $\omega \in \tilde{\Omega}^*$ and $(v, \eta) \in \mathcal{U}(\omega)$ there exists $\tau_1 = \tau_1(v, \eta, \omega) > 0$ and a positive δ_1 , independent of (v, η, ω) such that

$$||X(t,(v,\eta),\omega)||_{M_2} \ge ||(v,\eta)||_{M_2} e^{\delta_1 t}, \quad t \ge \tau_1.$$

(iv) For each $\omega \in \tilde{\Omega}^*$ and $(v, \eta) \in S(\omega)$ there exists $\tau_2 = \tau_2(v, \eta, \omega) > 0$ and a positive δ_2 , independent of (v, η, ω) such that

$$\|X(t,(v,\eta),\omega)\|_{M_2} \le \|(v,\eta)\|_{M_2} e^{-\delta_2 t}, \quad t \ge \tau_2.$$

(v) For each $t \ge 0$ and $\omega \in \tilde{\Omega}^*$,

$$X(t,\omega,\cdot)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t,\omega)),$$
$$X(t,\omega,\cdot)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)).$$

In particular, the restriction

$$X(t,\omega,\cdot) | \mathcal{U}(\omega) : \mathcal{U}(\omega) \to \mathcal{U}(\theta(t,\omega))$$

is a linear homeomorphism onto.

Proof.

[Mo], Stochastics, 1990, Corollary 2, pp. 127-130.

The Stable Manifold Theorem

5. Regular Linear Systems. Helix Noise

$$dx(t) = \left\{ \int_{[-r,0]} \nu(t)(ds) \, x(t+s) \right\} dt + dN(t) \, \int_{-r}^{0} K(t)(s) \, x(t+s) \, ds + dL(t) \, x(t-), \quad t > 0 \\ (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d) \right\}$$
(IX)

Linear systems driven by helix semimartingale noise, and memory driven by a measure-valued process ν on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}}, P)$.

Hypotheses (C)

(i) The processes ν , K are stationary ergodic in the sense that there is a measurable ergodic P-preserving flow $\theta : \mathbf{R} \times \Omega \rightarrow \Omega$ such that for each $t \in \mathbf{R}, \ \mathcal{F}_t = \theta(t, \cdot)^{-1}(\mathcal{F}_0)$ and

$$\nu(t,\omega) = \nu(0,\theta(t,\omega)), \quad t \in \mathbf{R}, \ \omega \in \Omega$$
$$K(t,\omega) = K(0,\theta(t,\omega)), \quad t \in \mathbf{R}, \ \omega \in \Omega.$$

(ii) L = M + V, M continuos local martingale, V B.V. process. The processes N, L, M have jointly stationary ergodic increments:

$$N(t+h,\omega) - N(t,\omega) = N(h,\theta(t,\omega)),$$
$$L(t+h,\omega) - L(t,\omega) = L(h,\theta(t,\omega)),$$
$$M(t+h,\omega) - M(t,\omega) = M(h,\theta(t,\omega)),$$

for
$$t \in \mathbf{R}$$
, $\omega \in \Omega$.

Semimartingales satisfying Hypothesis (C)(ii) were studied by de Sam Lazaro and P.A. Meyer ([S-M], 1971, 1976), Çinlar, Jacod, Protter and Sharpe [CJPS], Protter [P], 1986.

Equation (IX) is regular w.r.t. M_2 with a measurable flow X: $\mathbf{R}^+ \times M_2 \times \Omega \to M_2$. This flow satisfies Theorems III.4 and the cocycle property. This is achieved via a construction in ([M-S], AIHP, 1996) based on the following consequence of Hypothesis (C)(ii):

Theorem IV.6 ([Mo], Survey paper, 1992, [M-S], AIHP, 1996)

Suppose M satisfies Hypothesis (C)(ii). Then there is an $(\mathcal{F}_t)_{t\geq 0}$ -adapted version $\phi: \mathbf{R}^+ \times \Omega \to \mathbf{R}^{d \times d}$ of the solution to the matrix equation

$$\begin{cases} d\phi(t) = dM(t)\phi(t) & t > 0 \\ \phi(0) = I \in \mathbf{R}^{d \times d} \end{cases}$$
 (X)

and a set $\Omega_1 \in \mathcal{F}$ such that

(i) P(Ω₁) = 1;
(ii) θ(t, ·)(Ω₁) ⊆ Ω₁ for all t ≥ 0;
(iii) φ(t₁ + t₂, ω) = φ(t₂, θ(t₁, ω))φ(t₁, ω) for all t₁, t₂ ∈ **R**⁺ and every ω ∈ Ω₁;
(iv) φ(·, ω) is continuous for every ω ∈ Ω₁.

A proof of Theorem IV.6 is given in ([Mo], Survey, 1992; [M-S], AIHP, 1996): either by a double-approximation argument or via perfection techniques.

The existence of a discrete non-random Lyapunov spectrum $\{\lambda_i\}_{i=1}^{\infty}$ for the sfde (IX) is proved via Ruelle-Oseledec multiplicative ergodic theorem which requires the integrability property (Lemma 2):

$$E \sup_{0 \le t_1, t_2 \le r} \log^+ \|X(t_1, \theta(t_2, \cdot), \cdot)\|_{L(M_2)} < \infty.$$

The above integrability property is established under the following set of hypotheses on ν , K, N, L:

Hypotheses (I)

(i)

$$\sup_{\substack{-r \leq s \leq 2r \\ 0 \leq t \leq 2r, -r \leq s \leq 0}} \left\| \frac{d\bar{\nu}(\cdot)(s)}{ds} \right\|^2, \quad \sup_{\substack{0 \leq t \leq 2r, -r \leq s \leq 0 \\ 0 \leq t \leq 2r, -r \leq s \leq 0}} \|\frac{\partial}{\partial t} K(t, \cdot)(s)\|^3, \quad \sup_{\substack{0 \leq t \leq 2r, -r \leq s \leq 0 \\ \{|V|(2r, \cdot)\}^4,}} \|\frac{\partial}{\partial s} K(t, \cdot)(s)\|^3,$$

are all integrable, where

$$\bar{\nu}(\omega)(A) := \int_0^\infty |\nu(t,\omega)| \{ (A-t) \cap [-r,0] \} dt, \quad A \in Borel[-r,\infty)$$

has a locally (essentially) bounded density $\frac{d\bar{\nu}(\cdot)(s)}{ds}$; and |V| = total variation of V w.r.t. the Euclidean norm $\|\cdot\|$ on $\mathbf{R}^{d\times d}$.

(ii) Let $N = N^0 + V^0$ where the local $(\mathcal{F}_t)_{t \ge 0}$ -martingale $N^0 = (N_{ij}^0)_{i,j=1}^d$ and the bounded variation process

 $V^0 = (V^0_{ij})^d_{i,j=1}$ are such that

$$\{[N_{ij}^0](2r,\cdot)\}^2, \{|V_{ij}^0|(2r,\cdot)\}^4, i, j = 1, 2, \dots, d\}$$

are integrable.

 $|V_{ij}^0|(2r, \cdot) = \text{total variation of } V_{ij}^0 \text{ over } [0, 2r].$

(iii) $[M_{ij}](1) \in L^{\infty}(\Omega, \mathbf{R}), \quad i, j = 1, 2, \dots, d.$

The integrability property of the cocycle (X,θ) is a consequence of

$$E \log^{+} \sup_{0 \le t_{1}, t_{2} \le r, \, \|(v,\eta)\| \le 1} |x(t_{1}, (v,\eta), \theta(t_{2}, \cdot))| < \infty$$

Proof of latter property uses lengthy argument based on establishing the existence of suitable higher order moments for the coefficients of an associated random integral equation. (See Lemmas (5.1)-(5.5) in [M-S],I, AIHP, 1996.)

Since θ is ergodic, the multiplicative ergodic theorem (Theorem IV.3, Ruelle) now gives a fixed discrete set of Lyapunov exponents

Theorem IV.7 ([Mo], Survey, 1992; [M-S], AIHP, 1996)

Under Hypotheses (C) & (I), the statements of Theorems IV.2 and IV.5 hold true for the linear sfde (IX).

Note that the Lyapunov spectrum of (IX) does not change if one uses the state space $D([-r, 0], \mathbf{R}^d)$ with the supremum norm $\|\cdot\|_{\infty}$ ([M-S], AIHP 1996).