III. REGULARITY

CLASSIFICATION OF SFDE'S

Geilo, Norway

Wednesday, July 31, 1996

10:30-11:20

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III. REGULARITY. CLASSIFICATION OF SFDE'S

Denote the state space by E where $E = C$ or $M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d)$. Most results hold for either choice of state space.

Objectives

To study regularity properties of the trajectory of a sfde as a random field $X := \{{}^\eta x_t : t \geq 0, \eta \in C\}$ in the variables (t,η,ω) $(E = C)$ or $(t,(v,\eta),\omega)$ $(E = M_2)$:

(i) Pathwise regularity of trajectories in the time variable.

(ii) Regularity of trajectories (in probability or pathwise) in the initial state $\eta \in C$ or $(v,\eta) \in M_2$.

(iii) Classification of sfde's into regular and singular types.

Denote by $C^\alpha := C^\alpha([-r,0],\mathbf{R}^d)$ the (separable) Banach space of α-Hölder continuous paths $\eta : [-r,0] \to \mathbf{R}^d$ with the Hölder norm

$$\|\eta\|_\alpha := \|\eta\|_C + \sup\left\{\frac{|\eta(s_1) - \eta(s_2)|}{|s_1 - s_2|^\alpha} : s_1, s_2 \in [-r,0], s_1 \neq s_2\right\}.$$

C^α can be constructed in a *separable manner* by completing the space of smooth paths $[-r,0] \to \mathbf{R}^d$ with respect to the above norm (Tromba [Tr], JFA, 1972). First step is to think of ${}^\eta x_t(\omega)$ as a measurable mapping $X : \mathbf{R}^+ \times C \times \Omega \to C$ in the three variables (t,η,ω) simultaneously:

Theorem III.1([Mo], Pitman Books, 1984)

In the sfde

$$\left.\begin{aligned} dx(t) &= H(t,x_t)\,dt + G(t,x_t)\,dW(t) \quad t > 0 \\ x_0 &= \eta \in C \end{aligned}\right\} \qquad (XIII)$$

assume that the coefficients H, G are (jointly) continuous and globally Lipschitz in the second variable uniformly wrt the first. Then *(i) For any $0 < \alpha < \frac{1}{2}$, and each initial path $\eta \in C$,*

$$P({}^{\eta}x_t \in C^{\alpha}, \textit{for all} \ \ t \geq r) = 1.$$

(ii) the trajectory field has a measurable version

$$X : \mathbf{R}^+ \times C \times \Omega \to C.$$

(iii) The trajectory field ${}^{\eta}x_t, t \geq r,\ \eta \in C$, admits a measurable version

$$[r, \infty) \times C \times \Omega \to C^{\alpha}.$$

Remark.

Similar statements hold for $E = M_2$.

Give $L^0(\Omega, E)$ the complete (psuedo)metric

$$d_E(\theta_1, \theta_2) := \inf_{\epsilon > 0} [\epsilon + P(\|\theta_1 - \theta_2\|_E \geq \epsilon)], \quad \theta_1, \theta_2 \in L^0(\Omega, E),$$

(which corresponds to convergence in probability, Dunford and Schwartz [D-S], Lemma III.2.7, p. 104).

Proof of Theorem III.1.

(i) Sufficient to show that

$$P\big({}^{\eta}x|[0,a] \in C^{\alpha}([0,a], \mathbf{R}^d)\big) = 1$$

by using the estimate

$$P\Bigg(\sup_{0 \leq t_1, t_2 \leq a, t_1 \neq t_1} \frac{|{}^{\eta}x(t_1) - {}^{\eta}x(t_2)|}{|t_1 - t_2|^{\alpha}} \geq N\Bigg) \leq C_k^1 (1 + \|\eta\|_C^{2k}) \frac{1}{N^{2k}},$$

for all integers $k > (1 - 2\alpha)^{-1}$, and the Borel-Cantelli lemma. Above estimate is proved using Gronwall's lemma, Chebyshev's inequality, and Garsia-Rodemick-Rumsey lemma ([Mo], Pitman

Books, 1984, Theorem 4.1, p. 150; [Mo], Pitman Books, 1984, Theorem 4.4, pp.152-154.)

(ii) By mean-square Lipschitz dependence ([Mo], Pitman Books, 1984, Theorem 3.1, p. 41), the trajectory

$$\begin{aligned} [0,a] \times C &\to L^2(\Omega, C) \subset L^0(\Omega, C) \\ (t,\eta) &\mapsto {}^\eta x_t \end{aligned}$$

is globally Lipschitz in η uniformly wrt t in compact sets, and is continuous in t for fixed η. Therefore it is jointly continuous in (t,η) as a map

$$[0,a] \times C \ni (t,\eta) \mapsto {}^\eta x_t \in L^0(\Omega, C).$$

Then apply the Cohn-Hoffman-Jϕrgensen theorem:
If T, E are complete separable metric spaces, then each Borel map $X : T \to L^0(\Omega, E; \mathcal{F})$ admits a measurable version

$$T \times \Omega \to E$$

to the trajectory field to get measurability in (t,η). (Take $T = [0,a] \times C$, $E = C$ ([Mo], Pitman Books, 1984, p. 16).)

(iii) Use the estimate

$$P\big(\|{}^{\eta_1}x_t - {}^{\eta_2}x_t\|_{C^\alpha} \geq N\big) \leq \frac{C_k^2}{N^{2k}}\|\eta_1 - \eta_2\|_C^{2k}$$

for $t \in [r,a], N > 0$, ([Mo], 1984, Theorem 4.7, pp.158-162) to prove joint continuity of the trajectory

$$\begin{aligned} [r,a] \times C &\to L^0(\Omega, C^\alpha) \\ (t,\eta) &\mapsto {}^\eta x_t \end{aligned}$$

([Mo], Theorem 4.7, pp. 158-162) viewed as a process with values in the separable Banach space C^α. Again apply the Cohn-Hoffman-Jϕrgensen theorem. $\square$

As we have seen in Lecture I, the trajectory of a sfde possesses good regularity properties *in the mean-square.* The following theorem shows good behavior in distribution.

Theorem III.2. ([Mo], Pitman Books, 1984)

Suppose the coefficients H, G are globally Lipschitz in the second variable uniformly with respect to the first. Let $\alpha \in (0, 1/2)$ and k be any integer such that $k > (1-2\alpha)^{-1}$. Then there are positive constants C_k^3, C_k^4, C_k^5 such that

$$d_C({}^{\eta_1}x_t, {}^{\eta_2}x_t) \leq C_k^3 \|\eta_1 - \eta_2\|_C^{2k/(2k+1)} \qquad t \in [0, a]$$

$$d_{C^\alpha}({}^{\eta_1}x_t, {}^{\eta_2}x_t) \leq C_k^4 \|\eta_1 - \eta_2\|_C^{2k/(2k+1)} \qquad t \in [r, a]$$

$$P\Big(\|{}^{\eta}x_t\|_{C^\alpha} \geq N\Big) \leq C_k^5 (1 + \|\eta\|_C^{2k}) \frac{1}{N^{2k}}, \qquad t \in [r, a], \quad N > 0.$$

In particular the transition probabilities

$$\begin{aligned} [r, a] \times C &\to \mathcal{M}_p(C) \\ (t, \eta) &\mapsto p(0, \eta, t, \cdot) \end{aligned}$$

take bounded sets into relatively weak compact sets in the space $\mathcal{M}_p(C)$ of probability measures on C.*

Proof of Theorem III.2.

Proofs of the estimates use Gronwall's lemma, Chebyshev's inequality, and Garsia-Rodemick-Rumsey lemma ([Mo], 1984, Theorem 4.1, p. 150; [Mo], 1984, Theorem 4.7, pp.159-162.) The weak* compactness assertion follows from the last estimate, Prohorov's theorem and the compactness of the embedding $C^\alpha \hookrightarrow C$ ([Mo], 1984, Theorem 4.6, pp. 156-158). $\square$

Erratic Behavior. The Noisy Loop Revisited

Definition.

A sfde is *regular* with respect to M_2 if its trajectory random field $\{(x(t), x_t) : (x(0), x_0) = (v, \eta) \in M_2,\ t \geq 0\}$ admits a $(Borel\, \mathbf{R}^+ \otimes Borel\, M_2 \otimes \mathcal{F},\ Borel\, M_2)$-measurable version $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ with a.a. sample functions continuous on $\mathbf{R}^+ \times M_2$. The sfde is said to be *singular* otherwise. Similarly for regularity with respect to C.

Consider the one-dimensional linear sdde with a *positive delay* r

$$\left.\begin{aligned} dx(t) &= \sigma x(t-r))\, dW(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R} \times L^2([-r, 0], \mathbf{R}), \end{aligned}\right\} \qquad (I)$$

driven by a Wiener process W.

Theorem III.3 below implies that (I) is singular with respect to M_2 (and C).(See also [Mo], Stochastics, 1986).

Consider the regularity of the more general one-dimen-sional linear sfde:

$$\left.\begin{aligned} dx(t) &= \int_{-r}^{0} x(t+s) d\nu(s)\, dW(t), \quad t > 0 \\ (x(0), x_0) &\in M_2 := \mathbf{R} \times L^2([-r, 0], \mathbf{R}) \end{aligned}\right\} \qquad (II')$$

where W is a Wiener process and ν is a fixed finite real-valued Borel measure on $[-r, 0]$.

Exercise:

(II$'$) is regular if ν has a C^1 (or even L_1^2) density with respect to Lebesgue measure on $[-r, 0]$. (Hint: Use integration by parts to eliminate the Itô integral!)

The following theorem gives conditions on the measure ν under which (II$'$) is singular.

Theorem III.3 ([M-S], II, 1996)

Let $r > 0$, and suppose that there exists $\epsilon \in (0, r)$ such that supp $\nu \subset [-r, -\epsilon]$. *Suppose $0 < t_0 \leq \epsilon$. For each $k \geq 1$, set*

$$\nu_k := \sqrt{t_0} \left| \int_{[-r,0]} e^{2\pi iks/t_0} \, d\nu(s) \right|.$$

Assume that

$$\sum_{k=1}^{\infty} \nu_k x^{1/\nu_k^2} = \infty \tag{1}$$

for all $x \in (0, 1)$. Let $Y : [0, \epsilon] \times M_2 \times \Omega \to \mathbf{R}$ be any Borel-measurable version of the solution field $\{x(t) : 0 \leq t \leq \epsilon, \ (x(0), x_0) = (v, \eta) \in M_2\}$ of (II$'$). Then for a.a. *$\omega \in \Omega$, the map $Y(t_0, \cdot, \omega) : M_2 \to \mathbf{R}$ is unbounded in every neighborhood of every point in M_2, and (hence) non-linear.*

Corollary. ([Mo], Pitman Books, 1984)

Suppose $r > 0, \sigma \neq 0$ *in (I). Then the trajectory* $\{{}^\eta x_t : 0 \leq t \leq r, \eta \in C\}$ *of (I) has a measurable version* $X : \mathbf{R}^+ \times C \times \Omega \to C$ *s.t. for every* $t \in (0, r]$

$$P\Big(X(t, \eta_1 + \lambda\eta_2, \cdot) = X(t, \eta_1, \cdot) + \lambda X(t, \eta_2, \cdot) \quad \text{for all } \lambda \in \mathbf{R}, \eta_1, \eta_2 \in C\Big) = 0.$$

But

$$P\Big(X(t, \eta_1 + \lambda\eta_2, \cdot) = X(t, \eta_1, \cdot) + \lambda X(t, \eta_2, \cdot)\Big) = 1.$$

for all $\lambda \in \mathbf{R}, \eta_1, \eta_2 \in C$.

Remark.

(i) Condition (1) of the theorem is implied by

$$\lim_{k\to\infty} \nu_k \sqrt{\log k} = \infty.$$

(ii) For the delay equation (I), $\nu = \sigma\delta_{-r}, \ \epsilon = r$. In this case condition (1) is satisfied for *every* $t_0 \in (0, r]$.

(iii) Theorem III.3 also holds for state space C since every bound-ed set in C is also bounded in $L^2([-r, 0], \mathbf{R})$.

Proof of Theorem III.3.

Joint work with V. Mizel.

Main idea is to track the solution random field of (a complexified version of) (II′) along the classical Fourier basis

$$\eta_k(s) = \mathrm{e}^{2\pi i k s/t_0} \quad , \qquad -r \le s \le 0, \qquad k \ge 1 \tag{2}$$

in $L^2([-r,0],\mathbf{C})$. On this basis, the solution field gives an infinite family of independent Gaussian random variables. This allows us to show that no Borel measurable version of the solution field can be bounded with positive probability on an arbitrarily small neighborhood of 0 in M_2, and hence on any neighborhood of any point in M_2 (cf. [Mo], Pitman Books, 1984; [Mo], Stochastics, 1986). For simplicity of computations, complexify the state space in (II′) by allowing (v,η) to belong to $M_2^C := \mathbf{C} \times L^2([-r,0],\mathbf{C})$. Thus consider the sfde

$$\left.\begin{aligned} dx(t) &= \int_{[-r,0]} x(t+s)d\nu(s)\, dW(t), t>0, \\ (x(0),x_0) &= (v,\eta) \in M_2^C \end{aligned}\right\} \qquad (II'-C))$$

where $x(t) \in \mathbf{C}$, $t \ge -r$, and ν, W are real-valued.

Use contradiction. Let $Y : [0,\epsilon] \times M_2 \times \Omega \to \mathbf{R}$ be any Borel-measurable version of the solution field $\{x(t) : 0 \le t \le \epsilon,\ (x(0),x_0) = (v,\eta) \in M_2\}$ of (II′). Suppose, if possible, that there exists a set $\Omega_0 \in \mathcal{F}$ of positive P-measure, $(v_0,\eta_0) \in M_2$ and a positive δ such that for all $\omega \in \Omega_0$, $Y(t_0,\cdot,\omega)$ is bounded on the open ball $B((v_0,\eta_0),\delta)$ in M_2 of center (v_0,η_0) and radius δ. Define the complexification $Z(\cdot,\omega) : M_2^C \to \mathbf{C}$ of $Y(t_0,\cdot,\omega) : M_2 \to \mathbf{R}$ by

$$Z(\xi_1 + i\xi_2,\omega) := Y(t_0,\xi_1,\omega) + i\,Y(t_0,\xi_2,\omega), \qquad i = \sqrt{-1},$$

for all $\xi_1, \xi_2 \in M_2$, $\omega \in \Omega$. Let $(v_0,\eta_0)^C$ denote the complexification $(v_0,\eta_0)^C := (v_0,\eta_0) + i(v_0,\eta_0)$. Clearly $Z(\cdot,\omega)$ is bounded on the complex ball $B((v_0,\eta_0)^C,\delta)$ in M_2^C for all $\omega \in \Omega_0$. Define the sequence of complex random variables $\{Z_k\}_{k=1}^\infty$ by

$$Z_k(\omega) := Z((\eta_k(0),\eta_k),\omega) - \eta_k(0), \qquad \omega \in \Omega, \quad k \geq 1.$$

Then

$$Z_k = \int_0^{t_0} \int_{[-r,-\epsilon]} \eta_k(u+s)\, d\nu(s)\, dW(u), \quad k \geq 1.$$

By standard properties of the Itô integral, and Fubini's theorem,

$$EZ_k\overline{Z_l} = \int_{[-r,-\epsilon]} \int_{[-r,-\epsilon]} \int_0^{t_0} \eta_k(u+s)\overline{\eta_l(u+s')}\, du\, d\nu(s)\, d\nu(s') = 0$$

for $k \neq l$, because

$$\int_0^{t_0} \eta_k(u+s)\overline{\eta_l(u+s')}\ du = 0$$

whenever $k \neq l$, for all $s, s' \in [-r,0]$. Furthermore

$$\int_0^{t_0} \eta_k(u+s)\overline{\eta_k(u+s')}\ du = t_0 \mathrm{e}^{2\pi i k(s-s')/t_0}$$

for all $s, s' \in [-r,0]$. Hence

$$\begin{aligned} E|Z_k|^2 &= \int_{[-r,-\epsilon]} \int_{[-r,-\epsilon]} t_0 \mathrm{e}^{2\pi i k(s-s')/t_0}\ d\nu(s)\, d\nu(s') \\ &= t_0 \left| \int_{[-r,0]} \mathrm{e}^{2\pi i k s/t_0}\ d\nu(s) \right|^2 \\ &= \nu_k^2. \end{aligned}$$

$Z(\cdot,\omega) : M_2^C \to \mathbf{C}$ is bounded on $B((v_0,\eta_0)^C,\delta)$ for all $\omega \in \Omega_0$, and $\|(\eta_k(0),\eta_k)\| = \sqrt{r+1}$ for all $k \geq 1$. By the linearity property

$$\begin{aligned} &Z\bigg((v_0,\eta_0)^C + \frac{\delta}{2\sqrt{r+1}}(\eta_k(0),\eta_k),\cdot\bigg) \\ &\qquad = Z((v_0,\eta_0)^C,\cdot) + \frac{\delta}{2\sqrt{r+1}} Z((\eta_k(0),\eta_k),\cdot),\ k \geq 1, \end{aligned}$$

a.s., it follows that

$$P\big(\sup_{k\geq 1}|Z_k|<\infty\big)>0. \tag{3}$$

It is easy to check that $\{ReZ_k,\, ImZ_k : k\geq 1\}$ are independent $\mathcal{N}(0,\nu_k^2/2)$-distributed Gaussian random variables. Get a contradiction to (3):

For each integer $N\geq 1$,

$$\begin{aligned} P\Bigg(\sup_{k\geq 1}|Z_k|<N\Bigg) &\leq \prod_{k\geq 1} P\Bigg(|ReZ_k|<N\Bigg) \\ &= \prod_{k\geq 1}\left[1-\frac{2}{\sqrt{2\pi}}\int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty} \mathrm{e}^{-x^2/2}\,dx\right] \\ &\leq \exp\left\{-\frac{2}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty} \mathrm{e}^{-x^2/2}\,dx\right\}. \end{aligned} \tag{4}$$

There exists $N_0>1$ (independent of $k\geq 1$) such that

$$\int_{\frac{\sqrt{2}N}{\nu_k}}^{\infty} \mathrm{e}^{-x^2/2}\,dx \geq \frac{\nu_k}{2\sqrt{2}N}\mathrm{e}^{-\frac{N^2}{\nu_k^2}} \tag{5}$$

for all $N\geq N_0$ and all $k\geq 1$.

Combine (4) and (5) and use hypothesis (1) of the theorem to get

$$P\Bigg(\sup_{k\geq 1}|Z_k|<N\Bigg)=0$$

for all $N\geq N_0$. Hence

$$P\Bigg(\sup_{k\geq 1}|Z_k|<\infty\Bigg)=0.$$

This contradicts (3)(cf. Dudley [Du], JFA, 1967).

Since $Y(t_0,\cdot,\omega)$ is locally unbounded, it must be non-linear because of Douady's Theorem:

Every Borel measurable linear map between two Banach spaces is continuous.

(Schwartz [Sc], Radon Measures, Part II, 1973, pp. 155-160). □

Note that the pathological phenomenon in Theorem III.3 is peculiar to the delay case $r > 0$. The proof of the theorem suggests that this pathology is due to the *Gaussian nature* of the Wiener process W coupled with the *infinite-dimensionality* of the state space M_2. Because of this, one may expect similar difficulties in certain types of linear spde's driven by *multi-dimensional* white noise (Flandoli and Schaumlöffel [F-S], Stochastics, 1990).

Problem.

Classify all finite signed measures ν on $[-r, 0]$ for which (II') is regular.

Note that (I) automatically satisfies the conditions of Theorem III.3, and hence its trajectory field *explodes on every small neighborhood of* $0 \in M_2$. Because of the singular nature of (I), it is surprising that the maximal exponential growth rate of the trajectory of (I) is *negative* for small σ and is bounded away from zero *independently of the choice of the initial path* in M_2. This will be shown later in Lecture V (Theorem V.1).

Regular Linear Systems. White Noise

SDE's on $\mathbf{R}^d$ driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$, with smooth coefficients.

$$\left.\begin{aligned} dx(t) &= H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\ &\qquad + \sum_{i=1}^{m} g_i x(t)\, dW_i(t), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (VIII)$$

(VIII) is defined on

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) =$ canonical complete filtered Wiener space:

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R}^+ \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F} :=$ completed Borel σ-field of Ω;

$\mathcal{F}_t :=$ completed sub-σ-field of $\mathcal{F}$ generated by the evaluations $\omega \to \omega(u)$, $0 \le u \le t$, $\quad t \ge 0$;

$P :=$ Wiener measure on Ω;

$dW_i(t) =$ Itô stochastic differentials, $1 \le i \le m$.

Several finite delays $0 < d_1 < d_2 < \cdots < d_N \le r$ in drift term; *no delays in diffusion coefficient.*

$H : (\mathbf{R}^d)^{N+1} \times L^2([-r, 0], \mathbf{R}^d) \to \mathbf{R}^d$ is a fixed continuous linear map; g_i, $i = 1, 2, \ldots, m$, fixed (deterministic) $d \times d$-matrices.

Theorem III.4.([Mo], Stochastics, 1990])

(VIII) is regular with respect to the state space $M_2 = \mathbf{R}^d \times \mathbf{L}^2([-r, 0], \mathbf{R}^d)$. There is a measurable version $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ of the trajectory field $\{(x(t), x_t) : t \in \mathbf{R}^+, (x(0), x_0) = (v, \eta) \in M_2\}$ with the following properties:

- *(i) For each $(v, \eta) \in M_2$ and $t \in \mathbf{R}^+$, $X(t, (v, \eta), \cdot) = (x(t), x_t)$ a.s., is $\mathcal{F}_t$-measurable and belongs to $L^2(\Omega, M_2; P)$.*
- *(ii) There exists $\Omega_0 \in \mathcal{F}$ of full measure such that, for all $\omega \in \Omega_0$, the map $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times M_2 \to M_2$ is continuous.*
- *(iii) For each $t \in \mathbf{R}^+$ and every $\omega \in \Omega_0$, the map $X(t, \cdot, \omega) : M_2 \to M_2$ is continuous linear; for each $\omega \in \Omega_0$, the map $\mathbf{R}^+ \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ is measurable and locally bounded in the uniform operator norm on $L(M_2)$. The map $[r, \infty) \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ is continuous for all $\omega \in \Omega_0$.*
- *(iv) For each $t \ge r$ and all $\omega \in \Omega_0$, the map*

$$X(t, \cdot, \omega) : M_2 \to M_2$$

is compact.

Proof uses variational technique to reduce the problem to the solution of a random family of classical integral equations involving *no stochastic integrals.*

Compactness of semi-flow for $t \geq r$ will be used later to define hyperbolicity for (VIII) and the associated exponential dichotomies (Lecture IV).

Regular Linear Systems. Semimartingale Noise

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ a complete filtered probability space satisfying the usual conditions.

Linear systems driven by semimartingale noise, and memory driven by a measure-valued process
$\nu : \mathbf{R} \times \Omega \to \mathcal{M}([-r, 0], \mathbf{R}^{d\times d})$, where $\mathcal{M}([-r, 0], \mathbf{R}^{d\times d})$ is the space of all $d \times d$-matrix-valued Borel measures on $[-r, 0]$ (or $R^{d\times d}$-valued functions of bounded variation on $[-r, 0]$). This space is given the σ-algebra generated by all evaluations. The space $\mathbf{R}^{d\times d}$ of all $d \times d$-matrices is given the Euclidean norm $\|\cdot\|$.

$$\left.\begin{aligned} dx(t) = \Big\{\int_{[-r,0]} \nu(t)(ds)\, x(t+s)\Big\}\, dt + dN(t) \int_{-r}^{0} K(t)(s)\, x(t+s)\, ds \\ + dL(t)\, x(t-), \quad t > 0 \\ (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (IX)$$

Hypotheses (R)

(i) The process $\nu : \mathbf{R}\times\Omega \to \mathcal{M}([-r, 0], \mathbf{R}^{d\times d})$ is measurable and $(\mathcal{F}_t)_{t\geq 0}$)-adapted. For each $\omega \in \Omega$ and $t \geq 0$ define the positive measure $\bar{\nu}(t, \omega)$ on $[-r, \infty)$ by

$$\bar{\nu}(t, \omega)(A) := |\nu|(t, \omega)\{(A - t) \cap [-r, 0]\}$$

for all Borel subsets A of $[-r, \infty)$, where $|\nu|$ is the total variation measure of ν wrt the Euclidean norm on $\mathbf{R}^{d\times d}$. Therefore the equation

$$\mu(\omega)(\cdot) := \int_0^\infty \bar{\nu}(t, \omega)(\cdot)\, dt$$

defines a positive measure on $[-r, \infty)$. For each $\omega \in \Omega$ suppose that $\mu(\omega)$ has a density wrt Lebesgue measure which is locally essentially bounded.
(*Exercise:* This condition is automatically satisfied if $\nu(t, \omega)$ is independent of (t, ω).)

(ii) $K : \mathbf{R} \times \Omega \to L^\infty([-r, o], \mathbf{R}^{d\times d})$ is measurable and $(\mathcal{F}_t)_{t\geq 0}$- adapted. Define the random field $\tilde{K}(t, s, \omega)$ by
$\tilde{K}(t, s, \omega) := K(t, \omega)(s-t)$ for $t \geq 0$, $-r \leq s-t \leq 0$. Assume that $\tilde{K}(t, s, \omega)$ is absolutely continuous in t for Lebesgue a.a. s and all $\omega \in \Omega$. For every $\omega \in \Omega$, $\frac{\partial \tilde{K}}{\partial t}(t, s, \omega)$ and $\tilde{K}(t, s, \omega)$ are locally essentially bounded in (t, s). $\frac{\partial \tilde{K}}{\partial t}(t, s, \omega)$ is jointly measurable.

(iii) $L = M + V$, M continuos local martingale, V B.V. process.

Theorem III.5. ([M-S], I, AIHP, 1996)

Under hypotheses (R), equation (IX) is regular w.r.t. M_2 with a measurable flow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$. This flow satisfies Theorem III.4.

Proof.

This is achieved via a construction in ([M-S], I, AIHP, 1996) which reduces (IX) to a random linear integral equation with *no stochastic integrals* ([M-S], AIHP, 1996, pp. 85-96). Do a complicated pathwise analysis on the integral equation to establish existence and regularity properties of the semiflow. □

Regular Non-linear Systems

(a) SFDE's with Ordinary Diffusion Coefficients

In the sfde,

$$\left.\begin{aligned} dx(t) &= H(x_t)dt + \sum_{i=1}^{m} g_i(x(t))dW_i(t) \\ x_0 &= \eta \in C \end{aligned}\right\} \qquad (XV)$$

let $H : C \to \mathbf{R}^d$ be globally Lipschitz and $g_i : \mathbf{R}^d \to \mathbf{R}^d$ C^2-bounded maps satisfying the Frobenius condition (vanishing Lie brackets):

$$Dg_i(v)g_j(v) = Dg_j(v)g_i(v), \quad 1 \leq i, j \leq m, \ v \in \mathbf{R}^d;$$

and $W := (W_1, W_2, \cdots, W_m)$ is m-dimensional Brownian motion. Note that the diffusion coefficient in (XV) has no memory.

Theorem III.6 ([Mo], Pitman Books, 1984)

Suppose the above conditions hold. Then the trajectory field $\{{}^\eta x_t : t \geq 0, \eta \in C\}$ *of (XV) has a measurable version* $X : \mathbf{R}^+ \times C \times \Omega \to C$ *satisfying the following properties. For each* $\alpha \in (0, 1/2)$*, there is a set* $\Omega_\alpha \subset \Omega$ *of full measure such that for every* $\omega \in \Omega_\alpha$

- *(i)* $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times C \to C$ *is continuous;*
- *(ii)* $X(\cdot, \cdot, \omega) : [r, \infty) \times C \to C^\alpha$ *is continuous;*
- *(iii) for each* $t \geq r$, $X(t, \cdot, \omega) : C \to C$ *is compact;*
- *(iv) for each* $t \geq r$, $X(t, \cdot, \omega) : C \to C^\alpha$ *is Lipschitz on every bounded set in* C*, with a Lipschitz constant independent of* t *in compact sets. Hence each map* $X(t, \cdot, \omega) : C \to C$ *is compact: viz. takes bounded sets into relatively compact sets.*

Proof of Theorem III.6.

([Mo], Pitman Books, 1984, Theorem (2.1), Chapter (V), §2, p. 121). This latter result is proved using a non-linear variational method originally due to Sussman ([Su], Ann. Prob., 1978) and Doss ([Do], AIHP, 1977) in the non-delay case $r = 0$. Write $g := (g_1, g_2, \cdots, g_m) : \mathbf{R}^d \to \mathbf{R}^{d \times m}$. By the Frobenius condition, there is a C^2 map $F : \mathbf{R}^m \times \mathbf{R}^d \to \mathbf{R}^d$ such that $\{F(\underline{t}, \cdot) : \underline{t} \in \mathbf{R}^m\}$ is a group of C^2 diffeomorphisms $\mathbf{R}^d \to \mathbf{R}^d$ satisfying

$$\begin{aligned} D_1 F(\underline{t}, x) &= g(F(\underline{t}, x)), \\ F(\underline{0}, x) &= x \end{aligned}$$

for all $\underline{t} \in \mathbf{R}^m, x \in \mathbf{R}^d$.

Define

$$W^0(t) := \begin{cases} W(t) - W(0), & t \geq 0 \\ 0 & -r \leq t < 0 \end{cases}$$

and $\tilde{H} : \mathbf{R}^+ \times C \times \Omega \to \mathbf{R}^d$, by

$$\begin{aligned} \tilde{H}(t, \eta, \cdot) :=& D_2 F(W^0(t), \eta(0))^{-1} \big\{ H[F \circ (W_t^0, \eta)] \\ & - \frac{1}{2} \text{trace}\big(Dg[F(W^0(t), \eta(0))] \circ g[F(W^0(t), \eta(0))] \big) \big\} \end{aligned}$$

where the expression under the "trace" is viewed as a bilinear form $\mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^d$, and the trace has values in $\mathbf{R}^d$. Then for each ω, $\tilde{H}(t, \eta, \omega)$ is jointly continuous, Lipschitz in η in bounded subsets of C uniformly for t in compact sets, and satisfies a global linear growth condition in η ([Mo], Pitman Books, 1984, pp. 114-126).

Therefore solve the fde

$$\begin{aligned} &{}^{\eta}\xi_t' = \tilde{H}(t, {}^{\eta}\xi_t, \cdot) \qquad t \geq 0 \\ &{}^{\eta}\xi_0 = \eta. \end{aligned}$$

Define the semiflow

$$X(t, \eta, \omega) = F \circ \big(W_t^0(\omega), {}^{\eta}x_t(\omega)\big).$$

Check that X satisfies all assertions of theorem ([Mo], 1984, pp.126-133). $\square$

(b) SFDE's with Smooth Memory

$$\left.\begin{aligned} dx(t) = H(dt, x(t), x_t) + G(dt, x(t), g(x_t)), \quad t > 0 \\ (x(0), x_0) = (v, eta) \in M_2 \end{aligned}\right\} \qquad (XVI)$$

Coefficients H and G in (XVI) are semimartingale-valued random fields on $M_2 = \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$ and $\mathbf{R}^d \times \mathbf{R}^m$, respectively. The memory is driven by a functional $g : L^2([-r, 0], \mathbf{R}^d) \to \mathbf{R}^m$ with the smoothness property that the process $t \mapsto g(x_t)$ has absolutely continuous paths for each adapted process x. Under (technical) but general regularity and boundedness conditions on the characteristics of H and G, equation (XVI) is regular:

Theorem III.7 ([M-S], 1996)

Let

$$\Delta := \{(t_0, t) \in \mathbf{R}^2 : t_0 \leq t\}.$$

Under suitable regularity conditions on H, G, g in (XVI), there exists a random field $X : \Delta \times M_2 \times \Omega \to M_2$ satisfying the following properties:

(i) For each $(v, \eta) \in M_2$, $(t_0, t) \in \Delta$, $X(t_0, t, (v, \eta), \cdot) = (x^{t_0,(v,\eta)}(t), x_t^{t_0,(v,\eta)})$ a.s., where $x^{t_0,(v,\eta)}$ is the unique solution of (XVI) with $x_{t_0}^{t_0,(v,\eta)} = (v, \eta)$.

(ii) For each $(t_0, t, \omega) \in \Delta \times \Omega$, the map

$$X(t_0, t, \cdot, \omega) : M_2 \to M_2$$

is C^∞.

(iii) For each $\omega \in \Omega$ and $(t_0, t) \in \Delta$ with $t > t_0 + r$, the map

$$X(t_0, t, \cdot, \omega) : M_2 \to M_2$$

carries bounded sets into relatively compact sets.