LOBACHÉVSKI.

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ELEMENTARY GEOMETRY has been the most stable part of all science. Firm fixed as the Egyptian pyramids for two thousand years, how strange that in our century it should melt at a thought, be transformed utterly, present us with a potential new universe, while the old remains as nothing more than a special case of the new!

For philosophy the question of space has ever been fundamental. How shall she envisage this present of a completely new kind of universal space? Kenlore began with the geometric axioms. As typical of absolute *a priori* knowledge was our certainty that of two intersecting straight lines one at least would meet any third straight line coplanar with them. Of imperishable interest is the Russian who dared to doubt this axiom, and who made good his position against an adverse world. He shattered an enchanted barrier which had ever held imprisoned the human mind. Vast consequences resulted. To-day we are all his disciples.

Nicolái Ivánovich Lobachévski was born October 22 (November 2), 1793, in the town of Makariev, about forty miles above Nijni-Novgorod on the Volga. His father, an architect, died in 1797, leaving in poverty his widow with two small sons. The widow removed to Kazan, and succeeded in placing her sons at the cost of the state in the gymnasium there, and afterward in the university.

Lobachévski was admitted to the gymnasium in 1802, and was received at the university in 1807. The records of the inspector attest that in the sciences he outstripped his comrades. Nevertheless his disobedience and wilfulness often drew upon him the displeasure of the faculty. Once he was menaced with expulsion from the university, escaping only because of the protection and inter-

vention of the professor of mathematics, Bartels, toward whom Lobachévski was ever profoundly grateful to the end of his life.

This outspoken and passionate youth in a young university just opened in a half-wild country, in the *ultima Musarum Thule*, as the first professors, Germans, called it, was typical of the there prevailing ardent desire for knowledge, enthusiasm for study, for progress. With this fire of spirit there reigned among the pupils, as says S. T. Aksakov in his *Family-Chronicle*, "complete contempt for everything bad and low, and deep veneration for everything honest and noble, even if it were unreasonable."

Due to Bartels, the teaching of pure mathematics in the University of Kazan was placed on the same level with the teaching in the best universities of Germany. All the classical works of that time, the *Differential and Integral Calculus* of Euler, the *Méchanique Analytique* of Lagrange, the *Applications d'Analyse à Géométrie* of Monge, the *Disquisitiones Arithmeticæ* of Gauss were interpreted by the talented and erudite Bartels. From his own syllabus, Bartels taught the history of mathematics, unfolding before his audience the grand picture of the progress of human thought in this domain.

In spite of "bad deportment," Lobachévski received the grade of "Magister" July 10, 1811, "for extraordinary advance and talent in mathematical and physical sciences," and upon his thesis, "Theory of elliptic movement of the heavenly bodies." Then, four hours weekly at Bartels's home, under his direction, he studied the *Disquisitiones Arithmeticæ* and the first volume of Laplace's *Mécanique Céleste*. In 1813 he presented a paper "On the Solution of the Algebraic Equation $x^n-1=0$." In 1814 he was made adjunct professor of mathematics in the university.

Thus far the intellectual life of this luminous epoch inaugurated in Kazan, the ancient capital of Mohammedan Tatars, the gifted professors awakening ardent new spirits to the light of science, made a mental atmosphere well adapted to the nurture of our idealistic Lobachévski, with his longing for truth, his freedom, his liberty of spirit. Well-knit, well-nourished, well-matured was thus his genius, his character, his scientific ardor, before began the heavy shocks, the opposition, the indifferentism, the ironies of contemporaries, through which he was to persist in trying throughout life to teach a most unwilling world. He begins now his independent intellectual work rich in enthusiasm, energy, idealistic hopes.

Within two years we know he was deep in the fight with the

theory of parallels, where for a decade he struggled on and up through a series of failures. A manuscript exists written from the lectures which Lobachévski gave at the University of Kazan in 1815 and 1816. In this are presented three systematic treatments of the parallel-theory, each a wholly different attempt at its establishment. In one the idea of direction is presumed as fundamental, in the second the consideration of infinite biangles (Bertrand, 1778) is introduced, the third joins on to Legendre's proof that the sum of the angles of a triangle cannot be greater than two right angles. In this latter investigation occur proofs of theorems such as "the angle-sum is two right angles in every triangle if it is in any," "the angle-sum of a triangle contained in another having in common one side and angle, is greater than the angle-sum of the greater triangle," theorems which belong permanently to non-Euclidean geometry. The lecture-notes show that Bartels never touched upon this subject and gave no help. Lobachévski was unconsciously repeating the experience of the ages.

In 1816 he was made full professor. But now a dark epoch overshadows the University of Kazan. Magnitski became curator, "a hypocrite and malicious enemy of science and human reason." Yet to this monster in 1823 Lobachévski presented a manual of geometry by himself, hoping to have it printed at public cost. Magnitski sent the book over to the academician Nicolaus Fuss. Fuss judged the work very severely, finding "that if the author thinks it can be used as a manual, thereby he shows he has no right idea of the requirements for a text-book, that is, no idea of the completeness of the geometric truths making the content of an elementary course of the science, of the mathematical methods, of the need of sharp and clear definitions of all ideas, of the logical order and methodical division of the matter, of the requisite gradation of geometric truths, of the inevitable and, if possible, pure geometric rigor in their demonstration. Of all these necessary qualities there is not a trace in the geometry examined by me."

Fuss, pitiless in his judgment, could not foresee that to-day the whole learned world would rejoice could this lost manuscript-geometry be recovered. Lobachévski worked but waited. It seems more than a coincidence that on February 8, 1826, began the "revision" of Major-General Jeltuhin which consigned Magnitski to prison, while three days after, February 11, 1826, the physico-mathematical faculty examined Lobachévski's paper, "Exposition succincte des principes de la Géométrie avec une démonstration rigoureuse du théorème des parallèles."

The title is unfortunate, especially so, as this essay never was published and does not exist even in manuscript. His memoir, published in 1829 in the *Kazan Courier* under the title "On the Foundations of Geometry," establishes a non-Euclidean geometry, a geometry independent of Euclid's celebrated axiom, the parallel-postulate. The last page gives a way of turning any relation in the non-Euclidean into a relation of the ordinary Euclidean geometry, proof final that the non-Euclidean can never lead to any contradiction. Thus at a stroke the mathematician shattered the reasoning of the deepest of philosophers.

The philosopher had perceived that certain axioms and theorems in Euclid could never have been gotten by experience or observation, since they have absolute metric precision and generality, while results of any observations have definite limits of precision and particular conditions. Thus the theorem that the angle-sum of every rectilineal triangle is two right angles, could never be obtained by experience. Therefore it was supposed that geometry gave certain knowledge about the real world, independently of experience. As explanation given for this, we have the doctrine of space as an entirely subjective transcendental form of our intuition, *a priori,* preceding any experience, independent of all experience.

The crushing answer of Lobachévski is that there exists no real necessity for any of these exact theorems in geometry. He presents a perfect geometry wherein the angle-sum of a rectilineal triangle can be less than two right angles, varying with the size of the triangle; a system involving a constant, a parameter to be fixed approximately by actual observation and experiment, and so always ready to fit any results given by more precise observation. Henceforth there is an empirical element in geometry. He says: "Accordingly it can have nothing contradictory for our mind, if we admit that some forces in nature follow one, others another special geometry." He would not be confounded even if measuring triangles in England gave no defect of angle-sum, while in Germany a recognisable defect appeared.

In a word Lobachévski's account of space is by evolution. In this evolution the evolving mind bears a creative part, but unconsciously. He says explicitly, "Geometric ideas are artificial products of mind." The principle of economy, of parsimony, would settle upon the simpler of forms representing empirical data with requisite approximation.

After seventy years the non-Euclidean geometry has won the expert world. Says Klein in 1897:

"Ein Mathematiker, der die nichteuklidischen Theorieen kennt, wird kaum noch die Meinung früherer Zeiten festhalten wollen, als seien die Axiome nach ihrem concreten Inhalte Nothwendigkeiten der inneren Anschauung: was dem Laien als solche Nothwendigkeit erscheint, erweist sich bei längerer Beschäftigung mit den nichteuklidischen Problemen als Resultat sehr zusammengesetzter Processe, insbesondere auch der Erziehung und der Gewöhnung."

On May 3, 1827, at the early age of thirty-three, Lobachévski was made rector of the University of Kazan, a place he held for nineteen years. On July 5, 1828, he pronounced a remarkable address "On the Principle Objects of Education," from which I quote the following:

"Imagine the condition of a man separated from human society, left at the will of wild nature. Turn then the thoughts to a man who in the midst of an organised and cultivated citizenship of these last enlightened centuries redounds with his deep science to the honor and glory of his country. What a difference! What immeasurable distance separates one from the other.

"This difference is made by education. Education begins in the cradle. First by imitation alone it is acquired. Gradually develop reason, memory, imagination, the sense of beauty; then awakes the love of self, of one's neighbor, the love of glory, the sense of honor, the desire to enjoy life. All the capabilities of the spirit, all the talents, all the passions, are perfected by education and joined in one harmonious whole, and the man, as if new-born, appears as a perfect creation."

But education must not suppress and destroy the passions in a man and his inborn desires.

"All that must be retained in him; otherwise we will mar his nature, hurt its power and injure his happiness.

"Nothing is more usual than to hear complaints about the passions, but how justly has Mably said: the stronger the passions, the more useful are they to society; only their misdirection can be injurious.

"But intellectual culture by itself does not yet finish education. While a man enriches his spirit with knowledge, he must also learn to know how to enjoy life. Thereby I refer to the culture of taste.

"To live means to feel, to enjoy life, continually to feel something new which reminds us that we live.

"Nothing so much contracts the stream of life as ignorance; it guides life on a dead and straight way from the cradle to the grave.

"In the lower classes, exhausting necessary labor alternating with rest, may satisfy the spirit of the farm-hand, the workman; but you whose existence unjust fate has imposed as a heavy burden on others, you whose spirit is dulled and feelings extinguished, you do not enjoy life. For you Nature is dead, the beauties of poetry foreign, architecture has no charm, no magnificence, the world's history no interest.

"I console myself with the idea that from our university will never go out such products of a vegetable nature; even that they will not come here if unluckily born to such a fate. They will not come here, I repeat, for here reigns the love of glory, the sense of honor and of inner merit.

"Nature, having dowered man so generously at his birth, seems not yet satisfied, and so has inspired in every one the wish to surpass others, to be known, to be an object of admiration, to become celebrated; and in this way has she imposed on man the duty to care for his own perfecting.

"In unceasing activity aspires the spirit to win honors, to elevate itself; the whole human race advances from perfection to perfection—and where is an ending visible?"

This address of the young rector may be interpreted as autobiographic. He exemplified it in a beautiful life rich and full with work for science and his university.

But the extraordinary paper published in 1829–30 attracted no attention. In 1835 he restated the matter in a paper entitled "Imaginary Geometry" in the *Scientific Memoirs of the University of Kazan.* This same year he started to publish his largest work, *New Elements of Geometry with Complete Theory of Parallels.* This also remained unnoticed. In 1897 the Introduction was published in English. This is all that has ever appeared in any language but Russian.

In 1836 appeared *Application of Imaginary Geometry to Some Integrals.* In 1837, having failed of any recognition in Russia, Lobachévski published in *Crelle's Journal,* Vol. 17, a French translation of his *Imaginary Geometry.* This offered the matter to the attention of the whole learned world, but without result. In 1840 Lobachévski published in Berlin in German a little book on the subject, but again without avail.

After fifteen years, after terrible misfortunes, public, family, personal, including loss of sight, the end drew near. But the unconquerable scientist, progressive to the last, substituted for the designation *Imaginary Geometry* the better name *Pangeometry,* and under that title dictated a treatise in Russian and also in French, published in the year of his death, 1856.

For ten years after his death the dust of oblivion settled over his grave. Not even one of his pupils worked at his ideas, or appeared as their convinced defender. The academician V. Bunyakovski in his work *Parallel Lines* (1853) does not mention the investigations of Lobachévski. All seemed ended forever, forgotten forever. But in 1866 Baltzer brought to the attention of Hoüel, who had been working on parallels, the *Geometrische Untersuchungen zur Theorie der Parallellinien* of Lobachévski. Hoüel published a French translation of it, saying in his preface: "In spite of the high value of these researches, they have not hitherto attracted the attention of any geometer."

This was the bugle-call, heralding the new day. In a dozen years the literature had so grown as to warrant a "Bibliography of non-Euclidean Geometry," by Halsted (*American Journal of Mathematics*, 1878). This was reproduced in Russia by Vashtchenko-Zaharchenko at Kiev in 1880, and was instrumental in inducing the University of Kazan to issue an edition of the geometric works of Lobachévski (1883–86).

At the centenary of his birth an international committee composed of the foremost mathematicians of the world raised a memorial fund, establishing the Lobachévski prize, five hundred roubles, given every three years for work in geometry, preferably non-Euclidean geometry. It has just been awarded to Sophus Lie.

From the surplus of this fund a marble bust of Lobachévski has been placed in the hall of the university, and a bronze statue[1] in the square now named for the great geometer, facing the university with which his noble life is so inseparably connected.

Henceforth this very monument must aid in what he taught as the aim of the university: not only to enlighten the spirit with knowledge, but also to inculcate virtues, to implant a desire for glory, a feeling of nobility, justice, honor, a sacred honesty resisting all temptation, apart from any punishment.

[1] The frontispiece to the present *Open Court* is a reproduction of the bust of this statue.