# SFDE'S AS DYNAMICAL SYSTEMS: THE LINEAR CASE

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## 1. Regular Linear SFDE's-Ergodic Theory.

Linear sfde's on $\mathbf{R}^d$ driven by $m$-dimensional Brownian motion $W := (W_1, \cdots, W_m)$.

$$\left.\begin{aligned} dx(t) &= H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\ &\qquad\qquad + \sum_{i=1}^{m} g_i x(t)\, dW_i(t), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \quad (I)$$

(I) is defined on

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}}, P) =$ canonical complete filtered Wiener space.

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R} \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F} :=$(completed) Borel $\sigma$-field of $\Omega$;

$\mathcal{F}_t :=$ (completed) sub-$\sigma$-field of $\mathcal{F}$ generated by the evaluations $\omega \to \omega(u), \;\; u \leq t, \quad t \in \mathbf{R}$.

$P :=$ Wiener measure on $\Omega$.

$dW_i(t) =$ Itô stochastic differentials.

Several finite delays $0 < d_1 < d_2 < \cdots < d_N \leq r$ in drift term; *no delays in diffusion coefficient.*

$H : (\mathbf{R}^d)^{N+1} \times L^2([-r, 0], \mathbf{R}^d) \to \mathbf{R}^d$ is a fixed continuous linear map, $g_i$, $i = 1, 2, \ldots, m$, fixed (deterministic) $d \times d$-matrices.

## 2. Plan

Use state space $M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$. For (I) consider the following themes:

I) Existence of a "perfect" cocycle on $M_2$-a modification of the trajectory field $(x(t), x_t) \in M_2$.

II) Existence of almost sure Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|(x(t), x_t)\|_{M_2}$$

Multiplicative ergodic theorem and *hyperbolicity* of cocycle.

III) *"Random Saddle-Point Property"* in hyperbolic case.

## 3. Regularity

Say SFDE (I) is *regular* (wrt. $M_2$) if trajectory $\{(x(t), x_t) : (x(0), x_0) = (v, \eta) \in M_2\}$ admits a measurable modification $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ such that $X(\cdot, \cdot, \omega)$ is continuous for a.a. $\omega \in \Omega$.

**Theorem 1.**([Mo], 1990])

*(I) is regular with respect to state space* $M_2 = \mathbf{R}^d \times \mathbf{L}^2([-r, 0], \mathbf{R}^d)$. *There is a measurable version* $X : \mathbf{R}^+ \times$

*$M_2 \times \Omega \to M_2$ of the trajectory field $\{(x(t), x_t) : t \in \mathbf{R}^+,\ (x(0), x_0) = (v, \eta) \in M_2\}$ of (I) with the following properties:*

(i) *For each $(v, \eta) \in M_2$ and $t \in \mathbf{R}^+$, $X(t, (v, \eta), \cdot) = (x(t), x_t)$ a.s., is $\mathcal{F}_t$-measurable and belongs to $L^2(\Omega, M_2; P)$.*

(ii) *There exists $\Omega_0 \in \mathcal{F}$ of full measure such that, for all $\omega \in \Omega_0$, the map $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times M_2 \to M_2$ is continuous.*

(iii) *For each $t \in \mathbf{R}^+$ and every $\omega \in \Omega_0$, the map $X(t, \cdot, \omega) : M_2 \to M_2$ is continuous linear; for each $\omega \in \Omega_0$, the map $\mathbf{R}^+ \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ is measurable and locally bounded in the uniform operator norm on $L(M_2)$. The map $[r, \infty) \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ is continuous for all $\omega \in \Omega_0$.*

*(iv) For each $t \geq r$ and all $\omega \in \Omega_0$, the map*

$$X(t, \cdot, \omega) : M_2 \to M_2$$

*is compact.*

*Compactness of semi-flow for $t \geq r$ will be used to define hyperbolicity for (I) and the associated exponential dichotomies.*

*Example:* $dx(t) = x(t-1)\, dW(t)$ *is not regular (singular).*

## 4. Lyapunov Exponents. Hyperbolicity

Version $X$ of the trajectory field of (I) (in Theorem 1) is a multiplicative $L(M_2)$-valued linear cocycle over the canonical Brownian shift $\theta : \mathbf{R} \times \Omega \to \Omega$ on Wiener space:

$$\theta(t, \omega)(u) := \omega(t+u) - \omega(t), \quad u, t \in \mathbf{R}, \quad \omega \in \Omega.$$

I.e.

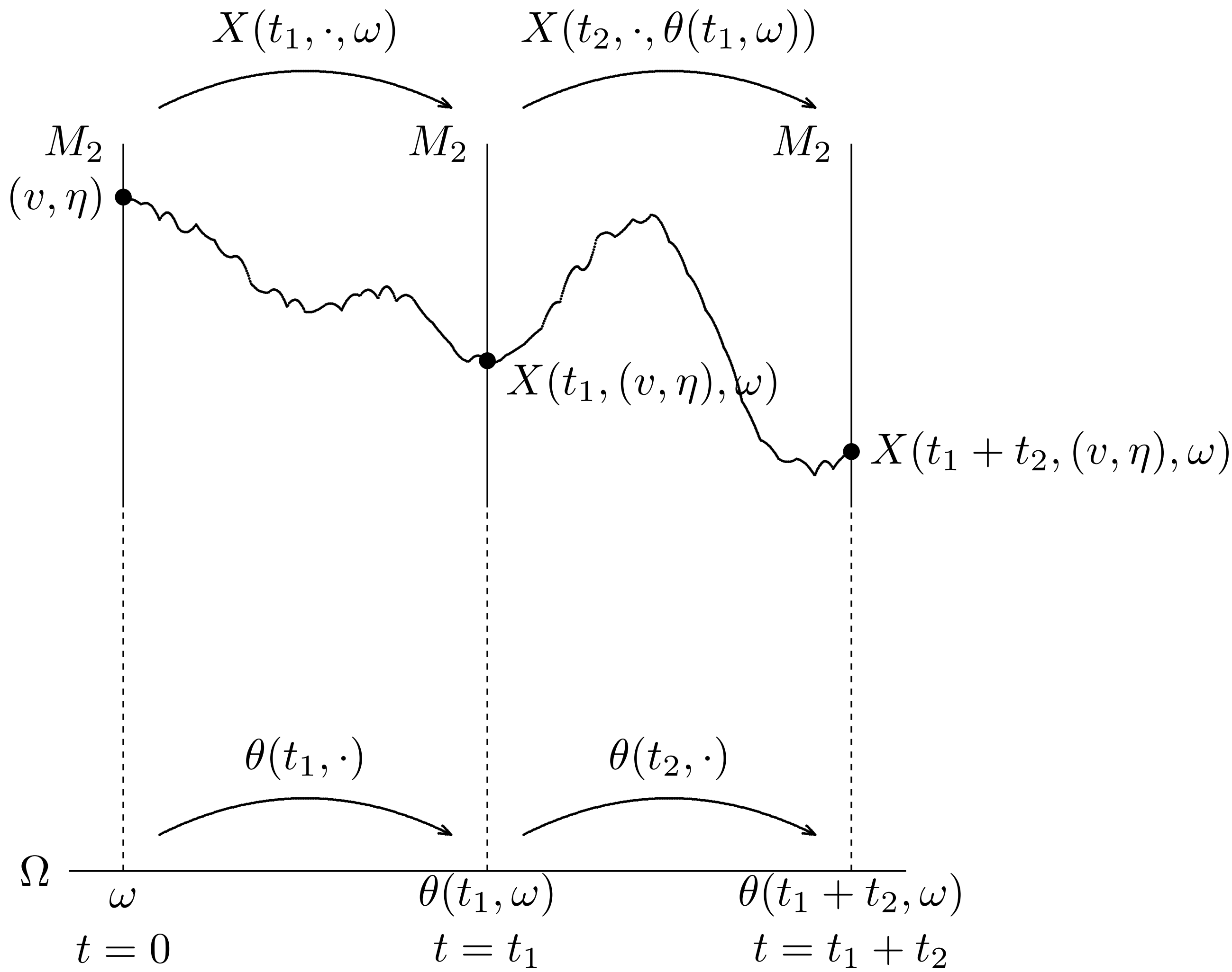
**Theorem 2**([Mo], 1990)

*There is an $\mathcal{F}$-measurable set $\hat{\Omega}$ of full $P$-measure such that $\theta(t,\cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$ for all $t \geq 0$ and*

$$X(t_2, \cdot, \theta(t_1, \omega)) \circ X(t_1, \cdot, \omega) = X(t_1 + t_2, \cdot, \omega)$$

*for all $\omega \in \hat{\Omega}$ and $t_1$, $t_2 \geq 0$.*

## *The Cocycle Property*




Vertical solid lines represent random fibers: copies of $M_2$. $(X, \theta)$ is a "vector-bundle morphism".

**Proof of Theorem 2.** (Sketch)

For simplicity consider case of a single delay $d_1$; i.e. $N = 1$ in (I).

*First step.*

Approximate the Brownian motion $W$ in (I) by smooth adapted processes $\{W^k\}_{k=1}^{\infty}$:

$$W^k(t) := k \int_{t-(1/k)}^{t} W(u)\,du - k \int_{-(1/k)}^{0} W(u)\,du, \qquad t \geq 0,\ k \geq 1. \tag{1}$$

Then each $W^k$ is a *helix* (i.e. has stationary increments):

$$W^k(t_1+t_2, \omega) - W^k(t_1, \omega) = W^k(t_2, \theta(t_1, \omega)), \qquad t_1, t_2 \in \mathbf{R},\ \omega \in \Omega. \tag{2}$$

Let $X^k : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ be the stochastic (semi)flow of the random fde's:

$$\left.\begin{aligned} dx^k(t) &= H(x^k(t-d_1), x^k(t), x^k_t)dt \\ &\qquad + \sum_{i=1}^m g_i x(t)(W_i^k)'(t)\, dt - \frac{1}{2}\sum_{i=1}^m g_i^2 x^k(t)\, dt \qquad t>0 \\ (x^k(0), x^k_0) &= (v,\eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d) \end{aligned}\right\} \qquad (I-k)$$

If $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ is the flow of (I) constructed in Theorem 1, then

$$\lim_{k\to\infty} \sup_{0\le t\le T} \|X^k(t,\cdot,\omega) - X(t,\cdot,\omega)\|_{L(M_2)} = 0 \qquad (3)$$

for every $0 < T < \infty$ and all $\omega$ in a Borel set $\hat{\Omega}$ of full Wiener measure which is invariant under $\theta(t,\cdot)$ for all $t \ge 0$ ([Mo], Stochastics, 1990). Prove (3) by stochastic variation:

Let $\phi : \mathbf{R}^+ \times \Omega \to \mathbf{R}^{d\times d}$ be the $d \times d$-matrix-valued solution of the linear Itô sode (without delay):

$$\left.\begin{aligned} d\phi(t) &= \sum_{i=1}^{m} g_i \phi(t)\, dW_i(t) \qquad\qquad t > 0 \\ \phi(0,\omega) &= I \in \mathbf{R}^{d\times d} \qquad\qquad \text{a.a. } \omega \end{aligned}\right\} \tag{4}$$

Denote by $\phi^k : \mathbf{R}^+ \times \Omega \to \mathbf{R}^{d\times d}$, $k \geq 1$, the $d \times d$-matrix solution of the random family of linear ode's:

$$\left.\begin{aligned} d\phi^k(t) &= \sum_{i=1}^{m} g_i \phi^k(t)(W_i^k)'(t) - \frac{1}{2}\sum_{i=1}^{m} g_i^2 \phi^k(t)\, dt \qquad\qquad t > 0 \\ \phi^k(0,\cdot) &= I \in \mathbf{R}^{d\times d}. \end{aligned}\right\} \tag{4'}$$

Let $\hat{\Omega}$ be the sure event of all $\omega \in \Omega$ such that

$$\phi(t,\omega) := \lim_{k\to\infty} \phi^k(t,\omega) \tag{5}$$

exists uniformly for $t$ in compact subsets of $\mathbf{R}^+$. Each $\phi^k$ is an $\mathbf{R}^{d\times d}$-valued *cocycle over* $\theta$, viz.

$$\phi^k(t_1+t_2,\omega) = \phi^k(t_2,\theta(t_1,\omega))\phi^k(t_1,\omega) \tag{6}$$

for all $t_1, t_2 \in \mathbf{R}^+$ and $\omega \in \Omega$. By definition of $\hat{\Omega}$ and passing to the limit in (6) as $k \to \infty$, conclude that $\{\phi(t,\omega) : t > 0,\ \omega \in \Omega\}$, is an $\mathbf{R}^{d\times d}$-valued *perfect* cocycle over $\theta$, viz.

(i) $P(\hat{\Omega}) = 1$;

(ii) $\theta(t,\cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$ for all $t \geq 0$;

(iii) $\phi(t_1+t_2,\omega) = \phi(t_2,\theta(t_1,\omega))\phi(t_1,\omega)$ for all $t_1, t_2 \in \mathbf{R}^+$ and *every* $\omega \in \hat{\Omega}$;

(iv) $\phi(\cdot,\omega)$ is continuous for every $\omega \in \hat{\Omega}$.

Alternatively use perfection theorem in ([M-S], AIHP, 1996, Theorem 3.1, p. 79-82) for crude cocycles with values in a metrizable second countable topological group. Observe that $\phi(t,\omega) \in GL(\mathbf{R}^d)$.

Define $\hat{H} : \mathbf{R}^+ \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ by

$$\hat{H}(t, v_1, v, \eta, \omega)$$
$$:= \phi(t,\omega)^{-1}[H(\phi_t(\cdot,\omega)(-d_1, v_1), \phi(t,\omega)(v), \phi_t(\cdot,\omega) \circ (id_J, \eta))] \tag{7}$$

for $\omega \in \Omega, t \geq 0,\ v, v_1 \in \mathbf{R}^d,\ \eta \in L^2([-r,0], \mathbf{R}^d),$ where

$$\phi_t(\cdot,\omega)(s,v) = \begin{cases} \phi(t+s,\omega)(v) & t+s \geq 0 \\ \\ v & -r \leq t+s < 0 \end{cases}$$

and

$$(id_J, \eta)(s) = (s, \eta(s)), \qquad s \in J.$$

Define $\hat{H}^k : \mathbf{R}^+ \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ by a relation similar to (7) with $\phi$ replaced by $\phi^k$. Then the random fde's

$$\left.\begin{aligned} y'(t) &= \hat{H}(t, y(t-d_1), y(t), y_t, \omega), \quad t > 0 \\ (y(0), y_0) &= (v, \eta) \in M_2 \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} {y^k}'(t) &= \hat{H}^k(t, y^k(t-d_1), y^k(t), y^k_t, \omega), \quad t > 0 \\ (y^k(0), y^k_0) &= (v, \eta) \in M_2 \end{aligned}\right\} \tag{9}$$

have unique *non-explosive* solutions

$$y,\, y^k : [-r, \infty) \times \Omega \to \mathbf{R}^d$$

([Mo], Stochastics, 1990, pp. 93-98). Itô's formula implies that

$$X(t, v, \eta, \omega) = (\phi(t, \omega)(y(t, \omega)), \phi_t(\cdot, \omega) \circ (id_J, y_t)) \tag{10}$$

The chain rule gives a similar relation for $X^k$ with $\phi$ replaced by $\phi^k$ ( [Mo], Stochastics, 1990, pp. 96-97).

Get the convergence

$$\lim_{k\to\infty} |\hat{H}^k(t, v_1, v, \eta, \omega) - \hat{H}(t, v_1, v, \eta, \omega)| = 0 \qquad (11)$$

uniformly for $(t, v_1, v, \eta)$ in bounded sets of $\mathbf{R}^+ \times \mathbf{R}^d \times M_2$. Use Gronwall's lemma and (11) to deduce (3).

*Second step.*

Fix $\omega \in \hat{\Omega}$ and use uniqueness of solutions to the approximating equation (I-k) and the helix property (2) of $W^k$ to obtain the cocycle property for $(X^k, \theta)$:

$$X^k(t_2, \cdot, \theta(t_1, \omega)) \circ X^k(t_1, \cdot, \omega) = X^k(t_1 + t_2, \cdot, \omega)$$

for all $\omega \in \hat{\Omega}$ and $t_1,\ t_2 \geq 0,\ k \geq 1$.

*Third step.*

Pass to limit as $k \to \infty$ in the above identity and use the convergence (3) *in operator norm* to get the perfect cocycle property for $X$. $\square$

The a.s. Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t, (v(\omega), \eta(\omega)), \omega)\|_{M_2},$$

(for a.a. $\omega \in \Omega$, $(v, \eta) \in L^2(\Omega, M_2)$) of the system (I) are characterized by the following "spectral theorem". Each $\theta(t, \cdot)$ is ergodic and preserves Wiener measure $P$. The proof of Theorem 3 below uses compactness of $X(t, \cdot, \omega) : M_2 \to M_2$, $t \geq r$, together with an infinite-dimensional version of Oseledec's multiplicative ergodic theorem due to Ruelle (1982).

**Theorem 3.** ([Mo], 1990)

*Let $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ be the flow of (I) given in Theorem 1. Then there exist*

*(a) an $\mathcal{F}$-measurable set $\Omega^* \subseteq \Omega$ such that $P(\Omega^*) = 1$ and $\theta(t, \cdot)(\Omega^*) \subseteq \Omega^*$ for all $t \geq 0$,*

*(b)* *a fixed (non-random) sequence of real numbers* $\{\lambda_i\}_{i=1}^{\infty}$*, and*

*(c)* *a random family* $\{E_i(\omega) : i \geq 1, \omega \in \Omega^*\}$ *of (closed) finite-codimensional subspaces of* $M_2$*, with the following properties:*

*(i) If the* **Lyapunov spectrum** $\{\lambda_i\}_{i=1}^{\infty}$ *is infinite, then* $\lambda_{i+1} < \lambda_i$ *for all* $i \geq 1$ *and* $\lim_{i \to \infty} \lambda_i = -\infty$*; otherwise there is a fixed (non-random) integer* $N \geq 1$ *such that* $\lambda_N = -\infty < \lambda_{N-1} < \cdots < \lambda_2 < \lambda_1$*;*

*(ii) each map* $\omega \mapsto E_i(\omega)$*,* $i \geq 1$*, is* $\mathcal{F}$*-measurable into the Grassmannian of* $M_2$*;*

*(iii)* $E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2$*,* $i \geq 1$*,* $\omega \in \Omega^*$*;*

*(iv) for each* $i \geq 1$*, codim* $E_i(\omega)$ *is fixed independently of* $\omega \in \Omega^*$*;*

*(v) for each* $\omega \in \Omega^*$ *and* $(v, \eta) \in E_i(\omega) \backslash E_{i+1}(\omega)$*,*

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} = \lambda_i, \ i \geq 1;$$

*(vi)* **Top Exponent***:*

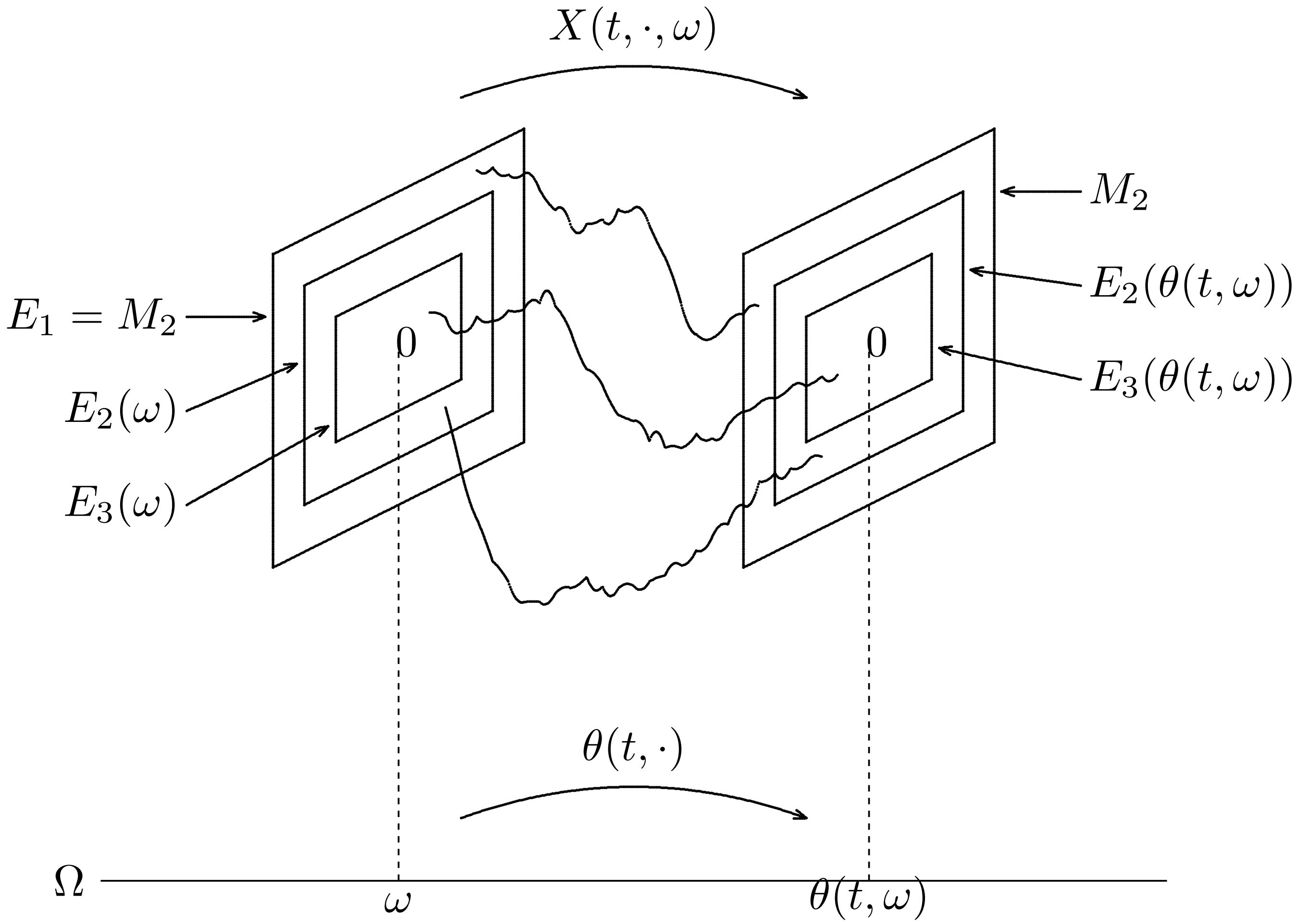
$$\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \log \|X(t, \cdot, \omega)\|_{L(M_2)} \quad \textit{for all } \omega \in \Omega^*;$$

*(vii)* **Invariance***:*

$$X(t, \cdot, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

*for all* $\omega \in \Omega^*, \ t \geq 0, \ i \geq 1.$

*Spectral Theorem*



Proof of Theorem 3 is based on Ruelle's discrete version of Oseledec's multiplicative ergodic theorem in Hilbert space ([Ru], Ann. of Math. 1982, Theorem (1.1), p. 248 and Corollary (2.2), p. 253):

**Theorem 4** ([Ru], 1982)

*Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and* $\tau : \Omega \to \Omega$ *a* $P$*-preserving transformation. Assume that* $H$ *is a separable Hilbert space and* $T : \Omega \to L(H)$ *a measurable map (w.r.t. the Borel field on the space of all bounded linear operators* $L(H)$*). Suppose that* $T(\omega)$ *is compact for almost all* $\omega \in \Omega$*, and* $E \log^+ \|T(\cdot)\| < \infty$*. Define the family of linear operators* $\{T^n(\omega) : \omega \in \Omega,\ n \geq 1\}$ *by*

$$T^n(\omega) := T(\tau^{n-1}(\omega)) \circ \cdots T(\tau(\omega)) \circ T(\omega)$$

*for* $\omega \in \Omega,\ n \geq 1$.

*Then there is a set $\Omega_0 \in \mathcal{F}$ of full $P$-measure such that $\tau(\Omega_0) \subseteq \Omega_0$, and for each $\omega \in \Omega_0$, the limit*

$$\lim_{n\to\infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

*exists in the uniform operator norm and is a positive compact self-adjoint operator on $H$. Furthermore, each $\Lambda(\omega)$ has a discrete spectrum*

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

*where the $\mu_i$'s are distinct. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at $\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigen-space $F_i(\omega)$, with $m_i(\omega) := \mathrm{dim} F_i(\omega)$. Define*

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \quad E_\infty(\omega) := \ker \Lambda(\omega).$$

*Then*

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = H$$

*and*

$$\lim_{n\to\infty} \frac{1}{n} \log \|T^n(\omega)x\|_H = \begin{cases} \mu_i(\omega), & \textit{if} \quad x \in E_i(\omega)\backslash E_{i+1}(\omega) \\ -\infty & \textit{if} \quad x \in \textit{ker}\ \Lambda(\omega). \end{cases}$$

**Proof.**

[Ru], Ann. of Math., 1982, pp. 248-254.

□

The following "perfect" version of Kingman's subadditive ergodic theorem is also used to construct the shift invariant set $\Omega^*$ appearing in Theorem 3 above.

**Theorem 5**([M], 1990)("Perfect" Subadditive Ergodic Theorem)

*Let* $f : \mathbf{R}^+ \times \Omega \to \mathbf{R} \cup \{-\infty\}$ *be a measurable process on the complete probability space* $(\Omega, \mathcal{F}, P)$ *such that*

(i) $E \sup_{0 \le u \le 1} f^+(u, \cdot) < \infty$, $E \sup_{0 \le u \le 1} f^+(1-u, \theta(u, \cdot)) < \infty$;

(ii) $f(t_1+t_2, \omega) \le f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ *for all* $t_1, t_2 \ge 0$ *and* **every** $\omega \in \Omega$.

*Then there exist a set* $\hat{\hat{\Omega}} \in \mathcal{F}$ *and a measurable* $\tilde{f} : \Omega \to \mathbf{R} \cup \{-\infty\}$ *with the properties:*

(a) $P(\hat{\hat{\Omega}}) = 1$, $\theta(t, \cdot)(\hat{\hat{\Omega}}) \subseteq \hat{\hat{\Omega}}$ *for all* $t \ge 0$;

(b) $\tilde{f}(\omega) = \tilde{f}(\theta(t, \omega))$ *for all* $\omega \in \hat{\hat{\Omega}}$ *and all* $t \ge 0$;

(c) $\tilde{f}^+ \in \mathbf{L}^1(\Omega, \mathbf{R}; P)$;

(d) $\lim_{t \to \infty} (1/t) f(t, \omega) = \tilde{f}(\omega)$ *for every* $\omega \in \hat{\hat{\Omega}}$.

*If* $\theta$ *is ergodic, then there exist* $f^* \in \mathbf{R} \cup \{-\infty\}$ *and* $\tilde{\tilde{\Omega}} \in \mathcal{F}$ *such that* *(a)'* $P(\tilde{\tilde{\Omega}}) = 1, \theta(t,\cdot)(\tilde{\tilde{\Omega}}) \subseteq \tilde{\tilde{\Omega}}$, $t \geq 0$;

*(b)'* $\tilde{f}(\omega) = f^* = \lim_{t\to\infty} (1/t) f(t,\omega)$ *for every* $\omega \in \tilde{\tilde{\Omega}}$.

**Proof.**

[Mo], Stochastics, 1990, Lemma 7, pp. 115–117. □

Proof of Theorem 3 is an application of Theorem 4. Requires Theorem 5 and the following sequence of lemmas.

## Lemma 1

*For each integer* $k \geq 1$ *and any* $0 < a < \infty$,

$$E \sup_{0\leq t\leq a} \|\phi(t,\omega)^{-1}\|^{2k} < \infty;$$

$$E \sup_{0\leq t_1,t_2\leq a} \|\phi(t_2,\theta(t_1,\cdot))\|^{2k} < \infty.$$

**Proof.**

Follows by standard sode estimates, the cocycle property for $\phi$ and Hölder's inequality. ([Mo], pp. 106-108). □

The next lemma is a crucial estimate needed to apply Ruelle-Oseledec theorem (Theorem 4).

**Lemma 2**

$$E \sup_{0\le t_1,t_2\le r} \log^+ \|X(t_2,\cdot,\theta(t_1,\cdot))\|_{L(M_2)} < \infty.$$

**Proof.**

If $y(t,(v,\eta),\omega)$ is the solution of the fde (8), then using Gronwall's inequality, taking $E \sup_{0\le t_1,t_2\le r} \log^+ \sup_{\|(v,\eta)\|\le 1}$ and applying Lemma 1, gives

$$E \sup_{0\le t_1,t_2\le r} \log^+ \sup_{\|(v,\eta)\|\le 1} \|(y(t_2,(v,\eta),\theta(t_1,\cdot)),y_{t_2}(\cdot,(v,\eta),\theta(t_1,\cdot)))\|_{M_2} < \infty.$$

Conclusion of lemma now follows by replacing $\omega'$ with $\theta(t_1, \omega)$ in the formula

$$X(t_2, (v, \eta), \omega')$$
$$= (\phi(t_2, \omega')(y(t_2, (v, \eta), \omega')), \phi_{t_2}(\cdot, \omega') \circ (id_J, y_{t_2}(\cdot, (v, \eta), \omega'))$$

and Lemma 1. $\square$

The existence of the Lyapunov exponents is obtained by interpolating the discrete limit

$$\frac{1}{r} \lim_{k \to \infty} \frac{1}{k} \log \|X(kr, (v(\omega), \eta(\omega)), \omega)\|_{M_2}, \qquad (12)$$

a.a. $\omega \in \Omega$, $(v, \eta) \in L^2(\Omega, M_2)$, between delay periods of length $r$. This requires the next two lemmas.

## Lemma 3

*Let* $h : \Omega \to \mathbf{R}^+$ *be* $\mathcal{F}$*-measurable and suppose* $E \sup_{0 \le u \le r} h(\theta(u, \cdot)$ *is finite. Then*

$$\Omega_1 := \big(\lim_{t \to \infty} \frac{1}{t} h(\theta(t, \cdot) = 0\big)$$

*is a sure event and* $\theta(t, \cdot)(\Omega_1) \subseteq \Omega_1$ *for all* $t \geq 0$*.*

## Proof.

Use interpolation between delay periods and the discrete ergodic theorem applied to the $L^1$ function

$$\hat{h} := \sup_{0 \le u \le r} h(\theta(u, \cdot).$$

([Mo], Stochastics, 1990, Lemma 5, pp. 111-113.) □

## Lemma 4

*Suppose there is a sure event $\Omega_2$ such that $\theta(t,\cdot)(\Omega_2) \subseteq \Omega_2$ for all $t \geq 0$, and the limit (12) exists (or equal to $-\infty$) for all $\omega \in \Omega_2$ and all $(v,\eta) \in M_2$. Then there is a sure event $\Omega_3$ such that $\theta(t,\cdot)(\Omega_3) \subseteq \Omega_3$ and*

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2} = \frac{1}{r} \lim_{k\to\infty} \frac{1}{k} \log \|X(kr,(v,\eta),\omega)\|_{M_2}, \tag{13}$$

*for all $\omega \in \Omega_3$ and all $(v,\eta) \in M_2$.*

## Proof:

Take $\Omega_3 := \hat{\Omega} \cap \Omega_1 \cap \Omega_2$. Use cocycle property for $X$, Lemma 2 and Lemma 3 to interpolate. ([Mo], Stochastics 1990, Lemma 6, pp. 113-114.)

$\square$

**Proof of Theorem 3.** (Sketch)

Apply Ruelle-Oseledec Theorem (Theorem 4) with

$T(\omega) := X(r,\omega) \in L(M_2)$, compact linear for $\omega \in \hat{\Omega}$;

$\tau : \Omega \to \Omega; \quad \tau := \theta(r,\cdot)$.

Then cocycle property for $X$ implies

$$\begin{aligned} X(kr,\omega,\cdot) &= T(\tau^{k-1}(\omega)) \circ T(\tau^{k-2}(\omega)) \circ \cdots \circ T(\tau(\omega)) \circ T(\omega) \\ &:= T^k(\omega) \end{aligned}$$

for all $\omega \in \hat{\Omega}$.

Lemma 2 implies

$$E \log^+ \|T(\cdot)\|_{L(M_2)} < \infty.$$

Theorem 4 gives a random family of compact self-adjoint positive linear operators $\{\Lambda(\omega) : \omega \in \Omega_4\}$ such that

$$\lim_{n\to\infty}[T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm for $\omega \in \Omega_4$, a (continuous) shift-invariant set of full measure. Furthermore each $\Lambda(\omega)$ has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

where the $\mu_i'$s are distinct, with no accumulation points except possibly $-\infty$. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at $\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigenspace $F_i(\omega)$, with $m_i(\omega) := dim F_i(\omega)$. Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \quad E_\infty(\omega) := \ker \Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2.$$

Note that $\operatorname{codim} E_i(\omega) = \sum_{j=1}^{i-1} m_j(\omega) < \infty$. Also

$$\lim_{k\to\infty} \frac{1}{k} \log \|X(kr, (v,\eta), \omega)\|_{M_2} = \begin{cases} \mu_i(\omega), & \text{if } (v,\eta) \in E_i(\omega) \backslash E_{i+1}(\omega) \\ -\infty & \text{if } (v,\eta) \in \ker \Lambda(\omega). \end{cases}$$

The functions

$$\omega \mapsto \mu_i(\omega), \quad \omega \mapsto m_i(\omega), \quad \omega \mapsto N(\omega)$$

are invariant under the ergodic shift $\theta(r,\cdot)$. Hence they take the fixed values $\mu_i,\ m_i,\ N$ almost surely, respectively.

Lemma 4 gives a continuous-shift-invariant sure event $\Omega^* \subseteq \Omega_4$ such that

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log \|X(t, (v,\eta), \omega)\|_{M_2} &= \frac{1}{r} \lim_{k\to\infty} \frac{1}{k} \log \|X(kr, (v,\eta), \omega)\|_{M_2} \\ &= \frac{\mu_i}{r} =: \lambda_i, \end{aligned}$$

for $(v,\eta) \in E_i(\omega)\backslash E_{i+1}(\omega),\ \omega \in \Omega^*, i \geq 1$.

$\{\lambda_i := \frac{\mu_i}{r} : i \geq 1\}$ is the *Lyapunov spectrum* of (I).

Since Lyapunov spectrum is discrete with no finite accumulation points, then $\{\lambda_i : \lambda_i > \lambda\}$ is finite for all $\lambda \in \mathbf{R}$.

To prove invariance of the Oseledec space $E_i(\omega)$ under the cocycle $(X, \theta)$ use the random field

$$\lambda((v,\eta),\omega) := \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2},\ (v,\eta) \in M_2,\ \omega \in \Omega^*$$

and the relations

$$E_i(\omega) := \{(v,\eta) \in M_2 : \lambda((v,\eta),\omega) \leq \lambda_i\},$$

$$\lambda(X(t,(v,\eta),\omega),\theta(t,\omega)) = \lambda((v,\eta),\omega), \quad \omega \in \Omega^*,\ t \geq 0$$

([Mo], Stochastics 1990, p. 122). $\square$

Lyapunov exponents $\{\lambda_i\}_{i=1}^{\infty}$ of (I) are non-random because $\theta$ is ergodic. Say (I) is *hyperbolic* if $\lambda_i \neq 0$ for all $i \geq 1$. When (I) is hyperbolic the flow satisfies a *stochastic saddle-point property* (or exponential dichotomy) (cf. the deterministic case with $E = C([-r, 0], \mathbf{R}^d)$, $g_i \equiv 0$, $i = 1, \ldots, m$, in Hale [H], Theorem 4.1, p. 181).

**Theorem 6** *(Random Saddles)*([Mo], 1990)

*Suppose the sfde (I) is hyperbolic. Then there exist*

*(a) a set* $\tilde{\Omega}^* \in \mathcal{F}$ *such that* $P(\tilde{\Omega}^*) = 1$, *and* $\theta(t, \cdot)(\tilde{\Omega}^*) = \tilde{\Omega}^*$ *for all* $t \in \mathbf{R}$,

*and*

*(b) a measurable splitting*

$$M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \tilde{\Omega}^*,$$

*with the following properties:*

(i) $\mathcal{U}(\omega)$, $\mathcal{S}(\omega)$, $\omega \in \tilde{\Omega}^*$, *are closed linear subspaces of* $M_2$, $\dim \mathcal{U}(\omega)$ *is finite and fixed independently of* $\omega \in \tilde{\Omega}^*$.

(ii) *The maps* $\omega \mapsto \mathcal{U}(\omega)$, $\omega \mapsto \mathcal{S}(\omega)$ *are* $\mathcal{F}$*-measurable into the Grassmannian of* $M_2$.

(iii) *For each* $\omega \in \tilde{\Omega}^*$ *and* $(v, \eta) \in \mathcal{S}(\omega)$ *there exists* $\tau_1 = \tau_1(v, \eta, \omega) > 0$ *and a positive* $\delta_1$, *independent of* $(v, \eta, \omega)$ *such that*

$$\|X(t, (v, \eta), \omega)\|_{M_2} \leq \|(v, \eta)\|_{M_2} e^{-\delta_1 t}, \quad t \geq \tau_1.$$

(iv) *For each* $\omega \in \tilde{\Omega}^*$ *and* $(v, \eta) \in \mathcal{U}(\omega)$ *there exists* $\tau_2 = \tau_2(v, \eta, \omega) > 0$ *and a positive* $\delta_2$, *independent of* $(v, \eta, \omega)$ *such that*

$$\|X(t, (v, \eta), \omega)\|_{M_2} \geq \|(v, \eta)\|_{M_2} e^{\delta_2 t}, \quad t \geq \tau_2.$$

*(v) For each $t \geq 0$ and $\omega \in \tilde{\Omega}^*$,*

$$X(t,\omega,\cdot)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t,\omega)),$$

$$X(t,\omega,\cdot)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)).$$

*In particular, the restriction*

$$X(t,\omega,\cdot)\,|\,\mathcal{U}(\omega) : \mathcal{U}(\omega) \to \mathcal{U}(\theta(t,\omega))$$

*is a linear homeomorphism onto.*

**Proof.**

[Mo], Stochastics, 1990, Corollary 2, pp. 127-130. □

# *The Saddle-Point Property*