The Linking Homomorphism of One-dimensional Minimal Sets

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Abstract: We introduce a way of characterizing the linking of one-dimensional minimal sets in three-dimensional flows and carry out the characterization for some minimal sets within flows modeled by templates.

The Paper

- link.ps
- lorenz.eps (Figure 1)
- 12363.ps (Figure 2)
- schematic.eps (Figure 3)
- tre323.ps (Figure 4)
- tre112.ps (Figure 5)

Minimal Sets

If f is a continuous flow or map on a space X, then a closed subset M of X is a **minimal set** of f if M is invariant under f but contains no proper, non-empty, closed set which is also invariant. Periodic orbits are minmal sets, but more complicated minimal sets exist.

References:

- Qualitative Theory of Differential Equations, Nemytskii & Stepanov, Princeton University Press, 1960.
- Lectures on Topological Dynamics, Robert Ellis, Benjamin, Inc., 1969.

Sturmian Minimal Sets

For each irrational number α in the open interval (0,1), a Strumian minimal set can be generated by the following algorythm. Let t be a real number. Let $\beta_n = t + n\alpha \bmod 1$. Then define x_n to be 0 when β_n is less than or equal to α or 1 otherwise. Then $(x_n)_{n>0}$ is a (one-sided) Sturmian minimal set. Sturmian minimal sets have "minimal block growth." Thus, they are the simplest minimal sets after periodic orbits.

References:

- Gottschalk, W. H., Hedlund, G., *Topological dynamics*, A. M. S. Colloquium Publications, Vol. 36., 1955.

- Coven, E. M.; Hedlund, G. A., *Sequences with minimal block growth*, Math. Systems Theory 7 (1973), 138--153.
- Paul, M. E., *Minimal symbolic flows having minimal block growth*, Math. Systems Theory 8 (1974/75), no. 4, 309--315.

Below is a maple procedure for generating Sturmian minimal sets:

```
> sturmian:=proc(alpha, t, numterms)
local i, word;
word:=[];
for i from 1 to numterms do
if (evalf(frac(t+i*alpha)) <= evalf(alpha)) then word:=[op(word),0] else word:=[op(word),1] fi;
od;
word;
end:

> sturmian(sqrt(1/2),0, 100);
```

[0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1]

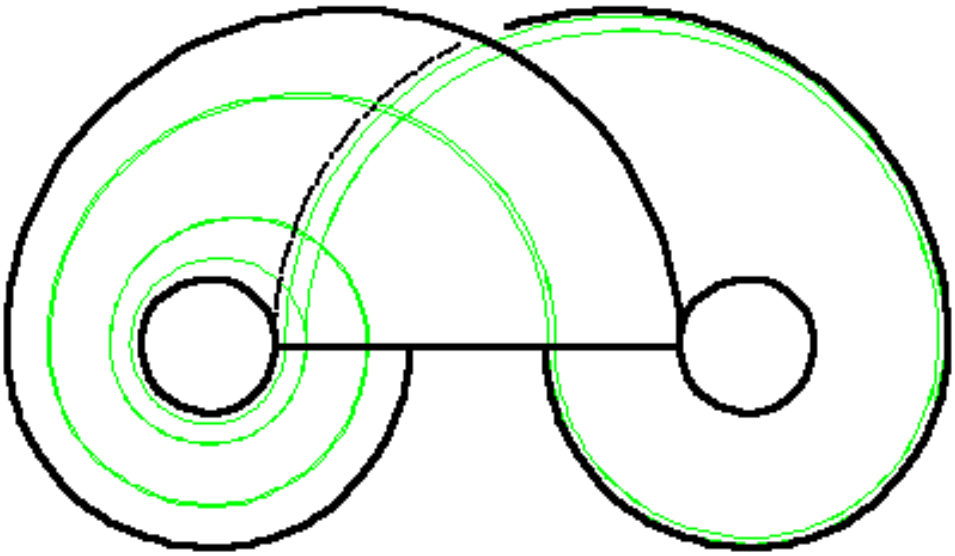
Lorenz Template

The Lorenz template is a branched 2-manifold with a semi-flow that gives a geometic model of the famous Lorenz attractor. Orbits in the Lorenz template are associated to the full 2-shift, and so contain Sturmian minimal sets.

Refernces:

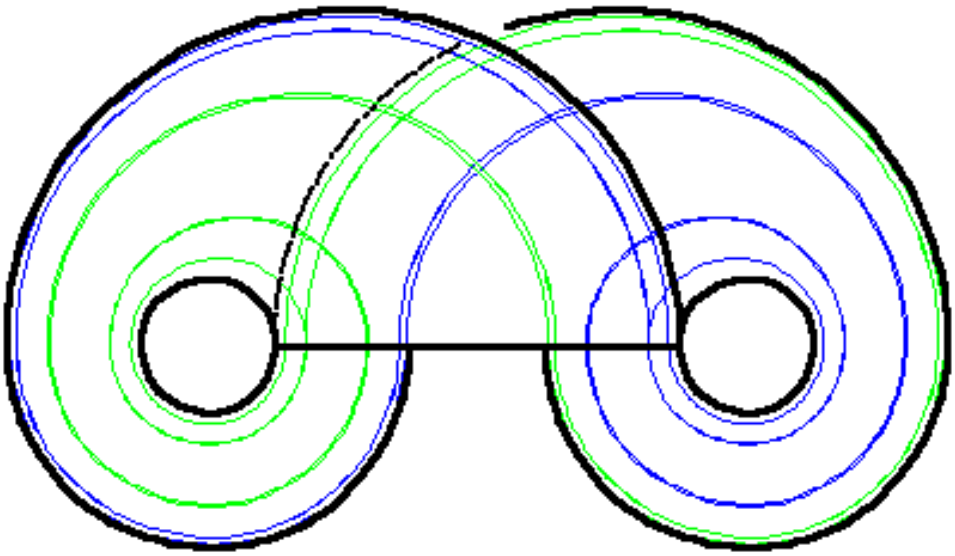
- Birman, J. S.; Williams, R. F., *Knotted periodic orbits in dynamical systems. I. Lorenz's equations.* Topology 22 (1983), no. 1, 47--82.
- Birman, J. S.; Williams, R. F., *Knotted periodic orbits in dynamical system. II. Knot holders for fibered knots*, Contemp. Math., 20, Amer. Math. Soc., (1983), 1--60.
- Ghrist, R.; Holmes, P. J.; Sullivan, M. C., *Knots and links in three-dimensional flows.* Lecture Notes in Mathematics, 1654 (1997), Springer-Verlag, Berlin.

We have used a Maple procedure to plot the Sturmian minimal set above in the Lorenz template.



Linking

In the above figure the minimal set appears to live in a small neighborhood of a wedge of two circles, a figure-8. The homology calculations in our paper confirm this. This allows one to talk about how two Sturmian minimal sets are linked. But instead of a linking number one gets a 2x2 linking matrix. Because one has some freedom in choosing bases for the neighborhoods the matrix itself is not an invariant, but its Smith normal form is. (See the ismith command in Maple.) Below is a plot of two Sturmian minimal sets. The the green one has α =1/sqrt(2), the blue one has α=1/sqrt(11). Both have t=0.



Careful inspection gives a preliminary linking matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, whose Smith normal form is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Other plots not in paper:

- ex3.gif blue alpha=1/sqrt{11} red alpha=1/sqrt{7}
- ex4.gif blue alpha_1=1/sqrt{11} red alpha_2=1/sqrt{6}
- 22225.ps
- 22ln22.ps
- cube1over9.ps
- morse.psMorse sequence
- 001.gifA generalized Morse sequence with seed word 001.
- 0011.gifA generalized Morse sequence with seed word 0011.

Maple programs used to make plots

- Lorenz.mws Maple release 7 worksheet.