SFDE'S
AS DYNAMICAL SYSTEMS
I: THE LINEAR CASE

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1. Regular Linear SFDE's-Ergodic Theory.

Linear sfde's on $\mathbf{R}^d$ driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$.

$$\left.\begin{aligned} dx(t) &= H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\ &\qquad + \sum_{i=1}^{m} g_i x(t)\, dW_i(t), \qquad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \quad (I)$$

(I) is defined on

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}}, P)$ = canonical complete filtered Wiener space.

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R} \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F}$:=(completed) Borel σ-field of Ω;

$\mathcal{F}_t :=$ (completed) sub-σ-field of $\mathcal{F}$ generated by the evaluations $\omega \to \omega(u),\ \ u \leq t, \quad t \in \mathbf{R}$.

$P :=$ Wiener measure on Ω.

$dW_i(t) =$ Itô stochastic differentials.

Several finite delays $0 < d_1 < d_2 < \cdots < d_N \leq r$ in drift term; *no delays in diffusion coefficient.*

$H : (\mathbf{R}^d)^{N+1} \times L^2([-r,0],\mathbf{R}^d) \to \mathbf{R}^d$ is a fixed continuous linear map, g_i, $i = 1, 2, \ldots, m$, fixed (deterministic) $d \times d$-matrices.

2. Plan

Use state space $M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d)$. For (I) consider the following themes:

I) Existence of a "perfect" cocycle on M_2-a modification of the trajectory field $(x(t), x_t) \in M_2$.

II) Existence of almost sure Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|(x(t), x_t)\|_{M_2}$$

Multiplicative ergodic theorem and *hyperbolicity* of cocycle.

III) *"Random Saddle-Point Property"* in hyperbolic case.

3. Regularity

Say SFDE (I) is *regular* (wrt. M_2) if trajectory $\{(x(t), x_t) : (x(0), x_0) = (v, \eta) \in M_2\}$ admits a measurable modification $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ such that $X(\cdot, \cdot, \omega)$ is continuous for a.a. $\omega \in \Omega$.

Theorem 1.([Mo], 1990])

(I) is regular with respect to state space $M_2 = \mathbf{R}^d \times \mathbf{L}^2([-r, 0], \mathbf{R}^d)$. There is a measurable version $X : \mathbf{R}^+ \times$

$M_2 \times \Omega \to M_2$ of the trajectory field $\{(x(t), x_t) : t \in \mathbf{R}^+,\ (x(0), x_0) = (v, \eta) \in M_2\}$ of (I) with the following properties:

- *(i) For each $(v, \eta) \in M_2$ and $t \in \mathbf{R}^+$, $X(t, (v, \eta), \cdot) = (x(t), x_t)$ a.s., is $\mathcal{F}_t$-measurable and belongs to $L^2(\Omega, M_2; P)$.*
- *(ii) There exists $\Omega_0 \in \mathcal{F}$ of full measure such that, for all $\omega \in \Omega_0$, the map $X(\cdot, \cdot, \omega) : \mathbf{R}^+ \times M_2 \to M_2$ is continuous.*
- *(iii) For each $t \in \mathbf{R}^+$ and every $\omega \in \Omega_0$, the map $X(t, \cdot, \omega) : M_2 \to M_2$ is continuous linear; for each $\omega \in \Omega_0$, the map $\mathbf{R}^+ \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ is measurable and locally bounded in the uniform operator norm on $L(M_2)$. The map $[r, \infty) \ni t \mapsto X(t, \cdot, \omega) \in L(M_2)$ is continuous for all $\omega \in \Omega_0$.*

(iv) For each $t \geq r$ and all $\omega \in \Omega_0$, the map

$$X(t, \cdot, \omega) : M_2 \to M_2$$

is compact.

Compactness of semi-flow for $t \geq r$ will be used to define hyperbolicity for (I) and the associated exponential dichotomies.

Example: $dx(t) = x(t-1)\, dW(t)$ *is not regular (singular).*

4. Lyapunov Exponents. Hyperbolicity

Version X of the trajectory field of (I) (in Theorem 1) is a multiplicative $L(M_2)$-valued linear cocycle over the canonical Brownian shift $\theta : \mathbf{R} \times \Omega \to \Omega$ on Wiener space:

$$\theta(t, \omega)(u) := \omega(t+u) - \omega(t), \quad u, t \in \mathbf{R}, \quad \omega \in \Omega.$$

I.e.

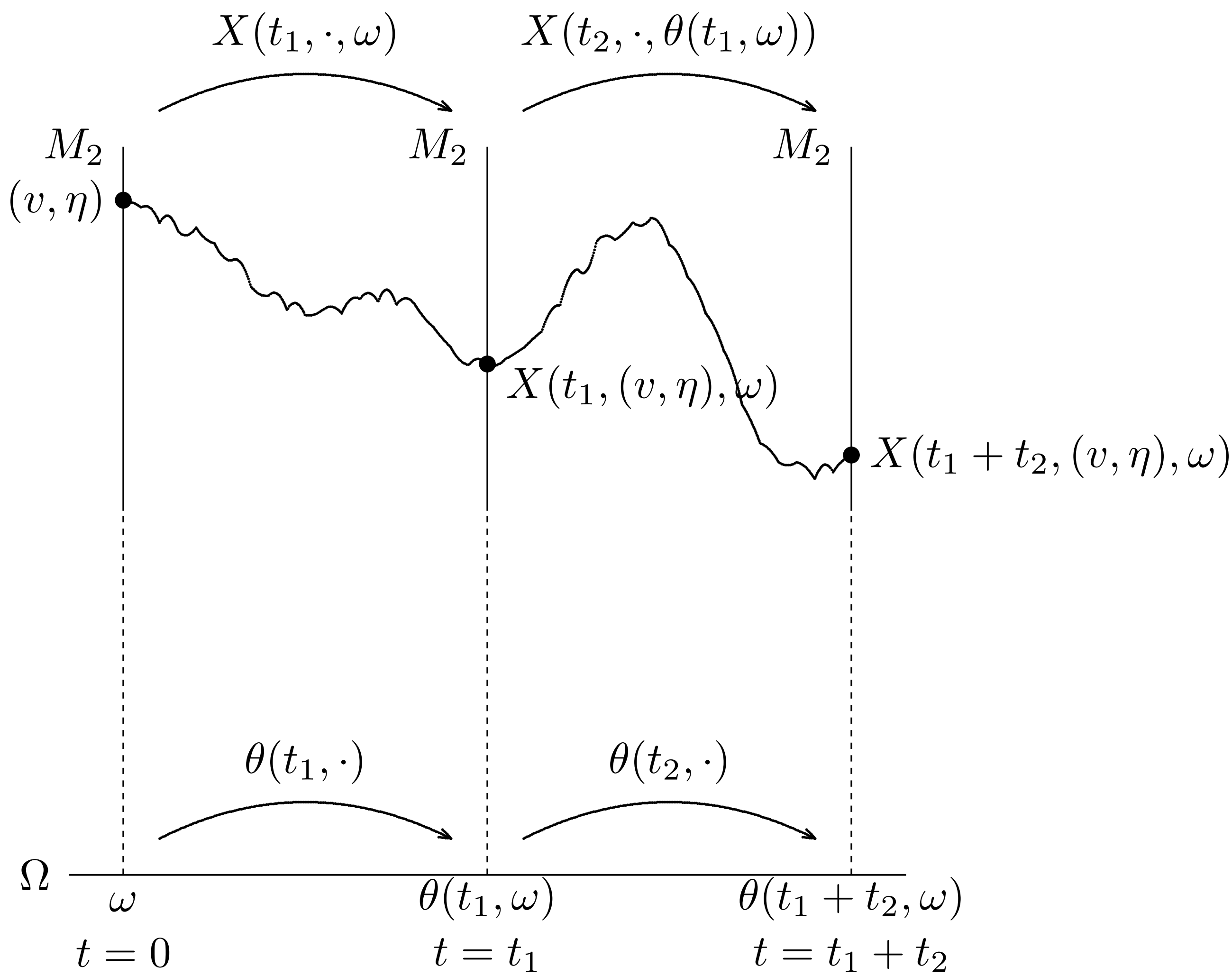
Theorem 2([Mo], 1990)

There is an $\mathcal{F}$-measurable set $\hat{\Omega}$ of full P-measure such that $\theta(t, \cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$ for all $t \geq 0$ and

$$X(t_2, \cdot, \theta(t_1, \omega)) \circ X(t_1, \cdot, \omega) = X(t_1 + t_2, \cdot, \omega)$$

for all $\omega \in \hat{\Omega}$ and t_1, $t_2 \geq 0$.

The Cocycle Property



Vertical solid lines represent random fibers: copies of M_2. (X, θ) is a "vector-bundle morphism".

The a.s. Lyapunov exponents

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v(\omega),\eta(\omega)),\omega)\|_{M_2},$$

(for a.a. $\omega \in \Omega$, $(v,\eta) \in L^2(\Omega, M_2)$) of the system (I) are characterized by the following "spectral theorem". Each $\theta(t,\cdot)$ is ergodic and preserves Wiener measure P. The proof of Theorem 3 below uses compactness of $X(t,\cdot,\omega) : M_2 \to M_2$, $t \geq r$, together with an infinite-dimensional version of Oseledec's multiplicative ergodic theorem due to Ruelle (1982).

Theorem 3. ([Mo], 1990)

Let $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ be the flow of (I) given in Theorem 1. Then there exist

(a) an $\mathcal{F}$-measurable set $\Omega^* \subseteq \Omega$ such that $P(\Omega^*) = 1$ and $\theta(t,\cdot)(\Omega^*) \subseteq \Omega^*$ for all $t \geq 0$,

(b) *a fixed (non-random) sequence of real numbers* $\{\lambda_i\}_{i=1}^{\infty}$*, and*

(c) *a random family* $\{E_i(\omega) : i \geq 1, \omega \in \Omega^*\}$ *of (closed) finite-codimensional subspaces of* M_2*, with the following properties:*

(i) If the **Lyapunov spectrum** $\{\lambda_i\}_{i=1}^{\infty}$ *is infinite, then* $\lambda_{i+1} < \lambda_i$ *for all* $i \geq 1$ *and* $\lim_{i\to\infty} \lambda_i = -\infty$*; otherwise there is a fixed (non-random) integer* $N \geq 1$ *such that* $\lambda_N = -\infty < \lambda_{N-1} < \cdots < \lambda_2 < \lambda_1$*;*

(ii) each map $\omega \mapsto E_i(\omega)$*,* $i \geq 1$*, is* $\mathcal{F}$*-measurable into the Grassmannian of* M_2*;*

(iii) $E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2$*,* $i \geq 1$*,* $\omega \in \Omega^*$*;*

(iv) for each $i \geq 1$*, codim* $E_i(\omega)$ *is fixed independently of* $\omega \in \Omega^*$*;*

(v) for each $\omega \in \Omega^*$ *and* $(v, \eta) \in E_i(\omega) \backslash E_{i+1}(\omega)$,

$$\lim_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2} = \lambda_i, \; i \geq 1;$$

(vi) **Top Exponent***:*

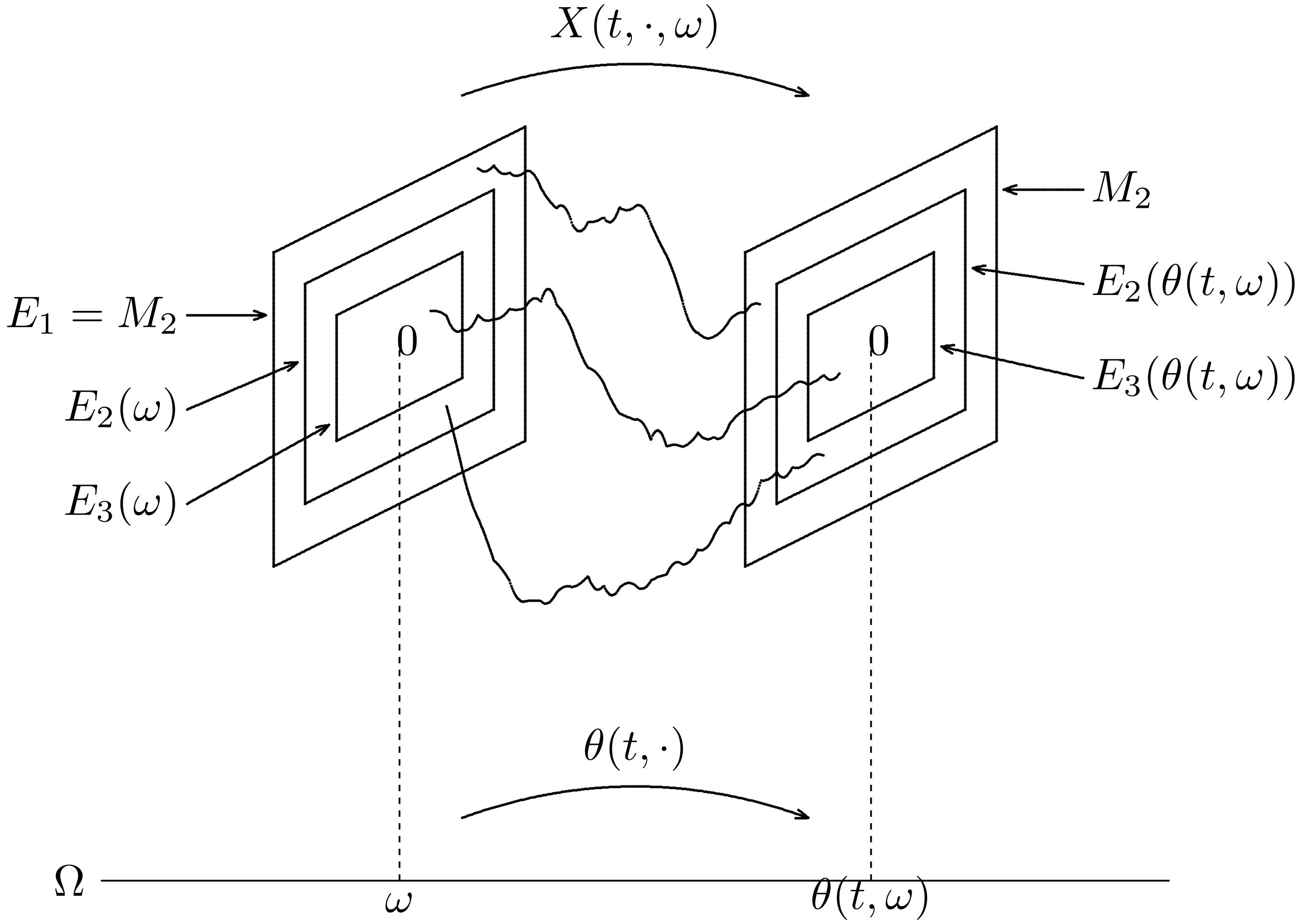
$$\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \log \|X(t, \cdot, \omega)\|_{L(M_2)} \quad \text{for all } \omega \in \Omega^*;$$

(vii) **Invariance***:*

$$X(t, \cdot, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

for all $\omega \in \Omega^*, \; t \geq 0, \; i \geq 1.$

Spectral Theorem



Proof of Theorem 3 is based on Ruelle's discrete version of Oseledec's multiplicative ergodic theorem in Hilbert space ([Ru], Ann. of Math. 1982, Theorem (1.1), p. 248 and Corollary (2.2), p. 253):

Theorem 4 ([Ru], 1982)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\tau : \Omega \to \Omega$ a P-preserving transformation. Assume that H is a separable Hilbert space and $T : \Omega \to L(H)$ a measurable map (w.r.t. the Borel field on the space of all bounded linear operators $L(H)$). Suppose that $T(\omega)$ is compact for almost all $\omega \in \Omega$, and $E \log^+ \|T(\cdot)\| < \infty$. Define the family of linear operators $\{T^n(\omega) : \omega \in \Omega,\ n \geq 1\}$ by

$$T^n(\omega) := T(\tau^{n-1}(\omega)) \circ \cdots T(\tau(\omega)) \circ T(\omega)$$

for $\omega \in \Omega,\ n \geq 1$.

Then there is a set $\Omega_0 \in \mathcal{F}$ of full P-measure such that $\tau(\Omega_0) \subseteq \Omega_0$, and for each $\omega \in \Omega_0$, the limit

$$\lim_{n\to\infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm and is a positive compact self-adjoint operator on H. Furthermore, each $\Lambda(\omega)$ has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

where the μ_i's are distinct. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at $\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigen-space $F_i(\omega)$, with $m_i(\omega) :=$ $\dim F_i(\omega)$. Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \quad E_\infty(\omega) := \ker \Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = H$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|T^n(\omega)x\|_H = \begin{cases} \mu_i(\omega), & \textit{if} \quad x \in E_i(\omega) \backslash E_{i+1}(\omega) \\ -\infty & \textit{if} \quad x \in \textit{ker}\ \Lambda(\omega). \end{cases}$$

Proof.

[Ru], Ann. of Math., 1982, pp. 248-254.

□

The following "perfect" version of Kingman's subadditive ergodic theorem is also used to construct the shift invariant set Ω^* appearing in Theorem 3 above.

Theorem 5([M], 1990)("Perfect" Subadditive Ergodic Theorem)

Let $f : \mathbf{R}^+ \times \Omega \to \mathbf{R} \cup \{-\infty\}$ *be a measurable process on the complete probability space* $(\Omega, \mathcal{F}, P)$ *such that*

(i) $E \sup\limits_{0 \leq u \leq 1} f^+(u, \cdot) < \infty, E \sup\limits_{0 \leq u \leq 1} f^+(1-u, \theta(u, \cdot)) < \infty;$

(ii) $f(t_1+t_2, \omega) \leq f(t_1, \omega)+f(t_2, \theta(t_1, \omega))$ *for all* $t_1, t_2 \geq 0$ *and* **every** $\omega \in \Omega$.

Then there exist a set $\hat{\hat{\Omega}} \in \mathcal{F}$ *and a measurable* $\tilde{f} : \Omega \to \mathbf{R} \cup \{-\infty\}$ *with the properties:*

(a) $P(\hat{\hat{\Omega}}) = 1, \ \theta(t, \cdot)(\hat{\hat{\Omega}}) \subseteq \hat{\hat{\Omega}}$ *for all* $t \geq 0$;

(b) $\tilde{f}(\omega) = \tilde{f}(\theta(t, \omega))$ *for all* $\omega \in \hat{\hat{\Omega}}$ *and all* $t \geq 0$;

(c) $\tilde{f}^+ \in \mathbf{L}^1(\Omega, \mathbf{R}; P)$;

(d) $\lim\limits_{t \to \infty} (1/t) f(t, \omega) = \tilde{f}(\omega)$ *for every* $\omega \in \hat{\hat{\Omega}}$.

If θ *is ergodic, then there exist* $f^* \in \mathbf{R} \cup \{-\infty\}$ *and* $\tilde{\tilde{\Omega}} \in \mathcal{F}$ *such that* *(a)′* $P(\tilde{\tilde{\Omega}}) = 1, \theta(t,\cdot)(\tilde{\tilde{\Omega}}) \subseteq \tilde{\tilde{\Omega}}$, $t \geq 0$;

(b)′ $\tilde{f}(\omega) = f^* = \lim_{t\to\infty}(1/t)f(t,\omega)$ *for every* $\omega \in \tilde{\tilde{\Omega}}$.

Proof.

[Mo], Stochastics, 1990, Lemma 7, pp. 115–117. □

Proof of Theorem 3 is an application of Theorem 4. Requires Theorem 5 and the following sequence of lemmas.

Lemma 1

For each integer $k \geq 1$ *and any* $0 < a < \infty$,

$$E \sup_{0\leq t\leq a} \|\phi(t,\omega)^{-1}\|^{2k} < \infty;$$

$$E \sup_{0\leq t_1,t_2\leq a} \|\phi(t_2,\theta(t_1,\cdot))\|^{2k} < \infty.$$

Proof.

Follows by standard sode estimates, the cocycle property for ϕ and Hölder's inequality. ([Mo], pp. 106-108). □

The next lemma is a crucial estimate needed to apply Ruelle-Oseledec theorem (Theorem 4).

Lemma 2

$$E \sup_{0 \le t_1, t_2 \le r} \log^+ \|X(t_2, \cdot, \theta(t_1, \cdot))\|_{L(M_2)} < \infty.$$

Proof.

If $y(t, (v, \eta), \omega)$ is the solution of the fde (8), then using Gronwall's inequality, taking $E \sup_{0 \le t_1, t_2 \le r} \log^+ \sup_{\|(v,\eta)\| \le 1}$ and applying Lemma 1, gives

$$E \sup_{0 \le t_1, t_2 \le r} \log^+ \sup_{\|(v,\eta)\| \le 1} \|(y(t_2, (v, \eta), \theta(t_1, \cdot)), y_{t_2}(\cdot, (v, \eta), \theta(t_1, \cdot)))\|_{M_2} < \infty.$$

Conclusion of lemma now follows by replacing ω' with $\theta(t_1, \omega)$ in the formula

$$X(t_2, (v, \eta), \omega')$$
$$= (\phi(t_2, \omega')(y(t_2, (v, \eta), \omega')), \phi_{t_2}(\cdot, \omega') \circ (id_J, y_{t_2}(\cdot, (v, \eta), \omega'))$$

and Lemma 1. □

The existence of the Lyapunov exponents is obtained by interpolating the discrete limit

$$\frac{1}{r} \lim_{k \to \infty} \frac{1}{k} \log \|X(kr, (v(\omega), \eta(\omega)), \omega)\|_{M_2}, \qquad (12)$$

a.a. $\omega \in \Omega$, $(v, \eta) \in L^2(\Omega, M_2)$, between delay periods of length r. This requires the next two lemmas.

Lemma 3

Let $h : \Omega \to \mathbf{R}^+$ *be* $\mathcal{F}$*-measurable and suppose* $E \sup_{0 \leq u \leq r} h(\theta(u, \cdot)$ *is finite. Then*

$$\Omega_1 := \big(\lim_{t \to \infty} \frac{1}{t} h(\theta(t, \cdot) = 0\big)$$

is a sure event and $\theta(t, \cdot)(\Omega_1) \subseteq \Omega_1$ *for all* $t \geq 0$*.*

Proof.

Use interpolation between delay periods and the discrete ergodic theorem applied to the L^1 function

$$\hat{h} := \sup_{0 \leq u \leq r} h(\theta(u, \cdot).$$

([Mo], Stochastics, 1990, Lemma 5, pp. 111-113.) □

Lemma 4

Suppose there is a sure event Ω_2 such that $\theta(t,\cdot)(\Omega_2) \subseteq \Omega_2$ for all $t \geq 0$, and the limit (12) exists (or equal to $-\infty$) for all $\omega \in \Omega_2$ and all $(v,\eta) \in M_2$. Then there is a sure event Ω_3 such that $\theta(t,\cdot)(\Omega_3) \subseteq \Omega_3$ and

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2} = \frac{1}{r} \lim_{k\to\infty} \frac{1}{k} \log \|X(kr,(v,\eta),\omega)\|_{M_2}, \tag{13}$$

for all $\omega \in \Omega_3$ and all $(v,\eta) \in M_2$.

Proof:

Take $\Omega_3 := \hat{\Omega} \cap \Omega_1 \cap \Omega_2$. Use cocycle property for X, Lemma 2 and Lemma 3 to interpolate. ([Mo], Stochastics 1990, Lemma 6, pp. 113-114.)

□

Proof of Theorem 3. (Sketch)

Apply Ruelle-Oseledec Theorem (Theorem 4) with

$T(\omega) := X(r, \omega) \in L(M_2)$, compact linear for $\omega \in \hat{\Omega}$;

$\tau : \Omega \to \Omega; \quad \tau := \theta(r, \cdot)$.

Then cocycle property for X implies

$$\begin{aligned} X(kr, \omega, \cdot) &= T(\tau^{k-1}(\omega)) \circ T(\tau^{k-2}(\omega)) \circ \cdots \circ T(\tau(\omega)) \circ T(\omega) \\ &:= T^k(\omega) \end{aligned}$$

for all $\omega \in \hat{\Omega}$.

Lemma 2 implies

$$E \log^+ \|T(\cdot)\|_{L(M_2)} < \infty.$$

Theorem 4 gives a random family of compact self-adjoint positive linear operators $\{\Lambda(\omega) : \omega \in \Omega_4\}$ such that

$$\lim_{n\to\infty} [T^n(\omega)^* \circ T^n(\omega)]^{1/(2n)} := \Lambda(\omega)$$

exists in the uniform operator norm for $\omega \in \Omega_4$, a (continuous) shift-invariant set of full measure. Furthermore each $\Lambda(\omega)$ has a discrete spectrum

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > e^{\mu_3(\omega)} > e^{\mu_4(\omega)} > \cdots$$

where the μ_i's are distinct, with no accumulation points except possibly $-\infty$. If $\{\mu_i\}_{i=1}^{\infty}$ is infinite, then $\mu_i \downarrow -\infty$; otherwise they terminate at $\mu_{N(\omega)} = -\infty$. If $\mu_i(\omega) > -\infty$, then $e^{\mu_i(\omega)}$ has finite multiplicity $m_i(\omega)$ and finite-dimensional eigenspace $F_i(\omega)$, with $m_i(\omega) := dim F_i(\omega)$. Define

$$E_1(\omega) := M_2, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \quad E_\infty(\omega) := \ker \Lambda(\omega).$$

Then

$$E_\infty(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2.$$

Note that $\operatorname{codim} E_i(\omega) = \sum_{j=1}^{i-1} m_j(\omega) < \infty$. Also

$$\lim_{k\to\infty} \frac{1}{k} \log \|X(kr, (v,\eta), \omega)\|_{M_2} = \begin{cases} \mu_i(\omega), & \text{if } (v,\eta) \in E_i(\omega) \backslash E_{i+1}(\omega) \\ -\infty & \text{if } (v,\eta) \in \ker \Lambda(\omega). \end{cases}$$

The functions

$$\omega \mapsto \mu_i(\omega), \quad \omega \mapsto m_i(\omega), \quad \omega \mapsto N(\omega)$$

are invariant under the ergodic shift $\theta(r,\cdot)$. Hence they take the fixed values $\mu_i,\ m_i,\ N$ almost surely, respectively.

Lemma 4 gives a continuous-shift-invariant sure event $\Omega^* \subseteq \Omega_4$ such that

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log \|X(t, (v,\eta), \omega)\|_{M_2} &= \frac{1}{r} \lim_{k\to\infty} \frac{1}{k} \log \|X(kr, (v,\eta), \omega)\|_{M_2} \\ &= \frac{\mu_i}{r} =: \lambda_i, \end{aligned}$$

for $(v,\eta) \in E_i(\omega)\backslash E_{i+1}(\omega),\ \omega \in \Omega^*, i \geq 1$.

$\{\lambda_i := \dfrac{\mu_i}{r} : i \geq 1\}$ is the *Lyapunov spectrum* of (I).

Since Lyapunov spectrum is discrete with no finite accumulation points, then $\{\lambda_i : \lambda_i > \lambda\}$ is finite for all $\lambda \in \mathbf{R}$.

To prove invariance of the Oseledec space $E_i(\omega)$ under the cocycle (X,θ) use the random field

$$\lambda((v,\eta),\omega) := \lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega)\|_{M_2},\ (v,\eta) \in M_2,\ \omega \in \Omega^*$$

and the relations

$$E_i(\omega) := \{(v,\eta) \in M_2 : \lambda((v,\eta),\omega) \leq \lambda_i\},$$

$$\lambda(X(t,(v,\eta),\omega),\theta(t,\omega)) = \lambda((v,\eta),\omega), \quad \omega \in \Omega^*,\ t \geq 0$$

([Mo], Stochastics 1990, p. 122). □

Lyapunov exponents $\{\lambda_i\}_{i=1}^{\infty}$ of (I) are non-random because θ is ergodic. Say (I) is *hyperbolic* if $\lambda_i \neq 0$ for all $i \geq 1$. When (I) is hyperbolic the flow satisfies a *stochastic saddle-point property* (or exponential dichotomy) (cf. the deterministic case with $E = C([-r, 0], \mathbf{R}^d)$, $g_i \equiv 0$, $i = 1, \ldots, m$, in Hale [H], Theorem 4.1, p. 181).

Theorem 6 *(Random Saddles)*([Mo], 1990)

Suppose the sfde (I) is hyperbolic. Then there exist

(a) a set $\tilde{\Omega}^* \in \mathcal{F}$ such that $P(\tilde{\Omega}^*) = 1$, and $\theta(t, \cdot)(\tilde{\Omega}^*) = \tilde{\Omega}^*$ for all $t \in \mathbf{R}$,

and

(b) a measurable splitting

$$M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \tilde{\Omega}^*,$$

with the following properties:

(i) $\mathcal{U}(\omega)$, $\mathcal{S}(\omega)$, $\omega \in \tilde{\Omega}^*$, *are closed linear subspaces of* M_2, $\dim \mathcal{U}(\omega)$ *is finite and fixed independently of* $\omega \in \tilde{\Omega}^*$.

(ii) *The maps* $\omega \mapsto \mathcal{U}(\omega)$, $\omega \mapsto \mathcal{S}(\omega)$ *are* $\mathcal{F}$*-measurable into the Grassmannian of* M_2.

(iii) *For each* $\omega \in \tilde{\Omega}^*$ *and* $(v, \eta) \in \mathcal{S}(\omega)$ *there exists* $\tau_1 = \tau_1(v, \eta, \omega) > 0$ *and a positive* δ_1, *independent of* (v, η, ω) *such that*

$$\|X(t, (v, \eta), \omega)\|_{M_2} \leq \|(v, \eta)\|_{M_2} e^{-\delta_1 t}, \quad t \geq \tau_1.$$

(iv) *For each* $\omega \in \tilde{\Omega}^*$ *and* $(v, \eta) \in \mathcal{U}(\omega)$ *there exists* $\tau_2 = \tau_2(v, \eta, \omega) > 0$ *and a positive* δ_2, *independent of* (v, η, ω) *such that*

$$\|X(t, (v, \eta), \omega)\|_{M_2} \geq \|(v, \eta)\|_{M_2} e^{\delta_2 t}, \quad t \geq \tau_2.$$

(v) For each $t \geq 0$ and $\omega \in \tilde{\Omega}^*$,

$$X(t,\omega,\cdot)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t,\omega)),$$

$$X(t,\omega,\cdot)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)).$$

In particular, the restriction

$$X(t,\omega,\cdot)\,|\,\mathcal{U}(\omega) : \mathcal{U}(\omega) \to \mathcal{U}(\theta(t,\omega))$$

is a linear homeomorphism onto.

Proof.

[Mo], Stochastics, 1990, Corollary 2, pp. 127-130. □

The Saddle-Point Property