STABILITY:

EXAMPLES AND CASE STUDIES

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STABILITY. EXAMPLES AND CASE STUDIES

1. Plan.

I) Estimates on the "maximal exponential growth rate" for the singular noisy feedback loop. Use of Lyapunov functionals.

II) Examples and case studies of linear sfde's: Existence of the stochastic semiflow and its Lyapunov spectrum.

III) Study almost sure asymptotic stability via upper bounds on the top Lyapunov exponent λ_1.

IV) Lyapunov spectrum for sdde's with Poisson noise.

Lyapunov exponents for linear sode's (without memory): studied by many authors: e.g. Arnold, Kliemann and Oeljeklaus, 1989, Arnold, Oeljeklaus and Pardoux, 1986, Baxendale, 1985, Pardoux and Wihstutz[PW1], 1988, Pinsky and Wihstutz [PW2], 1988, and the references therein.

Asymptotic stability of sfde's: treated in Kushner [K], JDE, 1968, Mizel and Trutzer [MT],1984, Mohammed [M1]-[M4], 1984, 1986, 1990, 1992, Mohammed and Scheutzow [MS], 1996, Scheutzow [S], 1988, Kolmanovskii and Nosov [KN], 1986. Mao ([Ma], 1994, Chapter 5) gives several results concerning top exponential growth rate for sdde's driven by C-valued semimartingales. Assumes that second-order characteristics of the driving semimartingales are *time-dependent* and *decay to zero exponentially fast in time, uniformly in the space variable.*

2. Noisy Feedback Loop Revisited Once More!

Noisy feedback loop is modelled by the one-dimensional linear sdde

$$\left.\begin{aligned} dx(t) &= \sigma x(t-r))\, dW(t), \quad t>0 \\ (x(0),x_0) &= (v,\eta)\in M_2 := \mathbf{R}\times L^2([-r,0],\mathbf{R}), \end{aligned}\right\} \quad (I)$$

driven by a Wiener process W with a *positive delay* r.

(I) is singular with respect to M_2 (Theorem III.3).

Consider the more general one-dimensional linear sfde:

$$\left.\begin{aligned} dx(t) &= \int_{-r}^{0} x(t+s)d\nu(s)\, dW(t), \quad t>0 \\ (x(0),x_0) &\in M_2 := \mathbf{R}\times L^2([-r,0],\mathbf{R}) \end{aligned}\right\} \quad (II')$$

where W is a Wiener process and ν is a fixed finite real-valued Borel measure on $[-r, 0]$.

(II') is regular if ν has a C^1 (or even L_1^2) density with respect to Lebesgue measure on $[-r, 0]$ ([M-S], I, 1996). If ν satisfies Theorem III.3, then (II') is singular.

In the singular case, there is no stochastic flow (Theorem III.3) and we do not know whether a (discrete) set of Lyapunov exponents

$$\lambda((v,\eta),\cdot) := \lim_{t\to\infty} \frac{1}{t} \log \|(x(t,(v,\eta)), x_t(\cdot,(v,\eta)))\|_{M_2}, \qquad (v,\eta) \in M_2$$

exists. Existence of Lyapunov exponents for singular equations is hard. But can still define the *maximal exponential growth rate*

$$\overline{\lambda}_1 := \sup_{(v,\eta)\in M_2} \limsup_{t\to\infty} \frac{1}{t} \log \|(x(t,(v,\eta)), x_t(\cdot,(v,\eta)))\|_{M_2}$$

for the trajectory random field $\{(x(t,(v,\eta)), x_t(\cdot,(v,\eta))):$ $t \geq 0, (v,\eta) \in M_2\}$. $\overline{\lambda}_1$ may depend on $\omega \in \Omega$. But $\overline{\lambda}_1 = \lambda_1$ in the regular case.

Inspite of the *extremely erratic dependence on the initial paths* of solutions of (I), it is shown in Theorem V.1 that for small noise variance, *uniform almost sure global asymptotic stability* still persists. For small σ, $\overline{\lambda}_1 \leq -\sigma^2/2 + o(\sigma^2)$ uniformly in the initial path (Theorem V.1, and Remark (iii)). For large $|\sigma|$ and $\nu = \delta_{-r}$,

$$\frac{1}{2r}\log|\sigma| + o(\log|\sigma|) \leq \overline{\lambda}_1 \leq \frac{1}{r}\log|\sigma|$$

([M-S], II, 1996, Remark (ii) after proof ofTheorem 2.3). This result is in sharp contrast with the non-delay case ($r=0$), where $\lambda_1 = -\sigma^2/2$ *for all* values of σ. Proofs of Theorems V.1, V.2

involve very delicate constructions of new types of Lyapunov functionals on the underlying state space.

Theorem V.1.([M-S], II, 1996).

Let ν be a probability measure on $[-r,0]$, $r>0$, and consider the sfde

$$\left.\begin{aligned} dx(t) &= \sigma\left(\int_{[-r,0]} x(t+s)\,d\nu(s)\right) dW(t), \quad t \geq 0 \\ (x(0),x_0) &= (v,\eta) \in M_2 \end{aligned}\right\} \qquad (II')$$

with $\sigma \in \mathbf{R}$, $(v,\eta) \in M_2$, W standard Brownian motion, and $x(\cdot,(v,\eta))$ the solution of (II') through $(v,\eta) \in M_2$. Then there exists $\sigma_0 > 0$ and a continuous strictly negative nonrandom function $\phi : (-\sigma_0,\sigma_0) \to \mathbf{R}^-$ (independent of $(v,\eta) \in M_2$ and ν) such that

$$P\left(\limsup_{t\to\infty} \frac{1}{t} \log \|(x(t,(v,\eta)), x_t(\cdot,(v,\eta)))\|_{M_2} \leq \phi(\sigma)\right) = 1.$$

for all $(v, \eta) \in M_2$ *and all* $-\sigma_0 < \sigma < \sigma_0$.

Remark:

Theorem also holds for state space C with $\|\cdot\|_\infty$.

Proof of Theorem V.1. (Sketch)

Sufficient to consider (II') on $C \equiv C([-r, 0], \mathbf{R})$, because C is continuously embedded in M_2. W.l.o.g., assume that $\sigma > 0$.

- Use Lyapunov functional $V : C \to \mathbf{R}^+$

$$V(\eta) := (R(\eta) \vee |\eta(0)|)^\alpha + \beta R(\eta)^\alpha, \quad \eta \in C.$$

where $R(\eta) := \overline{\eta} - \underline{\eta}$, the diameter of the range of η, $\overline{\eta} := \sup_{-r \le s \le 0} \eta(s)$ and $\underline{\eta} := \inf_{-r \le s \le 0} \eta(s)$.

- Fix $0 < \alpha < 1$ and *arrange* for $\beta = \beta(\sigma)$ for sufficiently small σ such that

$$E(V({}^{\eta}x_r)) \leq \delta(\sigma)V(\eta), \quad \eta \in C, \tag{1}$$

and $\delta(\sigma) \in (0,1)$ is a continuous function of σ defined near 0. There is a positive $K = K(\alpha)$ (independent of η, ν) such that $\delta(\sigma) \sim (1 - K\sigma^2)$. Set

$$\phi(\sigma) := \frac{1}{\alpha}\log\delta(\sigma).$$

Estimate (1) is hard ([M-S], II, 1996, pp. 12-18).

- $\{{}^{\eta}x_{nr}\}_{n=1}^{\infty}$ is a Markov process in C. So (1) implies that $\delta(\sigma)^{-n}V({}^{\eta}x_{nr})$, $n \geq 1$, is a non-negative $(\mathcal{F}_{nr})$ supermartingale.

- There exists $Z : \Omega \to [0, \infty)$ such that

$$\lim_{n\to\infty} \frac{V({}^{\eta}x_{nr})}{\delta(\sigma)^n} = Z \qquad \text{a.s.} \tag{2}$$

- Form of V and (2) imply

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log|x(t)| \leq \overline{\lim_{n\to\infty}}\frac{1}{nr}\log[|x(nr)|+R(x_{nr})]$$
$$= \frac{1}{\alpha}\overline{\lim_{n\to\infty}}\frac{1}{nr}\log V(x_{nr}) \leq \frac{1}{\alpha}\log\delta(\sigma) = \phi(\sigma) < 0.$$

- $\delta(\sigma)$, $\phi(\sigma)$ independent of η, ν. "Domain" of ϕ also independent of η, ν. □

Remarks.

(i) Choice of σ_0 in Theorem V.1 depends on r. In (I) the scaling $t \mapsto t/r$ has the effect of replacing r by 1 and σ by $\sigma\sqrt{r}$. If $\overline{\lambda}_1(r,\sigma)$ is the maximal exponential growth rate of (I), then $\overline{\lambda}_1(r,\sigma) = \frac{1}{r}\overline{\lambda}_1(1,\sigma\sqrt{r})$ (*Exercise*). Hence σ_0 decreases (like $\frac{1}{\sqrt{r}}$) as r increases . Thus (for a fixed σ), a *small delay* r tends to *stabilize* equation (I). A *large delay* in (I) has a *destabilizing* effect (Theorem V.2 below).

(ii) Using a Lyapunov function(al) argument, Theorem V.2 below shows that for sufficiently large σ, the singular delay equation (I) is unstable. Result is in sharp contrast with the non-delay case $r = 0$, where

$$\lim_{t\to\infty} \frac{1}{t} \log |x(t)| = -\sigma^2/2 < 0$$

for *all* $\sigma \in \mathbf{R}$ (even when σ is large).

(iii) The growth rate function ϕ in Theorem V.1 satisfies

$$\phi(\sigma) = -\sigma^2/2 + o(\sigma^2)$$

as $\sigma \to 0^+$. Agrees with non-delay case $r = 0$. Above relation follows by modifying proof of Theorem V.1.

Theorem V.2.

Consider the equation

$$\left.\begin{aligned} dx(t) = \sigma x(t-1)\,dW(t), \quad t > 0 \\ (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R} \times L^2([-r,0], \mathbf{R}), \end{aligned}\right\} \qquad (I)$$

driven by a standard Wiener process W *with a positive delay* r *and* $\sigma \in \mathbf{R}$ *. Then there exists a continuous function* $\psi : (0,\infty) \to \mathbf{R}$ *which is increasing to infinity such that*

$$P\left(\liminf_{t\to\infty} \frac{1}{t} \log \|(x(t,(v,\eta)), x_t(\cdot,(v,\eta))\|_{M_2} \geq \psi(|\sigma|)\right) = 1,$$

for all $(v,\eta) \in M_2 \backslash \{0\}$ *and all sufficiently large* $|\sigma|$*. The function* ψ *is independent of the choice of* $(v,\eta) \in M_2 \backslash \{0\}$*.*

Remarks.

(i) $\|\cdot\|_{M_2}$ can be replaced by the sup-norm on C.

(ii) Proof shows $\psi(\sigma) \sim \frac{1}{2}\log\sigma$ for large σ.

Proof of Theorem V.2.

Use the continuous Lyapunov functional

$$V : M_2\backslash\{0\} \to [0,\infty)$$

$$V((v,\eta)) := \left(v^2 + |\sigma| \int_{-1}^{0} \eta^2(s)\, ds\right)^{-1/4}$$

[M-S], Part II, 1996, pp. 20-24. □

3. Regular one-dimensional linear sfde's

To outline a general scheme for obtaining estimates on the top Lyapunov exponent for a class of one-dimensional regular linear sfde's. Then apply scheme to specific examples within the above class.

Scheme applies to multidimensional linear equations with multiple delays.

Note: Approach in ([Ku], JDE, 1968) uses Lyapunov functionals and yields strictly weaker estimates in all cases.

Consider the class of one-dimensional linear sfde's

$$\left.\begin{aligned} dx(t) = \Big\{\nu_1 x(t) + \mu_1 x(t-r) + \int_{-r}^{0} x(t+s)\sigma_1(s)\,ds\Big\}\,dt \\ + \Big\{\nu_2 x(t) + \int_{-r}^{0} x(t+s)\sigma_2(s)\,ds\Big\}\,dM(t), \end{aligned}\right\} \quad (XVII)$$

where $r > 0, \sigma_1, \sigma_2 \in C^1([-r,0], \mathbf{R})$, and M is a continuous helix local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with (stationary) ergodic increments. Ergodic theorem gives the a.s. deterministic limit $\beta := \lim_{t\to\infty} \frac{\langle M\rangle(t)}{t}$. Assume that $\beta < \infty$ and $\langle M\rangle(1) \in L^\infty(\Omega, \mathbf{R})$.

Hence (XVII) is regular with respect to M_2 and has a sample-continuous stochastic semiflow

$X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ (Theorem III.5). The stochastic semiflow X has a fixed (non-random) Lyapunov spectrum (Theorem IV.7). Let λ_1 be its top exponent. We wish to develop an upper bound for λ_1. By the spectral theorem (Theorem IV.7, cf. Theorem IV.2), there is a shift-invariant set $\Omega^* \in \mathcal{F}$ of full P-measure and a measurable random field $\lambda : M_2 \times \Omega \to \mathbf{R} \cup \{-\infty\}$,

$$\lambda((v,\eta),\omega) := \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2}, \quad (v,\eta) \in M_2,\ \omega \in \Omega^*, \tag{1}$$

giving the Lyapunov spectrum of (XVII).

Introduce family of equivalent norms

$$\|(v,\eta)\|_\alpha := \left\{ \alpha v^2 + \int_{-r}^0 \eta(s)^2\, ds \right\}^{1/2}, \quad (v,\eta) \in M_2, \quad \alpha > 0, \tag{2}$$

on M_2. Then

$$\lambda((v,\eta),\omega) = \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_\alpha, \quad (v,\eta) \in M_2,\ \omega \in \Omega^* \tag{3}$$

for all $\alpha > 0$; i.e. the Lyapunov spectrum of (XVII) with respect to $\|\cdot\|_\alpha$ is independent of $\alpha > 0$.

Let x be the solution of (XVII) starting at $(v,\eta) \in M_2$. Define

$$\rho_\alpha(t)^2 := \|X(t)\|_\alpha^2 = \alpha x(t)^2 + \int_{t-r}^{t} x(u)^2\, du, \quad t > 0, \quad \alpha > 0. \tag{4}$$

For each fixed $(v,\eta) \in M_2$, define the set $\Omega_0 \in \mathcal{F}$ by $\Omega_0 := \{\omega \in \Omega : \rho_\alpha(t,\omega) \neq 0 \text{ for all } t > 0\}$. If $P(\Omega_0) = 0$, then by uniqueness there is a random time τ_0 such that a.s. $X(t,(v,\eta),\cdot) = 0$ for all $t \geq \tau_0$.

Hence $\lambda_1 = -\infty$. So suppose that $P(\Omega_0) > 0$. Itô's formula implies

$$\log \rho_\alpha(t) = \log \rho_\alpha(0) + \int_0^t Q_\alpha(a(u), b(u), I_1(u))\, du + \int_0^t \tilde{Q}_\alpha(a(u), I_2(u))\, d\langle M\rangle(u) + \int_0^t R_\alpha(a(u), I_2(u))\, dM(u), \tag{5}$$

for $t > 0$, a.s. on Ω_0, where

$$\left.\begin{aligned} Q_\alpha(z_1, z_2, z_3) &:= \nu_1 z_1^2 + \sqrt{\alpha}\, \mu_1 z_1 z_2 + \sqrt{\alpha}\, z_1 z_3 + \tfrac{1}{2}\frac{z_1^2}{\alpha} - \tfrac{1}{2} z_2^2 \\ \tilde{Q}_\alpha(z_1, z_3') &:= \alpha(\tfrac{1}{2} - z_1^2)\left(\frac{\nu_2}{\sqrt{\alpha}}\, z_1 + z_3'\right)^2 \\ R_\alpha(z_1, z_3') &:= \nu_2 z_1^2 + \sqrt{\alpha} z_1 z_3', \quad \|\sigma_i\|_2 := \left\{\int_{-r}^0 \sigma_i(s)^2 ds\right\}^{1/2}, \end{aligned}\right\} \tag{6}$$

$i = 1, 2$, and

$$a(t) := \frac{\sqrt{\alpha} x(t)}{\rho_\alpha(t)}, \quad b(t) := \frac{x(t-r)}{\rho_\alpha(t)}, \quad I_i(t) := \frac{\int_{-r}^0 x(t+s)\sigma_i(s)\, ds}{\rho_\alpha(t)} \tag{7}$$

for $i = 1, 2$, $t > 0$, a.s. on Ω_0.

Since

$$|I_i(t)| \le \frac{1}{\rho_\alpha(t)} \left(\int_{-r}^{0} x(t+s)^2 \, ds \right)^{1/2} \|\sigma_i\|_2 = \sqrt{1 - a^2(t)} \, \|\sigma_i\|_2,$$

$i = 1, 2,$ a.s. on Ω_0 the variables z_1, z_2, z_3, z_3' in (6) must satisfy

$$|z_1| \le 1, \;\; z_2 \in \mathbf{R}, \;\; |z_3|^2 \le (1-z_1^2)\|\sigma_1\|_2^2, \;\; |z_3'|^2 \le (1-z_1^2)\|\sigma_2\|_2^2.$$

Let $\tau_1 := \inf\{t > 0 : \rho_\alpha(t) = 0\}$. Then the local martingale

$$\int_0^{t \wedge \tau_1} R_\alpha(a(u), I_2(u)) \, dM(u), \;\; t > 0,$$

is a time-changed (possibly stopped) Brownian motion. Since $|R_\alpha(a(u), I_2(u))| \le |\nu_2| + \sqrt{\alpha}\|\sigma_2\|_2$ for all $u \in [0, \tau_1)$, a.s., then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^{t \wedge \tau_1} R_\alpha(a(u), I_2(u)) \, dM(u) = 0 \qquad \text{a.s.} \tag{8}$$

Divide (5) by t, let $t \to \infty$, to get

$$\lambda((v,\eta),\omega) \le \limsup_{t\to\infty} \frac{1}{t} \int_0^t Q_\alpha(a(u), b(u), I_1(u))\, du + \limsup_{t\to\infty} \frac{1}{t} \int_0^t \tilde{Q}_\alpha(a(u), I_2(u))\, d\langle M\rangle(u). \tag{9}$$

a.s. on Ω_0, for all $\alpha > 0$.

Wish to develop upper bounds on λ_1 in the following cases.

One-dimensional linear sfde (smooth memory in white-noise term):

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}\, dt + \left\{\int_{-r}^0 x(t+s)\sigma_2(s)\, ds\right\} dW(t), \qquad t > 0 \tag{VII}$$

with real constants ν_1, μ_1 and $\sigma_2 \in C^1([-r,0],\mathbf{R})$. It is a special case of (XVII). Hence (VII) is regular with respect to M_2. The process $\int_{-r}^0 x(t+s)\sigma_2(s)\, ds$

has C^1 paths in t. Hence the stochastic differential dW in (VII) may be interpreted in the Itô or Stratonovich sense *without changing the solution* x.

Theorem V.3.

Suppose λ_1 *is the top a.s. Lyapunov exponent of (VII). Define the function*

$$\theta(\delta,\alpha) := -\delta + \left(\nu_1 + \delta + \frac{1}{2}\alpha\mu_1^2 e^{2\delta r} + \frac{1}{2\alpha}\right) \vee \left(\frac{\alpha}{2}\|\sigma_2\|_2^2 e^{2\delta^+ r}\right)$$

for all $\alpha \in \mathbf{R}^+, \delta \in \mathbf{R}$, *where* $\delta^+ := \max\{\delta, 0\}$.

Then

$$\lambda_1 \leq \inf\{\theta(\delta,\alpha) : \delta \in \mathbf{R},\ \alpha \in \mathbf{R}^+\}. \tag{10}$$

Proof.

Maximize the integrand on the right-hand-side of (9) (with $M = W$); then use exponential shift by δ to refine the resulting estimate. Then minimize over α, δ ([M-S], II, 1996, pp. 34-35).

□

Corollary below shows that the estimate in Theorem V.3 reduces to well-known estimate in deterministic case $\sigma_2 \equiv 0$ (Hale [Ha], pp.17-18).

Corollary V.3.1.

In (VII), suppose $\mu_1 \neq 0$ and let δ_0 be the unique real solution of the transcendental equation

$$\nu_1 + \delta + |\mu_1| e^{\delta r} = 0. \tag{11}$$

Then

$$\lambda_1 \leq -\delta_0 + \frac{1}{2} \frac{\|\sigma_2\|_2^2}{|\mu_1|} e^{|\delta_0| r}. \tag{12}$$

If $\mu_1 = 0$ *and* $\nu_1 \geq 0$*, then* $\lambda_1 \leq \frac{1}{2}\big(\nu_1 + \sqrt{\nu_1^2 + \|\sigma_2\|_2^2}\big)$*. If* $\mu_1 = 0$ *and* $\nu_1 < 0$*, then* $\lambda_1 \leq \nu_1 + \frac{1}{2}\|\sigma_2\|_2 e^{-\nu_1 r}$*.*

Proof.

Suppose $\mu_1 \neq 0$. Denote by $f(\delta)$, $\delta \in \mathbf{R}$, the left-hand-side of (11). Then $f(\delta)$ is an increasing function of δ. f has a unique real zero δ_0. Using (10), we may put $\delta = \delta_0$ and $\alpha = |\mu_1|^{-1}e^{-\delta_0 r}$ in the expression for $\theta(\delta, \alpha)$. This gives (12).

Suppose $\mu_1 = 0$. Put $\delta = (-\nu_1)^+$ in $\theta(\delta, \alpha)$ and minimize the resulting expression over all $\alpha > 0$. This proves the last two assertions of the corollary ([M-S], II, 1996, pp. 35-36). $\square$

Remarks.

(i) Upper bounds for λ_1 in Theorem (V.3) and Corollary V.3.1 agree with corresponding bounds in the deterministic case (for $\mu_1 \geq 0$), but are not optimal when $\mu_1 = 0$ and σ_2 is strictly positive and sufficiently small; cf. Theorem V.1 for small $\|\sigma_2\|_2$.

(ii) *Problem:* What are the asymptotics of λ_1 for small delays $r \downarrow 0$?

Our second example is the stochastic delay equation

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}\, dt + x(t) dM(t), \quad t > 0, \qquad (XVIII)$$

where M is the helix local martingale appearing in (XVII) and satisfying the conditions therein. Hence (XVIII) is regular with respect to M_2. Theorem below gives estimate on its top exponent.

Theorem V.4.

In (XVIII) define δ_0 as in Corollary V.3.1. Then the top a.s. Lyapunov exponent λ_1 of (XVIII) satisfies

$$\lambda_1 \leq -\delta_0 + \frac{\beta}{16}. \tag{13}$$

Proof.

Maximize the following functions separately over their appropriate ranges:

$$Q_\alpha(z_1, z_2) := \nu_1 z_1^2 + \sqrt{\alpha}\, \mu_1 z_1 z_2 + \tfrac{1}{2}\frac{z_1^2}{\alpha} - \tfrac{1}{2} z_2^2,$$
$$\tilde{Q}_\alpha(z_1) := (\tfrac{1}{2} - z_1^2) z_1^2, \quad |z_1| \leq 1,\ z_2 \in \mathbf{R}.$$

Then use an exponential shift of the Lyapunov spectrum by an amount δ. Minimize the resulting bound over all α (for fixed δ) and then over all $\delta \in \mathbf{R}$. This minimum is attained if δ solves the transcendental equation (11). Hence the conclusion of the theorem ([M-S], II, 1996, pp. 36-37).

□

Remark.

The above estimate for λ_1 is sharp in the deterministic case $\beta = 0$ and $\mu_1 \geq 0$, but is not sharp when $\beta \neq 0$; e.g. $M = W$, one-dimensional standard Brownian motion in the non-delay case ($\mu_1 = 0$). When $M = \nu_2 W$ for a fixed real ν_2, the above bound may be considerably sharpened as in Theorem V.5 below. The sdde in this theorem is a model of dye circulation in the blood stream (cf. Bailey and Williams [B-W], 1996; Lenhart and Travis, 1986).

Theorem V.5. *([M-S], II, 1996).*

For the equation

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}dt + \nu_2 x(t)\, dW(t) \quad (VI)$$

set

$$\phi(\delta) := -\delta + \frac{1}{4\nu_2^2}\left[\left(|\mu_1|e^{\delta r} + \nu_1 + \delta + \frac{1}{2}\nu_2^2\right)^+\right]^2, \quad (14)$$

for $\nu_2 \neq 0$. *Then*

$$\lambda_1 \leq \inf_{\delta \in \mathbf{R}} \phi(\delta). \quad (15)$$

In particular, if δ_0 *is the unique solution of the equation*

$$\nu_1 + \delta + |\mu_1|e^{\delta r} + \frac{1}{2}\nu_2^2 = 0, \quad (16)$$

then $\lambda_1 \leq -\delta_0$.

Proof.

Maximize

$$Q_\alpha(z_1, z_2, 0) + \tilde{Q}_\alpha(z_1, 0) = \left(\nu_1 + \frac{1}{2\alpha} + \frac{\nu_2^2}{2}\right) z_1^2 + \sqrt{\alpha}\, \mu_1 z_1 z_2 - \frac{1}{2} z_2^2 - \nu_2^2 z_1^4 \quad (17)$$

for $|z_1| \leq 1, z_2 \in \mathbf{R}$ and then minimize the resulting bound for λ_1 over $\alpha > 0$. Get

$$\lambda_1 \leq \frac{1}{16\nu_2^2}\left[(2\nu_1 + 2|\mu_1| + \nu_2^2)^+\right]^2.$$

The first assertion of the theorem follows from above estimate by applying an exponential shift to (VI). Last assertion of the theorem is obvious ([M-S], II, 1996, pp. 38-39.) $\square$

Problem: Is $\lambda_1 = \inf_{\delta \in \mathbf{R}} \phi(\delta)$?

Remark.

Estimate in Theorem V.5 agrees with the non-delay case $\mu_1 = 0$ whereby $\lambda_1 = \nu_1 - \frac{1}{2}\nu_2^2 = \inf_{\delta \in \mathbf{R}} \phi(\delta)$. Cf. also [AOP], 1986, [B], 1985, and [AKO], 1989.

4. SDDE with Poisson Noise.

Consider the one-dimensional linear delay equation

$$\left.\begin{aligned} dx(t) &= x((t-1)-)\,dN(t) \quad t>0 \\ x_0 &= \eta \in D := D([-1,0],\mathbf{R}). \end{aligned}\right\} \qquad (V)$$

The process $N(t) \in \mathbf{R}$ is a Poisson process with i.i.d. inter-arrival times $\{T_i\}_{i=1}^{\infty}$ which are exponentially distributed with the same parameter μ. The jumps $\{Y_i\}_{i=1}^{\infty}$ of N are i.i.d. and independent of all the T_i's. Let

$$j(t) := \sup\left\{ j \geq 0 : \sum_{i=1}^{j} T_i \leq t \right\}.$$

Then

$$N(t) = \sum_{i=1}^{j(t)} Y_i.$$

Equation (V) can be solved a.s. in forward steps of lengths 1, using the relation

$$x^{\eta}(t) = \eta(0) + \sum_{i=1}^{j(t)} Y_i x\left((\sum_{j=1}^{i} T_j - 1)- \right) \qquad \text{a.s.}$$

Trajectory $\{x_t : t \geq 0\}$ is a Markov process in the state space D (with the supremum norm $\|\cdot\|_\infty$). Furthermore, the above relation implies that (V) is regular in D; i.e., it admits a measurable flow $X : \mathbf{R}^+ \times D \times \Omega \to D$ with $X(t, \cdot, \omega) = {}^\eta x_t(\cdot, \omega)$, continuous linear in η for all $t \geq 0$ and a.a. $\omega \in \Omega$ (cf. the singular equation (I)).

The a.s. Lyapunov spectrum of (V) may be characterized directly (without appealing to the Oseledec Theorem) by interpolating between the sequence of random times:

$$\tau_0(\omega) := 0,$$

$$\tau_1(\omega) := \inf\left\{n \geq 1 : \sum_{j=1}^{k} T_j \notin [n-1, n] \quad \text{for all } k \geq 1\right\},$$

$$\tau_{i+1}(\omega) := \inf\left\{n > \tau_i(\omega) : \sum_{j=1}^{k} T_j \notin [n-1, n] \text{ for all } k \geq 1\right\}, \quad i \geq 1.$$

It is easy to see that $\{\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \cdots\}$ are i.i.d. and $E\tau_1 = \mathrm{e}^{\mu}$.

Theorem V.6. *([M-S], II, 1996)*

Let $\xi \in D$ be the constant path $\xi(s) = 1$ for all $s \in [-1, 0]$. Suppose $E \log \|X(\tau_1(\cdot), \xi, \cdot)\|_\infty$ exists (possibly $= +\infty$ or $-\infty$). Then the a.s. Lyapunov spectrum

$$\lambda(\eta) := \lim_{t\to\infty} \frac{1}{t} \log \|X(t, \eta, \omega)\|_\infty, \quad \eta \in D,\ \omega \in \Omega$$

of (V) is $\{-\infty, \lambda_1\}$ where

$$\lambda_1 = e^{-\mu}\, E \log \|X(\tau_1(\cdot), \xi, \cdot)\|_\infty.$$

In fact,

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t, \eta, \omega)\|_\infty = \begin{cases} \lambda_1 & \eta \notin \mathrm{Ker}\, X(\tau_1(\omega), \cdot, \omega) \\ -\infty & \eta \in \mathrm{Ker}\, X(\tau_1(\omega), \cdot, \omega). \end{cases}$$

Proof.

The i.i.d. sequence

$$S_i := \frac{\|(X(\tau_i, \xi, \cdot))\|}{\|(X(\tau_{i-1}, \xi, \cdot))\|} \qquad i = 1, 2, \ldots$$

and the LLN give

$$\lim_{n \to \infty} \frac{1}{\tau_n} \log \; \|(X(\tau_n, \xi, \omega))\| = e^{-\mu}(E \log S_1)$$

for a.a. $\omega \in \Omega$.

Interpolate between the times $\tau_1, \tau_2, \tau_3, \cdots$ to get the continuos limit ([M-S], II, 1996, pp. 27-28). □