

RENORMALIZATION AND *-PRODUCT OF THE HORSESHOE MAPS

by

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DISSERTATION APPROVAL

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AN ABSTRACT OF THE DISSERTATION OF

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TITLE: RENORMALIZATION AND *-PRODUCT OF THE HORSESHOE MAPS

MAJOR PROFESSOR: Prof. M. Sullivan

(The aim of this paper is to find the renormalization and *-product of the horseshoe maps, and to prove that the *-product of two horseshoe admissible pairs is a horseshoe admissible pair, then to find the relationship between the zeta function of the *-product of two $\mathcal{L}(n, m)$ admissible pairs and the zeta function of the originals.)

DEDICATION

To my father's soul who passed away when I was preparing for the qualifying exams. Who told me not to come without it. To my mother who was and is a symbol of the struggle for the sake of learning. Who is always there with her love, support and encouragement. To my Mother-in-Law who never stops praying for me. To my husband (I couldn't do it without you). To my kids even if some of them aren't kids any more. To my brothers and sisters (you are the best). To every one who helped me along the way, and all who waited for my success during this long and hopeful journey.

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INTRODUCTION

Edward Lorenz was a mathematician and meteorologist who introduced chaos theory and developed in 1963 a system of nonlinear equations;

$$\dot{x} = 10(y - x),$$

$$\dot{y} = 28x - y - xz,$$

$$\dot{z} = xy - 8/3z,$$

which become one of the most paradigmatic symbols in modern dynamical systems. The solution of these equations has the shape of a butterfly as in Fig 1.

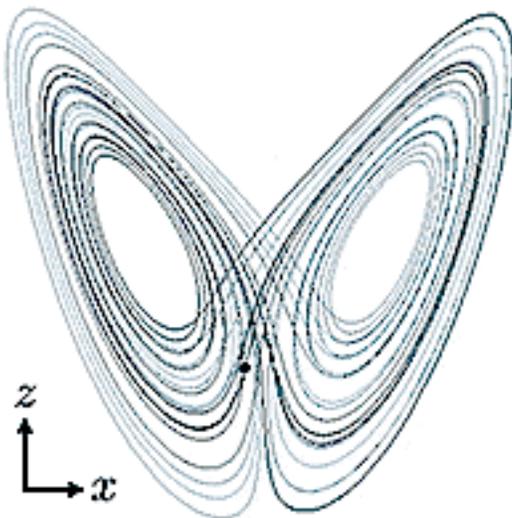


Figure 1. A view of the Lorenz Attractor [8]

In 1976 Williams showed that a non trivial knotting occurs in this system. The notion of template was introduced by Birmann and Williams in 1983. In 1995, Sullivan defined the zeta function that tracks periodic orbits according to the number of half twists in the stable bundles when n and m are nonnegative. Silva and Ramos in 2002 showed how this zeta function changes under renormalization for the case $\mathcal{L}(0, 0)$.

See Birman [2], Franco [4], Lorenz [7].

This paper gives an idea of Lorenz maps and attraction, and shows by studying Lorenz maps, their renormalization, and their $*$ -product, how we can find the renormalization and the $*$ -product of the horseshoe maps, $\mathcal{L}(0, 1)$.

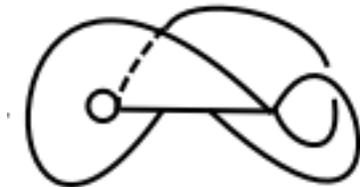


Figure 2. The horseshoe template $\mathcal{L}(0, 1)$

It also provides the zeta function and how to calculate it, and it takes the theorem of finding the zeta function of a $*$ -product of two admissible pairs by the zeta function of the sub-Lorenz of the original pairs which was given by Silva and Ramos [10] to a higher level.

Chapter 1 contains the background of Lorenz maps and horseshoe maps. It also deals with the ordering technique for both of them, then their admissible pairs and their $*$ -product, where we have our first theorem that proved that the $*$ -product of two horseshoe admissible pairs is a horseshoe admissible pair.

Chapter 2 deals with zeta function, how we can calculate it in general, then for $\mathcal{L}(0, 0)$, $\mathcal{L}(0, 1)$, $\mathcal{L}(n, m)$ and calculate the zeta function for their sub, and how this zeta function changes under renormalization for the case $\mathcal{L}(0, 0)$. In this paper we generalize their result to the case where n and m are even and nonnegative. This is our main theorem where we find the relationship between the zeta function of the sub- $\mathcal{L}(n, m)$ of the $*$ -product of two admissible $\mathcal{L}(n, m)$ pairs and the zeta function of the sub- $\mathcal{L}(n, m)$ of the originals, which is

$$\zeta_{T_{(X,Y)*}(S,W)}^S(t^{2+n}, t^{2+m}) = \zeta_{T_{(X,Y)}}^S(t^{2+n}, t^{2+m}) \zeta_{T_{(S,W)}}^S(t^{[(2+n)X_L+(2+m)X_R]}, t^{[(2+n)Y_L+(2+m)Y_R]}),$$

where n and m are even.

CHAPTER 1

LORENZ ATTRACTORS

1.1 INTRODUCTIONS TO LORENZ ATTRACTORS

In this section we introduce the Lorenz system, a system of nonlinear ordinary differential equations, which when plotted, resemble a butterfly or figure eight. That was discovered by Edward Lorenz in 1963.

E. Lorenz was a mathematician and meteorologist. When he was working on some atmospheric phenomena which were non-linear as other weather forecasting, he found a system with specific features that he wanted to retain. The modified equations governed fluid convection in a very thin disc, cooled above and heated below, system of 3 ODEs :

$$\begin{aligned}\dot{x} &= 10(y - x), \\ \dot{y} &= 28x - y - xz, \\ \dot{z} &= xy - 8/3z,\end{aligned}$$

where $x, y, z \in R$ and derivatives are with respect to time.

y = horizontal temperature variation.

x = vertical temperature variation.

z = rate of convective overturning.

These equations now known as Lorenz equations. Birman [2]

Numerical studies by Lorenz show all orbits appear to collapse quickly onto a particular subset $\mathcal{L} \subset R^3$, which is now called a *Lorenz attractor*. It has become one of the most paradigmatic symbols in modern dynamical systems

Figure 1.1 shows a plot of the solution of these equations in which the Lorenz attractor is

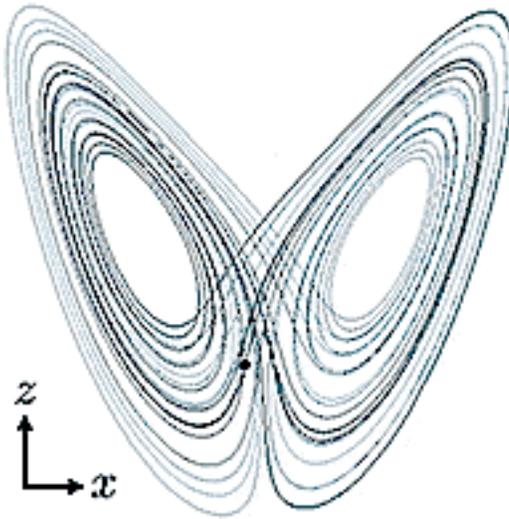


Figure 1.1. A view of the Lorenz Attractor [8]

present.

Definition. A *template* is a branched two-manifold with a semi-flow. They are used to study the periodic orbits of certain flows in 3-manifolds.

There are two classes of Lorenz templates. See Figure 1.2. We denote them by $\mathcal{L}(0,0)$. The one on the left has a saddle point and the left and right most point on the branch line

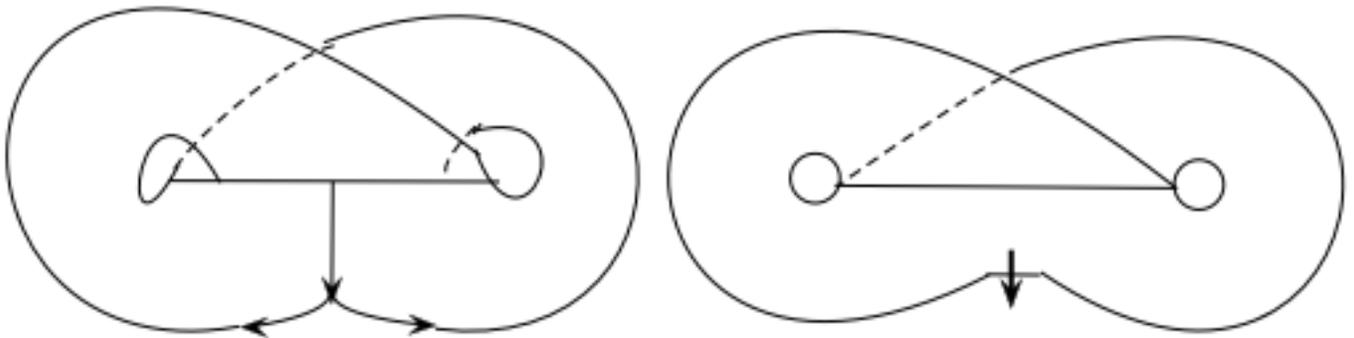


Figure 1.2. Lorenz templates

are not fixed points of the first return map. It is used as a geometric model of the Lorenz

equations.

The one on the right is an idealized version. It contains all possible orbits of any Lorenz system. However, orbits near the middle of the branch line exit. Thus it cannot be an attractor. It is however used to model saddle sets in certain flows. A cross section of the orbits that never exits is a conter set.

The left side one is the primary version used in this dissertation. It can be generalized by adding twists to either or both branches, as shown in Figure 1.3. We denote these by $\mathcal{L}(m, n)$.

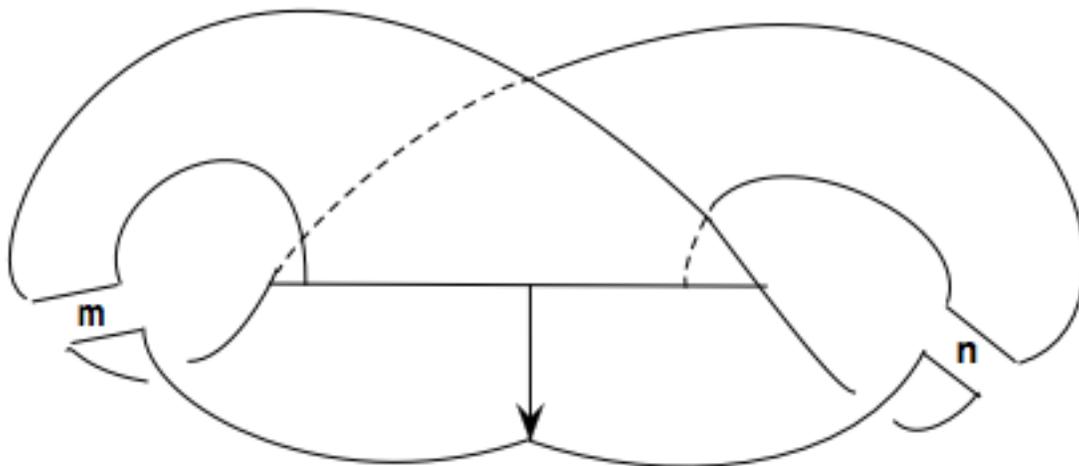


Figure 1.3. The Lorenz-like template $\mathcal{L}(m, n)$

Definition. Consider a map $f : I \times I \rightarrow R^2$ on the square given in Figure1.4 .The map acts on the horizontal strips labeled H_1 and H_2 , stretching by a factor $\lambda^u > 2$ in the vertical direction and compressing by $\lambda^s < 1/2$ in the horizontal direction, while bending the entire square into a "horseshoe."This give *Smale's horseshoe*. Ghrist [5].

The horseshoe map f acts horizontally on a square $I^2 \in R^2$ stretching then bending and suspending f yield a flow on a mapping torus $I^2 \times S^1$. Embedding this flow into R^3 gives a well-defined suspension flow as in Figure 1.5, and this is how we obtain $\mathcal{L}(0, 1)$ the

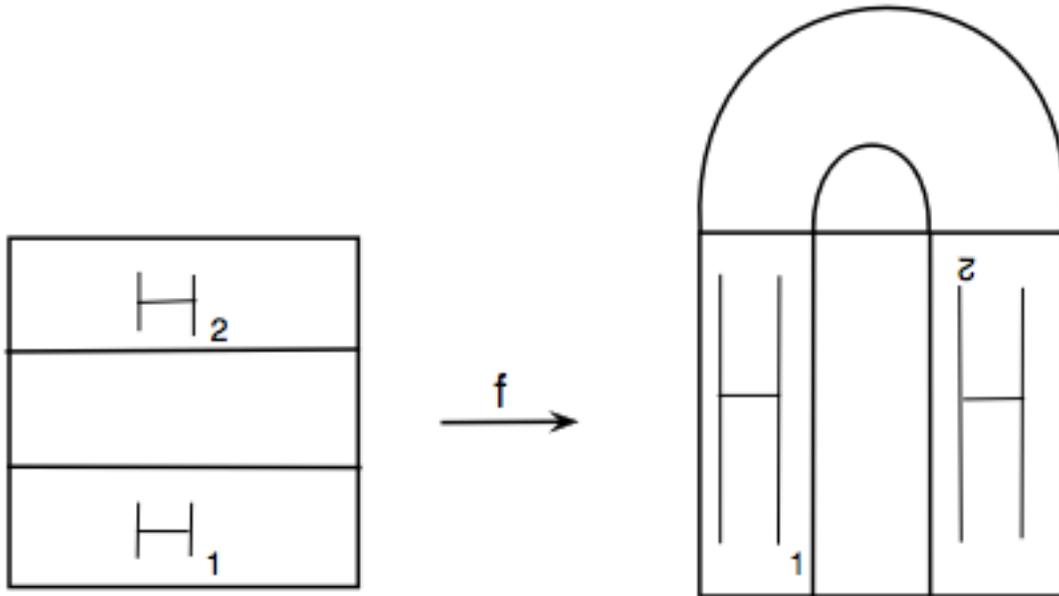


Figure 1.4. The Smale horseshoe map

horseshoe template.

Definition. Let $p < c < q$. A Lorenz map from $[-1, 1]$ to $[-1, 1]$ is a pair $f = (f_-, f_+)$, where

1. $f_- : [-1, 0) \rightarrow [-1, 1]$ and $f_+ : [0, 1] \rightarrow [-1, 1]$ are strictly increasing differentiable maps.
2. $f_-(-1) = -1$ and $f_+(1) = 1$

So

$$f(x) = \begin{cases} f_-(x) & \text{if } x < 0 \\ f_+(x) & \text{if } x > 0 \end{cases}$$

As in Silva [10].

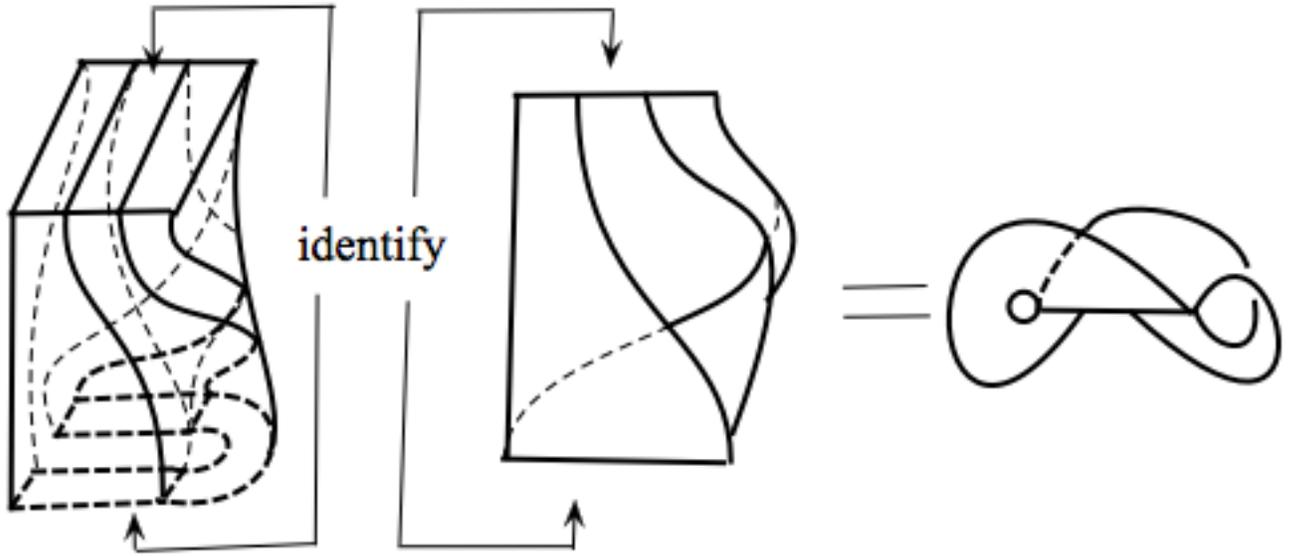


Figure 1.5. how to form the horseshoe template $\mathcal{L}(0,1)$ from the embedded suspension of the Smale horseshoe map.

1.2 THE ORDERING TECHNIQUE

In this section we give the technique we will use to order different examples of Lorenz and Horseshoe.

Remark. Using one way in ordering or the other is right as long as it is constant in the whole work.

Kneading theory:

Kneading theory is a standard tool for studying maps of the interval and has been developed for Lorenz maps in Rand [9]. Let $f : [-1, 1] \rightarrow [-1, 1]$ be a Lorenz map with discontinuity point 0. Let $x \in [-1, 0) \cup (0, 1]$ such that $f^n(x) \neq 0$ for all $n \in \mathbb{N}$. Define the Kneading sequence of x , $k(x) \in \{L, R\}^{\mathbb{N}}$, to be the sequence $k_0(x), k_1(x), k_2(x), \dots$, where

$$k_0(x) = \begin{cases} L & \text{if } x < 0 \\ R & \text{if } x > 0 \end{cases}$$

and $k_i(x) = k_0(f^i(x))$. Imposing the relation $L < R$, these sequences can be ordered using the standard lexicographical order, that is $k(x) < k(y)$ if and only if there exists an $r \geq 0$ such that $k_i(x) = k_i(y)$ for all $i < r$ and $k_r(x) < k_r(y)$.

Furthermore, in the topology induced by the standard metric

$$d(k(x), k(y)) = \sum |k_i(x) - k_i(y)|/2^i$$

where

$$|k_i(x) - k_i(y)| = \begin{cases} 0 & \text{if } k_i(x) = k_i(y) \\ 1 & \text{if } k_i(x) \neq k_i(y) \end{cases}$$

the limits

$$k(x^+) = \lim_{y \downarrow x} k(y)$$

and

$$k(x^-) = \lim_{y \uparrow x} k(y)$$

over the y 's such that $f^n(y) \neq 0$ for all $n \in N$, exist for all $x \in [-1, 1]$. The kneading invariant $k(f)$ of f is the pair $(k^+(f), k^-(f)) = (k(0^+), k(0^-))$ where we are in the context of Lorenz families we usually denote $k(f_{(a,b)})$ as $k(a, b)$. We have that:

- If $x \in [-1, 1]$ and $f^n(y) \neq 0$ for all $n \in N$ then $k(x^-) = k(x) = k(x^+)$.

- For all $x \in [-1, 1]$, $k(x^-) \leq k(x^+)$.
- $x, y \in [-1, 1]$ and $x < y \Rightarrow k(x^+) \leq k(y^-)$.

See Silva [11].

Example 1.2.1. Lets take the word $w = L^3R^3L^2R^2$, then the ten cyclic permutations of w are:

$$L^2R^3L^2R^2L \quad L^2R^2L^3R^3$$

$$LR^3L^2R^2L^2 \quad LR^2L^3R^3L$$

$$R^3L^2R^2L^3 \quad R^2L^3R^3L^2$$

$$R^2L^2R^2L^3R \quad RL^3R^3L^2R$$

$$RL^2R^2L^3R^2 \quad L^3R^3L^2R^2$$

Now the lexicographical order for these permutations is :

$$L^3R^3L^2R^2 < L^2R^2L^3R^3 < L^2R^3L^2R^2L < LR^2L^3R^3L < LR^3L^2R^2L^2 < RL^3R^3L^2R < RL^2R^2L^3R^2 < R^2L^3R^3L^2 < R^2L^2R^2L^3R < R^3L^2R^2L^3$$

which is the same order given by the lexicographical ordering in Figure 1.6.

Example 1.2.2. Let $w_1 = LR$ (the thin line) and $w_2 = LR^2$ (the thick line) see Figure 1.7. Hence $\overline{w_1} = LRLR\dots$, and $\overline{w_2} = LRRLRR\dots$

Therefore the set of permutations for w_1 is $\{LRLR\dots, RLRL\dots\}$ and for w_2 is $\{LRRLRR\dots, RRLRRL\dots, RLRRRL\dots\}$. And this will give the following order:
 $LRLR\dots < LRRLRR\dots < RLRL\dots < RLRRRL\dots < RRLRRL\dots$

1.2.1 Applying the kneading theory for the horseshoe template

Let $z \in I$ denote a point on the branch line . Let $\psi(z) \in \{L, R\}^{\mathbb{Z}^+}$ denote the semi-infinite sequence obtained by the rule

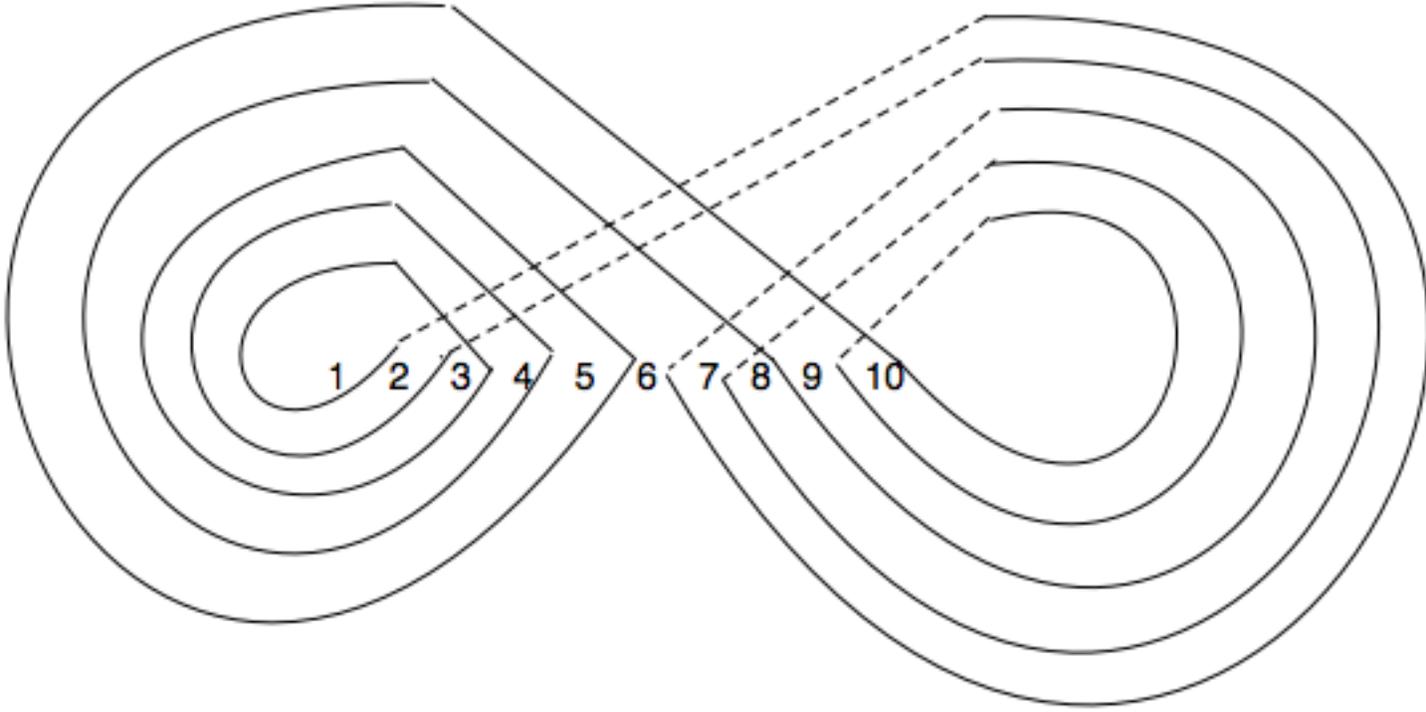


Figure 1.6. $w = L^3 R^3 L^2 R^2$

$$\psi_j(z) = \begin{cases} L & \text{if } f^j(z) < 0 \\ R & \text{if } f^j(z) > 0 \end{cases}$$

$\psi(z)$ is the itinerary of z . Let $\varepsilon(L) = +1, \varepsilon(R) = -1$, define the invariant coordinate of z as $\theta(z) = \{\theta_k\}_{k=0}^{\infty}$, where

$$\theta_k(z) = \varepsilon(\psi_j(z)), \text{ for } j = 0, \dots, k.$$

Let \triangleright denote lexicographical ordering on the invariant coordinate $(+1 \triangleright 0 \triangleright -1)$. Then for $z_1, z_2 \in I$, $z_1 < z_2$ if and only if $\theta(z_1) \triangleright \theta(z_2)$.

Example 1.2.3. The following example illustrates how this allows us to compute knot and link information and in our case the horseshoe examples.

Consider the words $w_1 = L^2 R L R, w_2 = R$. We list the shifts, the signed symbols and invariant coordinates below; writing ' $+ - \dots$ ' for ' $+ 1, -1, \dots$ '. etc. The order from left to

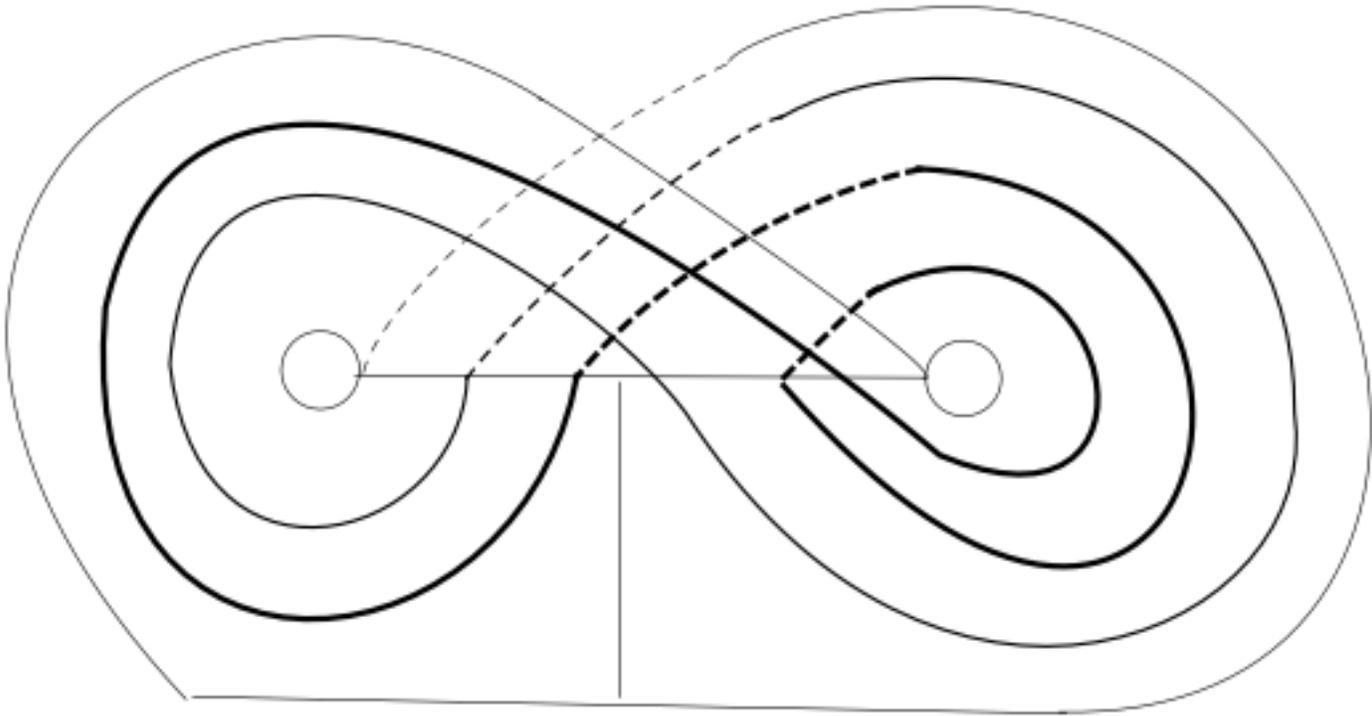


Figure 1.7. $w_1 = LR$ (the thin line) and $w_2 = LRR$ (the thick line)

right on the branch line (smallest to biggest). As in table 1.1.

1.3 THE ADMISSIBLE PAIRS

Definition. Let Σ be the symbolic space of sequences $k_0 \cdots k_n$ on the symbols $\{R, 0, L\}$, such that $k_i \neq 0$ for all $i < n$ and $n = \infty$ or $k_n = 0$. The kneading invariant $k(f)$ of the Lorenz map, f is the pair $(k^-(f), k^+(f)) = (k(0^-), k(0^+))$. We want to know which pairs of sequences can be the kneading invariant of some Lorenz map. We call this set *the set of admissible kneading invariants* and will denote it by Σ^+ .

Let σ be the usual shift operator, defined by

$$\sigma(k_0 k_1 \cdots) = k_1 k_2 k_3 \cdots .$$

Table 1.1. An example table showing how to order horseshoe words.

Word and shifts	Sings	Invariant coordinate	The order
$L^2RLR\dots$	$++-+-\dots$	$+- -- +\dots$	1
$LRLRL\dots$	$+ - + - +\dots$	$+ - - + +\dots$	2
$RLRL^2\dots$	$- + - + +\dots$	$- - + + +\dots$	5
$LRL^2R\dots$	$+ - + + -\dots$	$+ - - - +\dots$	3
$RL^2RL\dots$	$- + + - +\dots$	$- - - + +\dots$	6
$RRRR\dots$	$- - - - -\dots$	$- + - + -\dots$	4

Hubbard and Sparrow [6] show that if a pair of sequences (K^-, K^+) is admissible if and only if

$$K_0^- = L, K_0^+ = R,$$

$$\sigma(K^+) \leq \sigma^n(K^+) < \sigma(K^-), \quad \sigma(K^+) < \sigma^n(K^-) \leq \sigma(K^-) \quad \text{for all } n \in \mathbb{N}.$$

See Silva [11]. This can be restated as the following proposition;

Proposition 1.3.1. *Let $(X, Y) \in \Sigma \times \Sigma$, then $(X, Y) \in \Sigma^+$ if and only if $X_0 = L, Y_0 = R$ and, for $Z \in \{X, Y\}$ we have:*

1. *If $Z_i = L$ then $\sigma^i(Z) \leq X$;*
2. *if $Z_i = R$ then $\sigma^i(Z) \geq Y$; with inequality (1) (resp. (2)) strict if X (resp. Y) is finite,*

For an example of an admissible pair see Figure 1.8.

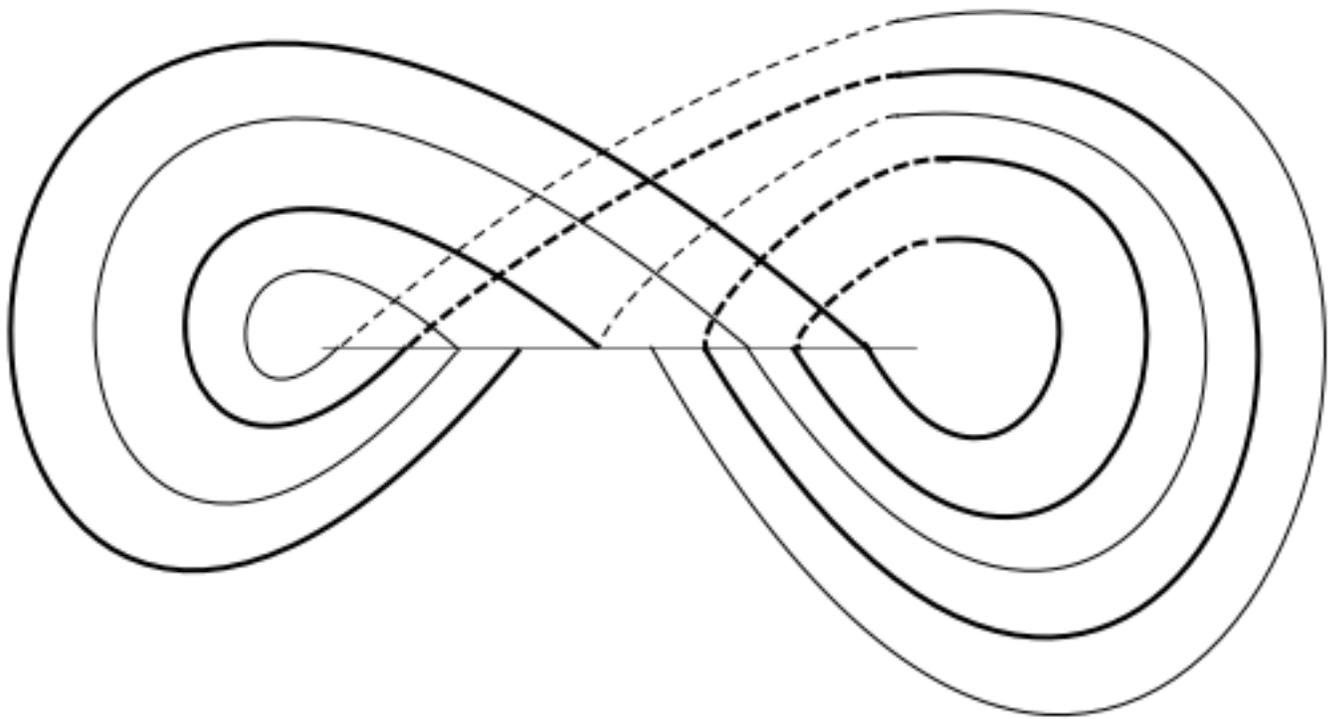


Figure 1.8. the Lorenz admissible pair ($LRRRL0, RLLRO$)

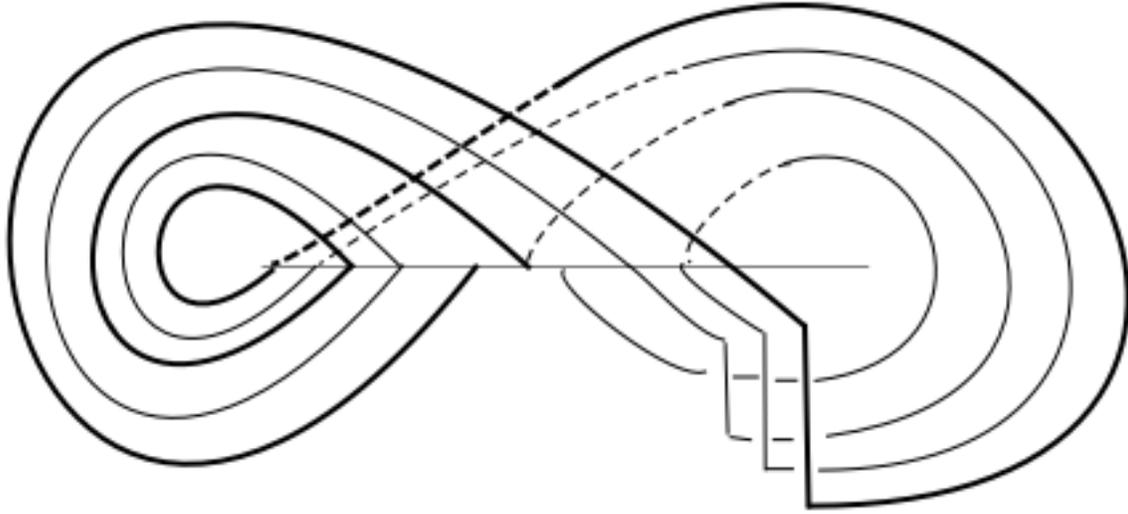


Figure 1.9. the horseshoe admissible pair $(LRLLO, RLLRO)$

1.3.1 The admissible pairs for the horseshoe maps

Following the same idea of the kneading invariant of Lorenz map we want to know which pairs of horseshoe sequences can be the kneading invariant of some horseshoe map. Since $\mathcal{L}(0,1)$ similar to $\mathcal{L}(0,0)$ in the left half, this we give us the upper bound for the different shifts of K^+ and K^- which is $\sigma(K^-)$. The half twist in the right half will force the first shift of K^+ to stay inside the right half and the shift of $\sigma(K^-)$ to go to the first point on the left of the template line which makes $\sigma^2(K^-)$ the lower bound of the different shifts. We can summarize this in the following:

A pair of horseshoe sequences (K^-, K^+) is admissible if and only if

$K_0^- = L$, $K_0^+ = R$, and

$$\sigma^2(K^-) \leq \sigma^n(K^+) < \sigma(K^-), \quad \sigma^2(K^-) < \sigma^n(K^-) \leq \sigma(K^-) \quad \text{for all } n \in \mathbb{N}.$$

Remark. The previous proposition holds for horseshoe admissible pairs and we can see easily that for any horseshoe admissible pairs K^- starts with LRL and K^+ starts with RR .

For an example of a horseshoe admissible pair see Figure 1.9.

1.4 RENORMALIZATION AND *-PRODUCT FOR LORENZ MAPS

Definition. Let f be a lorenz map, then we say that f is *renormalizable* if there exist $n, m \in \mathbb{N}$ with $n + m \geq 3$ and points $P < y_L < 0 < y_R < Q$ such that

$$g(x) = \begin{cases} f^n(x) & \text{if } y_L \leq x < 0 \\ f^m(x) & \text{if } 0 < x \leq y_R \end{cases}$$

is a Lorenz map.

The map $R_{(n,m)}(f) = g = (f^n, f^m)|_{[y_L, y_R]}$ is called the (n, m) -renormalization of f .

In Figure 1.10 and 1.11 we can see a graph of a Lorenz map and its renormalization.

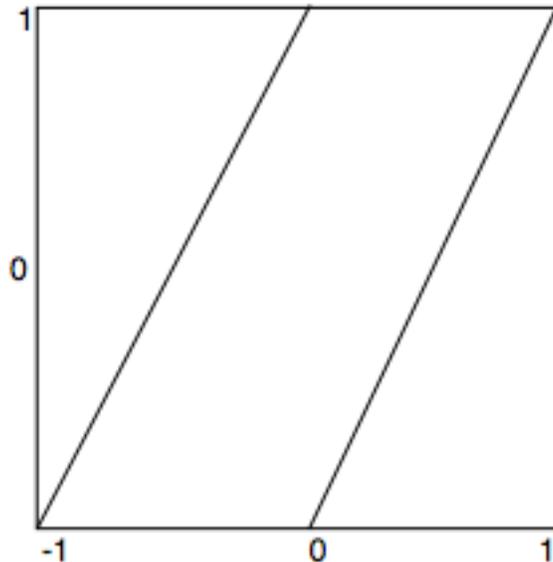


Figure 1.10. a graph of a Lorenz map

Let X be a finite sequence, $X = X_0 \cdots X_{|X|-1}0$, where $|X|$ denote the length of X . Denote $\bar{X} = X_0 \cdots X_{|X|-1}$.

Definition. We define *the *-product* between a pair of finite sequences $(X, Y) \in \Sigma \times \Sigma$, and a sequence $U \in \Sigma$ as

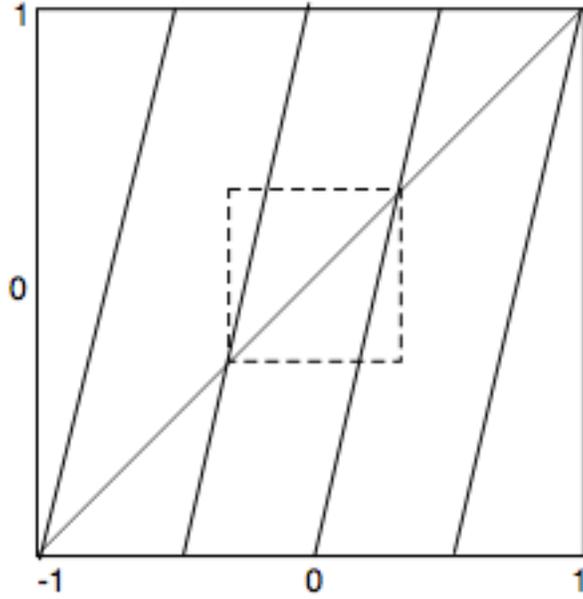


Figure 1.11. a graph of its renormalization

$$(X, Y) * U = \bar{U}_0 \bar{U}_1 \cdots \bar{U}_{|U|-1} 0,$$

where

$$\bar{U}_i = \begin{cases} \bar{X} & \text{if } U_i = L \\ \bar{Y} & \text{if } U_i = R \end{cases}$$

And define the $*$ -product between two pairs of sequences, $(X, Y), (U, T) \in \Sigma \times \Sigma$, X and Y finite, as

$$(X, Y) * (U, T) = ((X, Y) * U, (X, Y) * T).$$

Franco[4].

Example 1.4.1. Let $(X, Y) = (LRRRL0, RLLR0)$ and $(U, T) = (LRR0, RL0)$. Then $(X, Y) * (U, T) = (LRRRLRLLRLLR0, RLLRLRRRL0)$.

1.4.1 Renormalization and *-product for horseshoe maps

Definition. Let f be a horseshoe map, then we say that f is *renormalizable* if there exist $n, m \in \mathbb{N}$ with $n + m \geq 3$ and points $P < y_L < 0 < y_R < Q$ such that

$$g(x) = \begin{cases} f^n(x) & \text{if } y_L \leq x < 0 \\ f^m(x) & \text{if } 0 < x \leq y_R \end{cases}$$

is a horseshoe map.

The map $R_{(n,m)}(f) = g = (f^n, f^m)|_{[y_L, y_R]}$ is called the (n, m) -renormalization of f .

In Figure 1.12 and 1.13 we can see a graph of a horseshoe map and its renormalization.

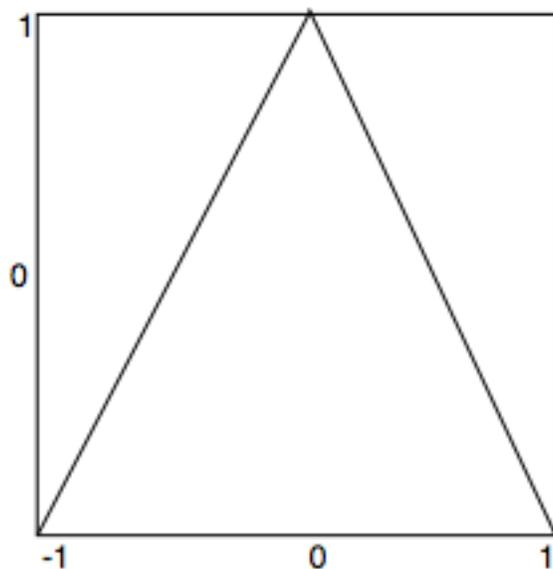


Figure 1.12. a graph of a horseshoe map

The *-product we have for Lorenz maps works fine for horseshoe maps. But it does not give an admissible pair when we multiply two admissible pairs. Therefore we will modify it to get the right one.

Definition. We define *the *-product* between two pairs of horseshoe finite sequence $(X, Y), (S, W) \in \Sigma \times \Sigma$, X and Y finite, X and S start with LR , Y and W start with

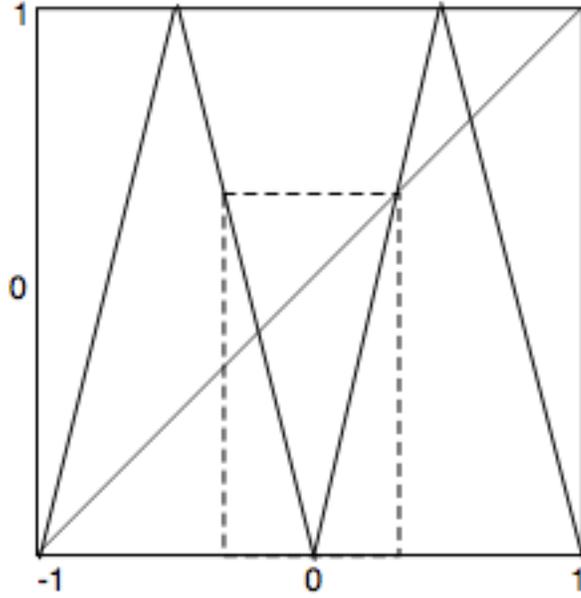


Figure 1.13. a graph of its renormalization

RR , as

$$(X, Y) * (S, W) = ((X, Y) * S, (X, Y) * W).$$

$$(X, Y) * S = \bar{S}_0 \bar{S}_1 \cdots \bar{S}_{|S|-1} 0,$$

$$(X, Y) * W = \bar{W}_0 \bar{W}_1 \cdots \bar{W}_{|W|-1} 0,$$

where

$$\bar{S}_i = \begin{cases} \overline{X_{+L}} & \text{if } S_i = S_0 \\ \bar{X} & \text{if } S_i = L \\ \bar{Y} & \text{if } S_i = R \end{cases}$$

and

$$\bar{W}_i = \begin{cases} \overline{Y_{+L}} & \text{if } W_i = W_0 \\ \bar{X} & \text{if } W_i = L \\ \bar{Y} & \text{if } W_i = R \end{cases}$$

Where X_{+L} and Y_{+L} defined as

If $X = x_0x_1x_2 \cdots x_{|X|-1}$, then $X_{+L} = x_0x_1Lx_2 \cdots x_{|X|-1}$. Similarly we can define Y_{+L} .

Example 1.4.2. Let $(X, Y) = (LRLLO, RRLLR0)$ and $(S, W) = (LRL0, RRLLR0)$. Then $(X, Y) * (S, W) = (LRLLLRLLRLRLL0, RRLLRRRLLRLRLLRRLLR0)$

Lemma 1.4.1. *If (X, Y) is a horseshoe admissible pair, then (X_{+L}, Y_{+L}) is an admissible horseshoe pair.*

Proof. Let $X = x_0x_1x_2 \cdots x_{|X|-1}$, $Y = y_0y_1y_2 \cdots y_{|Y|-1}$ where $x_i, y_i \in \{L, R\}$

(X, Y) is an admissible pair if and only if

$$x_0 = L, y_0 = R,$$

$\sigma^2(X) \leq \sigma^n(Y) < \sigma(X)$, $\sigma^2(X) < \sigma^n(X) \leq \sigma(X)$ for all $n \in N$. But we know that for a horseshoe admissible pair X and Y will be in the form:

$$X = LRLx_3x_4 \cdots x_{|X|-1}, Y = RRly_3y_4 \cdots y_{|Y|-1}$$

Part(II):

$$\sigma^2(X) \leq \sigma^n(Y) < \sigma(X)$$

Thus,

$$Lx_3x_4 \cdots x_{|X|-1} \leq y_jy_j + 1 \cdots y_{|Y|-1} \leq RLx_3x_4 \cdots x_{|X|-1}$$

Now,

$$Y_{+L} = RRLLy_3y_4 \cdots y_{|Y|-1}$$

and it is obvious that

$$\sigma^2(X_{+L}) = LLx_3x_4 \cdots x_{|X|-1} \leq Lx_3x_4 \cdots x_{|X|-1} = \sigma^2(X)$$

Also,

$$\sigma(X) = RLx_3x_4 \cdots x_{|X|-1} \leq RLLx_3x_4 \cdots x_{|X|-1} = \sigma(X_{+L})$$

Case (1): $j = 0$,

Since,

$$Lx_3x_4 \cdots x_{|X|-1} \leq RRLy_3y_4 \cdots y_{|Y|-1} \leq RLx_3x_4 \cdots x_{|X|-1}$$

then,

$$\begin{aligned} \sigma^2(X_{+L}) \leq \sigma^2(X) &\leq RRLy_3y_4 \cdots y_{|Y|-1} \leq RLLy_3y_4 \cdots y_{|Y|-1} \leq RLx_3x_4 \cdots x_{|X|-1} = \\ &\sigma(X) \leq \sigma(X_{+L}) \end{aligned}$$

Case (2): $j = 1$,

Since,

$$RLy_3y_4 \cdots y_{|Y|-1} \leq RLx_3x_4 \cdots x_{|X|-1}$$

then,

$$\begin{aligned} RLLy_3y_4 \cdots y_{|Y|-1} &\leq RLLx_3x_4 \cdots x_{|X|-1}, \\ \sigma^2(X_{+L}) \leq \sigma^2(X) &\leq RLy_3y_4 \cdots y_{|Y|-1} \leq RLLy_3y_4 \cdots y_{|Y|-1} \leq \sigma(X_{+L}) \end{aligned}$$

Case (3): $j = 2$,

Since,

$$Lx_3x_4 \cdots x_{|X|-1} \leq Ly_3y_4 \cdots y_{|Y|-1}$$

then,

$$LLx_3x_4 \cdots x_{|X|-1} \leq LLy_3y_4 \cdots y_{|Y|-1}$$

So,

$$\sigma^2(X_{+L}) \leq LLy_3y_4 \cdots y_{|Y|-1} \leq RLx_3x_4 \cdots x_{|X|-1} = \sigma(X) \leq \sigma(X_{+L})$$

Case (4): $j \geq 3$,

Since,

$$Lx_3x_4 \cdots x_{|X|-1} \leq y_3y_4 \cdots y_{|Y|-1}$$

then,

$$\begin{aligned} LLx_3x_4 \cdots x_{|X|-1} &\leq Ly_3y_4 \cdots y_{|Y|-1}, \\ \sigma^2(X_{+L}) &\leq Ly_3y_4 \cdots y_{|Y|-1} \leq RLx_3x_4 \cdots x_{|X|-1} = \sigma(X) \leq \sigma(X_{+L}) \end{aligned}$$

From the four cases we get,

$$\sigma^2(X_{+L}) \leq \sigma^n(Y_{+L}) < \sigma(X_{+L})$$

Part (II): By a similar argument we can prove that,

$$\sigma^2(X_{+L}) \leq \sigma^n(X_{+L}) < \sigma(X_{+L})$$

From part (I) and (II), (X_{+L}, Y_{+L}) is an admissible pair. □

Theorem 1.4.2. *If (X, Y) and (U, V) are horseshoe admissible pairs, then $(X, Y) * (U, V)$ is an admissible horseshoe pair.*

Proof. Let

$$(X, Y) = (k_0^- k_1^- \cdots k_{m_1-1}^-, k_0^+ k_1^+ \cdots k_{n_1-1}^+)$$

where $m_1 = |X|$ and $n_1 = |Y|$.

And

$$(U, V) = (s_0^- s_1^- \cdots s_{m_2-1}^-, s_0^+ s_1^+ \cdots s_{n_2-1}^+)$$

where $m_2 = |U|$ and $n_2 = |V|$.

Since (X, Y) and (U, V) are horseshoe admissible pairs, then

$$k_0^- k_1^- k_2^- = s_0^- s_1^- s_2^- = LRL$$

$$k_0^+ k_1^+ = s_0^+ s_1^+ = RR$$

and

$$k_2^- k_3^- \cdots k_{m_1-1}^- 0 \leq k_r^- k_{r+1}^- \cdots k_{m_1-1}^- 0 \leq k_1^- k_2^- \cdots k_{m_1-1}^- 0,$$

$$k_2^- k_3^- \cdots k_{m_1-1}^- 0 < k_r^+ k_{r+1}^+ \cdots k_{n_1-1}^+ 0 < k_1^- k_2^- \cdots k_{m_1-1}^- 0,$$

$$s_2^- s_3^- \cdots s_{m_2-1}^- 0 \leq s_r^- s_{r+1}^- \cdots s_{n_1-1}^- 0 \leq s_1^- s_2^- \cdots s_{m_2-1}^- 0,$$

$$s_2^- s_3^- \cdots s_{m_2-1}^- 0 < s_r^+ s_{r+1}^+ \cdots s_{n_2-1}^+ 0 < s_1^- s_2^- \cdots s_{m_2-1}^- 0.$$

Now let

$$(X, Y) * (S, W) = (a_0 a_1 \cdots 0, b_0 b_1 \cdots 0),$$

we want to prove that for all p,

$$a_2 a_3 \cdots 0 \leq a_p a_{p+1} \cdots 0 \leq a_1 a_2 \cdots 0,$$

$$a_2 a_3 \cdots 0 < b_p b_{p+1} \cdots 0 < a_1 a_2 \cdots 0.$$

Take

$$m_k^- = (\#\{i < k : s_i^- = L\})m_1 + (\#\{i < k : s_i^- = R\})n_1 + 1.$$

And

$$X_{R[0,i]} = \# \text{ of } R \text{ symbols in the sequence } k_0^- k_1^- \cdots k_{r-1}^-$$

We will just prove the first two inequalities for the case $s_k^- = L$, because the rest is completely analogous.

Part I: $p = m_k^-$

1. If $p = m_k^-$ and $s_k^- = L$

$$a_{p+1} a_{p+2} \cdots = k_0^- \cdots k_{m_1-1}^- \cdots = L \cdots < a_1 \cdots = R \cdots$$

To prove the other inequality, from the admissibility of (U, V) , $s_k^- \cdots \geq s_2^- \cdots$. Therefore, we have two cases:

(a) $s_k^- \cdots = s_2^- \cdots$, then

$$a_{p+1}a_{p+2} \cdots = k_0^- k_1^- \cdots k_{m_1-1}^- \cdots > k_2^- \cdots k_{m_1-1}^- \cdots > a_2 a_3 \cdots = L k_2^- k_3^- \cdots k_{m_1-1}^- \cdots$$

(b) $s_k^- \cdots > s_2^- \cdots$, then $\exists r$ such that $\forall_{0 \leq i < r} s_{k+i}^- = s_{2+i}^-$ and one of two cases:

i. $X_{R[0,i]}$ is even, $s_{k+r}^- = R, s_{1+r}^- = L$ or 0 and this gives

$$a_{p+1}a_{p+2} \cdots a_{m_k+r-1} k_0^+ \cdots > a_2 \cdots a_{m_1+r-1} k_0^- \cdots$$

or

$$a_{p+1}a_{p+2} \cdots a_{m_k+r-1} k_0^+ \cdots > a_2 \cdots a_{m_1+r-1} 0$$

ii. $X_{R[0,i]}$ is odd, $s_{k+r}^- = L$ or $0, s_{1+r}^- = R$ and this gives

$$a_{p+1}a_{p+2} \cdots a_{m_k+r-1} k_0^- \cdots > a_2 \cdots a_{m_1+r-1} k_0^+ \cdots$$

or

$$a_{p+1}a_{p+2} \cdots a_{m_k+r-1} 0 > a_2 \cdots a_{m_1+r-1} k_0^+ \cdots$$

2. If $p = m_k^-$ and $s_k^- = R$

$$a_{p+1}a_{p+2} \cdots = k_0^+ \cdots = R \cdots > a_2 \cdots = L \cdots$$

And from admissibility of (X, Y)

$$k_0^+ \cdots k_{n_1-1}^+ \cdots < k_1^- k_2^- \cdots k_{m_1-1}^- \cdots < a_1 a_2 \cdots$$

Part II: $m_{k-1}^- < p < m_k^-$

1. If $s_{k-1}^- = L$ (the prove of the second inequality),

$$a_{p+1} \cdots = k_{l+1}^- \cdots ,$$

for some

$$l \in \{-1, 1, 2, \dots, m_1 - 2\}.$$

Since

$$k_{l+1}^- \cdots k_{m_1-1}^- \cdots < k_1^- k_2^- \cdots k_{m_1-1}^- \cdots ,$$

there exists $r \leq m_1 - l$ such that $\forall_{i < r}, k_{l+i}^- = k_i^-$ and one of two cases:

- (a) $X_{R_{i < r}}$ is even, $k_{l+r}^- = L$ or 0 , $k_r^- = R$.
- (b) $X_{R_{i < r}}$ is odd, $k_{l+r}^- = R$, $k_r^- = L$ or 0 .

If $r < m_1 - l$, the proof follows immediatly.

If $r = m_1 - l$, there are four cases:

- i. If $X_{R_{i < r}}$ is odd ($k_{l+r}^- = L$ or 0) and $s_k^- = L$, then

$$k_{l+1}^- \cdots k_{m_1-1}^- k_0^+ \cdots < k_1^- k_2^- \cdots k_{r-1}^- k_r^- \cdots < a_1 a_2 \cdots ,$$

or

$$k_{l+1}^- \cdots k_{m_1-1}^- k_0^+ \cdots < k_1^- k_2^- \cdots k_{r-1}^- 0 < a_1 a_2 \cdots ,$$

- ii. If $X_{R_{i < r}}$ is odd ($k_{l+r}^- = L$ or 0) and $s_k^- = R$, so if $k_{l+r}^- = L$ we need to compare the two sequences;

$$k_{l+1}^- \cdots k_{m_1-1}^- k_0^- \cdots$$

$$k_1^- k_2^- \cdots k_{r-1}^- k_r^- \cdots$$

Thus we need to compare (since $r \leq m_1 - l$)

$$k_1^- k_2^- \cdots$$

$$k_{m_1-l+1}^- k_{m_1-l+2}^- \cdots$$

From admissibility of (X, Y)

$$k_{m_1-l+1}^- k_{m_1-l+2}^- \cdots < k_1^- k_2^- \cdots$$

And since $X_{R_{i < r}}$ is odd, then

$$k_{l+1}^- \cdots k_{m_1-1}^- k_0^- k_1^- \cdots < k_1^- k_2^- \cdots k_{r-1}^- k_r^- k_{r+1}^- \cdots$$

And if $k_{l+r}^- = 0$, then

$$k_{l+1}^- \cdots k_{m_1-1}^- k_0^- \cdots > k_1^- k_2^- \cdots k_{r-1}^- 0.$$

But the extra L we have in X_{+L} will fix that since

$$k_{l+1}^- \cdots k_{m_1-1}^- = k_1^- k_2^- \cdots k_{r-1}^- < k_1^- L k_2^- \cdots k_{r-1}^-,$$

then

$$k_{l+1}^- \cdots k_{m_1-1}^- k_0^- \cdots < a_1 a_2 \cdots$$

iii. $X_{R_{i < r}}$ is even, so $k_r^- = R$. If $s_k^- = L$, then

$$k_{l+1}^- \cdots k_{m_1-1}^- k_0^- \cdots < k_1^- k_2^- \cdots k_{r-1}^- k_r^- \cdots < a_1 a_2 \cdots,$$

iv. $X_{R_{i < r}}$ is even, so $k_r^- = R$. If $s_k^- = R$, then we need to compare

$$k_{l+1}^- \cdots k_{m_1-1}^- k_0^+ \cdots$$

$$k_1^- k_2^- \cdots k_{r-1}^- k_r^- \cdots$$

Thus we need to compare (since $r \leq m_1 - l$)

$$k_0^+ k_1^+ \cdots$$

$$k_{m_1-l+1}^- k_{m_1-l+2}^- \cdots$$

And this gives four cases:

- A. then $\exists r < \min\{n_1, l\}$ such that $k_r^+ < k_{m_1-l+r}$ and in this case the proof follows immediately.
- B. $l < n_1$ and $K_{m_1-l+1}^- \cdots k_{m_1-1}^- = k_1^+ \cdots k_{l-1}^+$ then $k_l^+ = L$ or 0 (otherwise we will have an equality in this part and we need to compare the rest with an odd case and since $k_1^+ = R$ this will make $s_1^- s_2^- < s_{k-1}^- s_k^-$). Therefore,

$$k_{l+1}^- \cdots k_0^+ k_1^+ \cdots k_{l+1}^+ k_l^+ \cdots < k_1^- \cdots k_{m_1-1}^- k_0^+ \cdots$$

- C. $l = n_1$ and $K_{m_1-l+1}^- \cdots k_{m_1-1}^- = k_1^+ \cdots k_{n_1-1}^+$. In this case the prove follows from the fact that $s_{k+1}^- s_{k+2}^- \cdots \leq s_1^- s_2^-$.
- D. $l < n_1$ and $K_{m_1-l+1}^- \cdots k_{m_1-l+n_1-1}^- = k_1^+ \cdots$. If $s_{k+1} = L$, as $s_1^- = R$, the proof follows immediately. If $s_{k+1} = R$ we want to compare the sequences;

$$k_{l+1}^- \cdots k_{m_1-1}^- k_0^+ \cdots k_{n_1-1}^+ k_0^+ \cdots$$

$$k_1^- k_2^- \cdots k_{m_1-l-1}^- k_{m_1-l}^- k_{m_1-l+1}^- \cdots k_{m_1-l+n_1-1}^- k_{m_1-l+n_1}^- \cdots$$

and that is completely analogous to (iv).

Thus in this case we will repeat the previous process, but since n_1 is finite there exists α (minimum) such that $m_1 - l + \alpha n_1 \geq n_1$ and so, after α steps we will have one of (i, ii, iii, A, B, C) situations. Hence the second inequality is proved.

2. The proof of the first inequality $a_{p+1} \cdots \geq a_2 \cdots$ is completely analogous to (1) in Part II.

□

CHAPTER 2

ZETA FUNCTION

A zeta function for a map $f : M \rightarrow M$ is a device for counting periodic orbits. M. Sullivan presented a formula for the Twist-zeta function which tracks periodic orbits according to the number of half twists in the stable bundles when n and m are nonnegative.

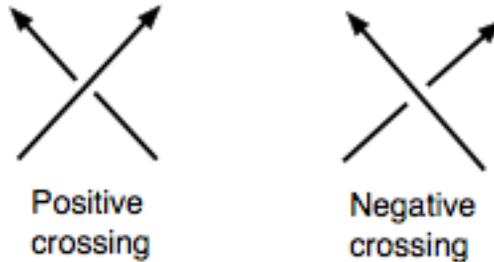


Figure 2.1. positive crossing and negative crossing

2.1 POSITIVE RIBBONS

A ribbon is an embedded annulus or Mobius band in S^3 . Ribbons can be braided like knots and templates.

A positive ribbon is a ribbon with a braid presentation where each crossing of one strand over another is positive and each twist in each strand is positive.

Let c be the sum of the crossing numbers of the core of a ribbon R , where $+1$ is a positive crossing and -1 is a negative crossing. Let n be the number of strands of the core and t the sum of the half twists in the strands of braid presentation of the ribbon $p(R)$. In this section we define three notation of twist for ribbons.

Definition. *the usual twist is $\tau_u = c + t/2$, the modified twist is $\tau_m = n - 1 + t/2$ and the computed twist is $\tau_c = 2n + t$.*

Lemma 2.1.1. τ_u is an isotopy invariant of ribbons over all braid presentations. τ_m and τ_c are isotopy invariants of positive ribbons over positive braid presentations.

Proof. The proof uses linking numbers and the genus. These are standard invariants of knot theory. The two components of a link is the sum of the crossing types as one of the two knots passes under the other. For an embedded annulus the linking number of the two boundary components is $c + t/2$. The same formula gives one half the linking number of an embedded Mobius band's boundary with its core. In both cases we find that τ_u is invariant.

The invariance of τ_m for positive ribbons follows from checking that

$$\tau_m = \tau_u - 2g,$$

Where $g = 1/2(c - n + 1)$ is the genus of the core of R . Here we have appealed . □

Lemma 2.1.2. For positive templates the number of closed orbits with a given computed twist is finite.

For the proof see Sullivan[12].

One of the tools we will use in the rest of our work is Markov partition. But first we will introduce some definitions.

Definition. Let $X \subset \Lambda$ be a subset of a hyperbolic invariant set of a flow ϕ_t on M . Then the *stable* and *unstable manifolds* of X in M are given by:

$$W^s(X) = \{y \in M : \lim_{t \rightarrow \infty} \|\phi_t(X) - \phi_t(y)\| = 0\},$$

$$W^u(X) = \{y \in M : \lim_{t \rightarrow -\infty} \|\phi_t(X) - \phi_t(y)\| = 0\}.$$

The *local* stable and unstable manifolds of X in M are given by:

$$W_{loc}^s(X) = \{y \in M : \lim_{t \rightarrow \infty} \|\phi_t(y) - \phi_t(X)\| = 0 \text{ and } \|\phi_t(y) - \phi_t(X)\| < \epsilon \forall t \geq 0\},$$

$$W_{loc}^u(X) = \{y \in M : \lim_{t \rightarrow -\infty} \|\phi_t(y) - \phi_t(X)\| = 0 \text{ and } \|\phi_t(y) - \phi_t(X)\| < \epsilon \forall t \leq 0\},$$

for ϵ an "appropriately" small positive number.

Definition. Let f be a diffeomorphism, Λ a hyperbolic basic set for f , and Ω a finite collection of rectangles R_i . Let $W^s(x, R_i) \equiv W_{loc}^s(x) \cap R_i$ and $W^u(x, R_i) \equiv W_{loc}^u(x) \cap R_i$. Then Ω is a *Markov partition* for f if:

1. $\Lambda = \bigcup_i R_i$;
2. $int(R_i) \cap int(R_j) = \emptyset$;
3. $x \in int(R_i)$ and $f(x) \in int(R_j)$,
 $f(W^s(x, R_i)) \subset W^s(f(x), R_j)$, $W^u(f(x), R_j) \subset f(W^u(x, R_i))$,

Where $W^s(x)$ and $W^u(x)$ are the stable and unstable manifolds of x . Same as that $W_{loc}^s(x)$ and $W_{loc}^u(x)$ are the local stable and unstable manifolds of x .

See Ghrist [4].

2.2 TWIST-ZETA FUNCTION

Definition. For a given positive template let $T_{q'}$ be the number of closed orbits with computed twist q' . Let $\mathcal{T}_q = \sum_{q'|q} q' T_{q'}$. Then we define the *zeta function* of the template to

be the exponential of a formal power series:

$$\zeta(t) = \exp\left(\sum_{q=2}^{\infty} \mathcal{T}_q \frac{t^q}{q}\right).$$

Next we define a *twist matrix*, $A(t) = [a_{ij}]$ whose entries are non-negative powers of t and 0's, by considering the contribution to τ_c as an orbit goes from one element of a Markov partition to another. Let $a_{ij} = 0$ if there is no branch going from the i -th to the j -th partition element and $a_{ij} = t^{q_{ij}}$ if there is such a branch, where q_{ij} is the amount of computed twist orbits pick up as they travel from the i -th to the j -th partition element.

Theorem 2.2.1. *For any template and any allowed choice of $A(t)$ we have*

$\zeta(t) = 1/\det(I - A(t))$. *Thus, the zeta function is rational.*

Example 2.2.1. The template and partiton in Figure 2.2 give

$$A(t) = \begin{bmatrix} 0 & 0 & 0 & t & t \\ 0 & 0 & 0 & 1 & 1 \\ 0 & t^2 & t^2 & 0 & 0 \\ t^2 & t^2 & t^2 & 0 & 0 \\ t^3 & t^3 & t^3 & 0 & 0 \end{bmatrix}$$

2.2.1 For $\mathcal{L}(0,0)$

For Lorenz template, $\mathcal{L}(0,0)$ lets go back to Figure 1.3. In the general case we can see that the choices we have are:

L can go to L

L can go to R

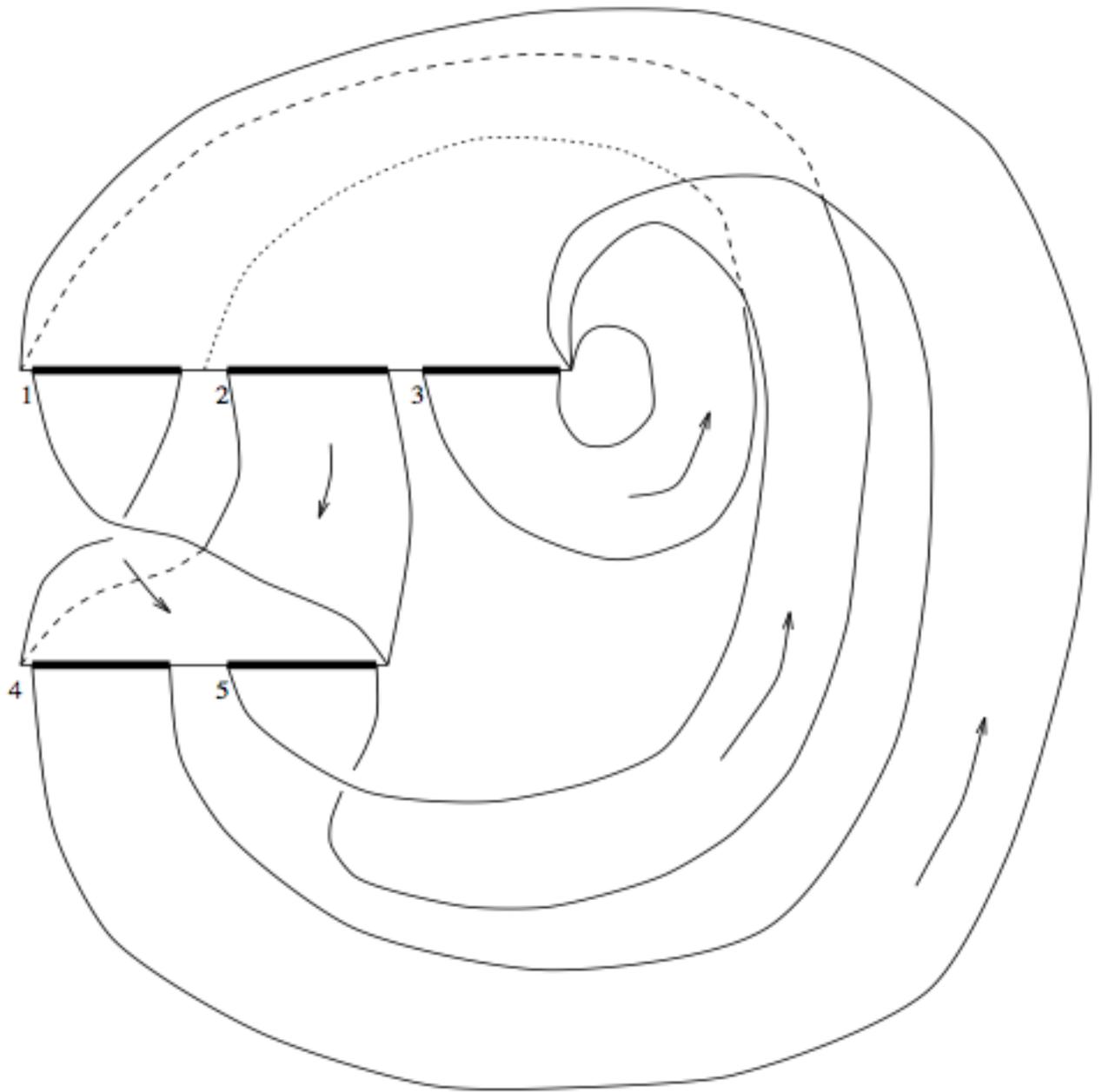


Figure 2.2. A template with a Markov partition indicated by thick lines [12]

R can go to L

R can go to R

And in all cases we will get a full twist which gives a t^2 . Hence the matrix $A(t)$ is:

$$A(t) = \begin{bmatrix} t^2 & t^2 \\ t^2 & t^2 \end{bmatrix}$$

Therefore;

$$\zeta(t) = 1/\det(I - A(t)) = 1/(1 - 2t^2)$$

2.2.2 For $\mathcal{L}(0,1)$

By the same way for the Horseshoe template, $\mathcal{L}(0,1)$ we can see from Figure 1.4 in the general case that we have the following choices:

L can go to L

L can go to R

R can go to L

R can go to R

But here the first two give a full twist (t^2) and the last two give three half twists (t^3).

Hence the matrix $A(t)$ is:

$$A(t) = \begin{bmatrix} t^2 & t^2 \\ t^3 & t^3 \end{bmatrix}$$

Therefore;

$$\zeta(t) = 1/\det(I - A(t)) = 1/(1 - t^2 - t^3)$$

Now by applying a standard matrix identity (see Sullivan [12]), we get

$$\frac{1}{\det(I - A(t))} = \exp\left(\sum_{n=1}^{\infty} \frac{\text{tr}A(t)^n}{n}\right)$$

By using this relationship we see the following;

$$\sum_{n=1}^{\infty} \frac{\text{tr}A(t)^n}{n} = \frac{t^2 + t^3}{1} + \frac{t^4 + 2t^5 + t^6}{2} + \frac{t^6 + 3t^7 + 3t^8 + t^9}{3} + \dots$$

By using x and y as the symbols for the left and right partition elements respectively for the word of orbits, we will analyze these first three terms. In the first term, the t^2 correspond to the orbit x and the t^3 correspond to the orbit y . In the second term the t^4 also correspond to x but this time it has been traversed twice (resp. is the t^6 to y). The $2t^5$ correspond to xy , and the 2 means that we have two of this orbit. By the same way $3t^7$ correspond to xyx and $3t^8$ to xyy . The t^6 and the t^9 correspond to three trips of x and y respectively. And this is how we count the orbits form this equation.

2.3 ZETA FUNCTION FOR THE SUB-TEMPLATES

2.3.1 the Sub-Lorenz template

In this section we will show how did Franco and Silva [4] developed a theorem to find the Zeta function of a star product of two Lorenz admissible pairs using their Zeta function throw their sublorenz templates.

Definition. We say that a Lorenz map f has a double saddle connection if $f^n(0^-) = f^m(0^+) = 0$ for some n, m .

In this case using the points $\{f^i(0^-), f^j(0^+) : 1 \leq i \leq n, 1 \leq j \leq m\}$ we can define a finite

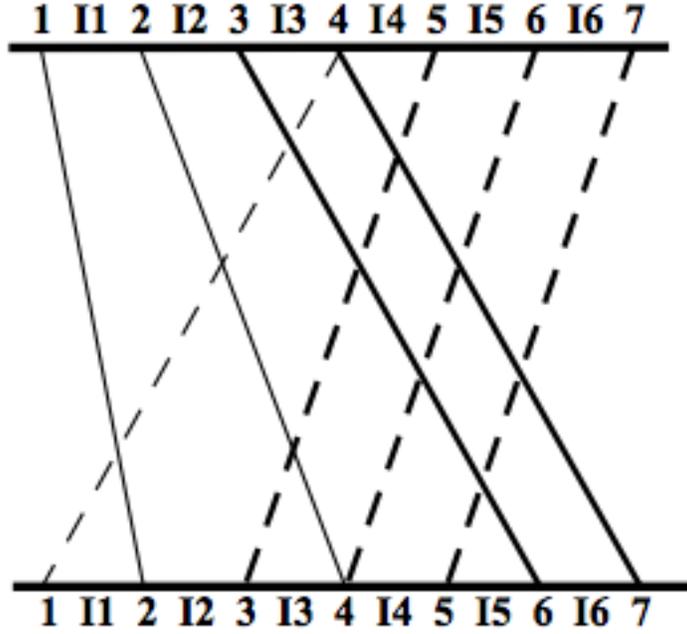


Figure 2.3. Sub-Lorenz template $T_{(LRRLR0, RLL0)}$.

Markov partition for the semiflow.

The restriction of the semiflow to this partition is called a *Sub-Lorenz template*.

Example 2.3.1. For $(X, Y) = (LRRLR0, RLL0)$, we construct the sub-Lorenz template $T_{(X, Y)}$ following the procedure of Figure 2.3 where;

$$\sigma(X) = RRLR0 \quad (7) \quad \sigma(Y) = LL0 \quad (1)$$

$$\sigma^2(X) = RLR0 \quad (5) \quad \sigma^2(Y) = L0 \quad (2)$$

$$\sigma^3(X) = LR0 \quad (3) \quad \sigma^3(Y) = 0 \quad (4)$$

$$\sigma^4(X) = R0 \quad (6)$$

$$\sigma^5(X) = 0 \quad (4)$$

We associate to the template the transition matrix $A_{T_{(X,Y)}}(t)$;

$$A_{T_{(X,Y)}}(t) = \begin{bmatrix} 0 & t^2 & t^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^2 & t^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & t^2 \\ t^2 & t^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^2 & 0 & 0 \end{bmatrix}$$

Lemma 2.3.1. *Let (X, Y) be one admissible pair of finite sequences, and $Z < Z'$, then $(X, Y) * Z < (X, Y) * Z'$.*

Lemma 2.3.2. *Let (X, Y) and (S, W) be admissible pairs and A and B be any two sequences in Σ such that $A \leq B$. Consider $Z \in \{X, Y\}$, then a sequence $K \in \Sigma\{Z_{[p,|Z|-1]}0\}$ belong to $[Z_{[p,|Z|-1]}(X, Y) * A, Z_{[p,|Z|-1]}(X, Y) * B] \cap \Sigma^+((X, Y) * (S, W))$ if and only if $K = Z_{[p,|Z|-1]}(X, Y) * C$, with $C \in [A, B] \cap \Sigma^+((S, W))$.*

Lemma 2.3.3. *Let (X, Y) be one admissible pair of finite sequence, $W, W' \in \{X, Y\}$. if $\sigma^p((\overline{W})^\infty) < \sigma^q((\overline{W}')^\infty)$ and $W_{[p,|W|-1]} \neq W'_{[p,|W'|-1]}$ then*

$$W_{[p,|W|-1]}(X, Y) * Z \leq W'_{[p,|W'|-1]}(X, Y) * Z'$$

for any sequences Z, Z' .

Next we will introduce some more notations:

For $l \leq p$,

$$Z_{[l,p]} = Z_l \cdots Z_p$$

$$m(A, B) = \min\{k \geq 0 : A_{|A|-1-k} \neq B_{|B|-k1}\}$$

$$\Sigma(A, B) = \{\sigma^n(A), \sigma^m(B) : 0 \leq n < |A|, 0 \leq m < |B|\},$$

and

$$\begin{aligned}\phi(A, B) &= \sum(A, B) \\ \sum(A, B) &= \sigma(A, B) \rightarrow \{1, \dots, |A| + |B|\},\end{aligned}$$

is the map that associates to each $X \in \sum(A, B)$, the position occupied by X in the lexicographic ordination of $\sum(A, B)$.

For each $1 \leq k \leq |S| + |W|$, denote

$$k_k = \begin{cases} [(X, Y) * \phi_{(S,W)}^{-1}(k), (X, Y) * \phi_{(S,W)}^{-1}(k+1)] & \text{if } m(X, Y) = 0, \\ [X_{[|X|-m(X,Y), |X|-1]}(X, Y) * \phi_{(S,W)}^{-1}(k), X_{[|X|-m(X,Y), |X|-1]}(X, Y) * \phi_{(S,W)}^{-1}(k+1)] & \text{if } m(X, Y) \neq 0. \end{cases}$$

Remark. We can get the following results from the last three lemmas which we will use to prove our main theorem;

1. If $p < m(X, Y)$, denote by I_{X_p} the set $\{X_{[p, |X|-1]}(X, Y) * \sigma^k(Z) : Z \in \{S, W\} \text{ and } Z_{k-1} = L\}$, from Lemma 1 and 2, $I_{X_p} = [X_{[p, |X|-1]}(X, Y) * \sigma^2(W), X_{[p, |X|-1]}(X, Y) * \sigma(S)] \cap \sum((X, Y) * (S, W))$, analogously denoting $I_{Y_p} = \{Y_{[p, |Y|-1]}(X, Y) * \sigma^k(Z) : Z \in \{S, W\} \text{ and } Z_{k-1} = R\}$. Thus $I_{Y_p} = [Y_{[p, |Y|-1]}(X, Y) * \sigma(W), Y_{[p, |Y|-1]}(X, Y) * \sigma^2(S)] \cap \sum((X, Y) * (S, W))$. On the other hand, if $p \geq m(X, Y)$, then $X_{[p, |X|-1]} = Y_{[p, |Y|-1]}$ and $[X_{[p, |X|-1]}(X, Y) * \sigma^2(W), X_{[p, |X|-1]}(X, Y) * \sigma(S)] \cap \sum((X, Y) * (S, W)) = \{X_{[p, |X|-1]}(X, Y) * \sigma^k(Z) : Z \in \{S, W\}\}$. Without risk of confusion, we will denote these sets by I_{X_p} .
2. The ordination of the elements of the sets I_{X_p} and I_{Y_q} is induced by the ordination of the sequences $\sigma^k(Z)$ such that $Z \in \{S, W\}$. This follows from Lemma 1 immediately.
3. For each $Z \in \{X, Y\}$, if $p \neq |Z| - m(X, Y) - 1$ then $\sigma(I_{Z_p}) = I_{Z_{p+1}}$. On the other hand, $\sigma(I_{X_{|X|-m(X,Y)-1}}) \cup \sigma(I_{Y_{|Y|-m(X,Y)-1}}) = I_{X_{|X|-m(X,Y)}}$. This follows from the definition.

4. Let $J_k = [\max I_{\phi_{(X,Y)}^{-1}(k)}, \min I_{\phi_{(X,Y)}^{-1}(k+1)}]$ and $H_k = [\phi_{(X,Y)}^{-1}(k), \phi_{(X,Y)}^{-1}(k+1)]$, it follows from Lemma 3 that $\sigma(J_{p_{k'}}) \subset J_{p_k}$ if and only if $\sigma(H_{k'}) \subset H_k$. Moreover, from (3) and Lemma 1, if $\phi_{(X,Y)}^{-1}(k) \notin \{\sigma^{|X|-m(X,Y)-1}(X), \sigma^{|Y|-m(X,Y)-1}(Y)\}$, then $\sigma(\max I_{\phi_{(X,Y)}^{-1}(k)}) = \max I_{\sigma(\phi_{(X,Y)}^{-1}(k))}$ and $\sigma(\min I_{\phi_{(X,Y)}^{-1}(k)}) = \min I_{\sigma(\phi_{(X,Y)}^{-1}(k))}$.
5. From (2) and (3), performing some straightforward computations with the lengths of $(X, Y) * S$ and $(X, Y) * W$ we see that, for $k \neq k'$, then $\sigma^n(I_k) \cap \sigma^m(I_{k'}) \neq \emptyset$ if and only if both $\sigma^n(I_k)$ and $\sigma^m(I_{k'})$ are contained in $I_{|X|-m(X,Y)}$ and $\sigma^p(P_k) \cap \sigma^q(P_{k'}) \neq \emptyset$, where $P_k = [\phi_{(S,W)}^{-1}(k), \phi_{(S,W)}^{-1}(k+1)]$ and p and q are such that
- $$n = |X|_{n_{L[0,p-1]}}(\phi_{(S,W)}^{-1}(k)) + |Y|_{n_{R[0,p-1]}}(\phi_{(S,W)}^{-1}(k)) \text{ and}$$
- $$m = |X|_{n_{L[0,p-1]}}(\phi_{(S,W)}^{-1}(k')) + |Y|_{n_{R[0,p-1]}}(\phi_{(S,W)}^{-1}(k')).$$

In the rest of our work we will use the following notation

$$A_{T(X,Y)}(t) = A_{T(X,Y)}(t^2, t^2).$$

since in the case of sub-Lorenz template $T_{(X,Y)}$ we have one curl in each ribbon from one partiton element to other.

Theorem 2.3.4. *For a reducible kneading pair $(X, Y) * (S, W)$ with both (X, Y) and (S, W) , admissible finite Lorenz pairs we have that*

$$\zeta_{T_{(X,Y) * (S,W)}}^S(t^2, t^2) = \zeta_{T_{(X,Y)}}^S(t^2, t^2) \zeta_{T_{(S,W)}}^S(t^{2|X|}, t^{2|Y|}).$$

For the proof see Franco [3].

Example 2.3.2. Let $(X, Y) = (LRRLR0, RLL0)$, as in the last example. And let $(S, W) = (LRR0, RL0)$, see Figure 2.4 for the template.

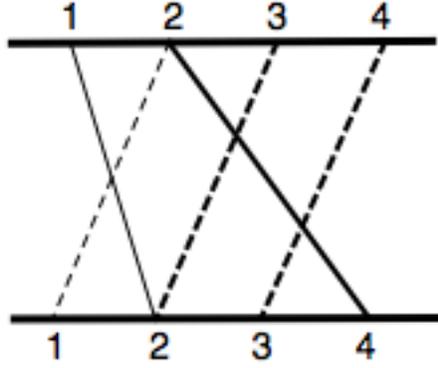


Figure 2.4. Sub-Lorenz template $T_{(LRR0,RL0)}$.

We already have $A_{T_{(x,y)}}(t^2, t^2)$ and,

$$A_{T_{(S,W)}}(t^2, t^2) = \begin{bmatrix} 0 & t^2 & t^2 \\ t^2 & 0 & 0 \\ 0 & t^2 & 0 \end{bmatrix}$$

Thus,

$$I - A_{T_{(x,y)}}(t^2, t^2) = \begin{bmatrix} 1 & -t^2 & -t^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t^2 & -t^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -t^2 \\ -t^2 & -t^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -t^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -t^2 & 0 & 1 \end{bmatrix}$$

$$\det(I - A_{T_{(x,y)}}(t^2, t^2)) = 1 - t^4 - t^6 - t^8 - t^{10} - t^{12}$$

And,

$$I - A_{T_{(S,W)}}(t^{2|5|}, t^{2|3|}) = \begin{bmatrix} 1 & -t^{10} & -t^{10} \\ -t^6 & 1 & 0 \\ 0 & -t^6 & 1 \end{bmatrix}$$

$$\det(I - A_{T_{(S,W)}}(t^{10}, t^6)) = 1 - t^{16} - t^{22}$$

$$(X, Y) * (S, W) = (LRRLRLLRLL0, RLLLRRLR0)$$

The template of this star product is in Figure 2.5. Now let $H = I - A_{T_{(X,Y)*(S,W)}}$, then

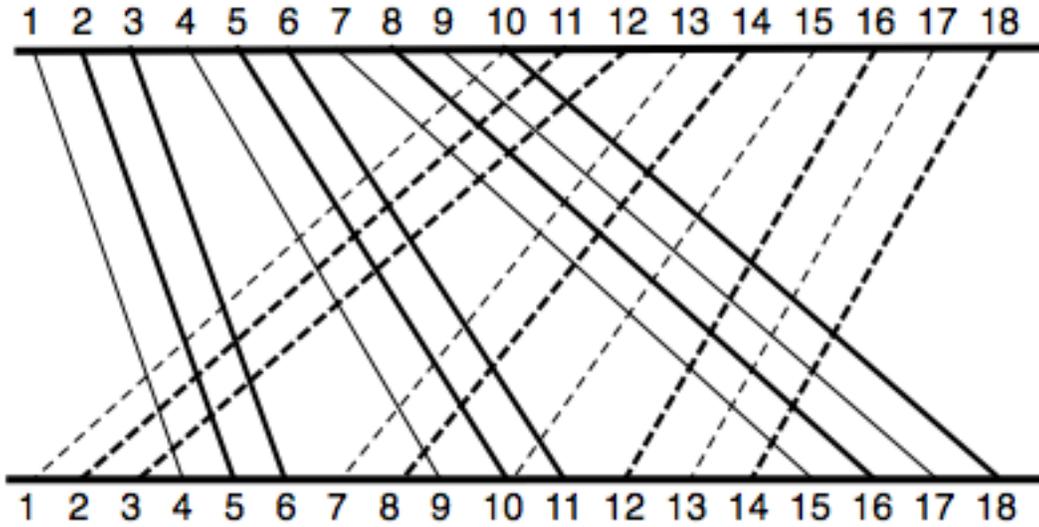


Figure 2.5. Sub-Lorenz template $T_{(X,Y)*(S,W)}$.

$$H = \begin{bmatrix} 1 & 0 & 0 & -t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -t^2 & -t^2 & -t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -t^2 & -t^2 & -t^2 & -t^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^2 \\ -t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t^2 & -t^2 & -t^2 & -t^2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -t^2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^2 & -t^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^2 & -t^2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(H) = \det(I - A_{T_{(X,Y)*(S,W)}}) = 1 - t^8 - t^4 + 2t^{26} - t^{12} + t^{32} - t^{16} + t^{34} + t^{30} + 2t^{28} + t^{24} + t^{20} - t^{10} - t^6$$

Therefore;

$$\det(I - A_{T_{(X,Y)*(S,W)}}) = \det(I - A_{T_{(X,Y)}}(t^2, t^2)) \det(I - A_{T_{(S,W)}}(t^{10}, t^6))$$

And we can see that;

$$\zeta_{T_{(X,Y)*(S,W)}}^S(t^2, t^2) = \zeta_{T_{(X,Y)}}^S(t^2, t^2) \zeta_{T_{(S,W)}}^S(t^{2|X|}, t^{2|Y|}).$$

2.3.2 the $\text{Sub-}\mathcal{L}(n, m)$ template

Definition. If we have a $\mathcal{L}(n, m)$ map f that has a double saddle connection, we can use the points $\{f^i(0^-), f^j(0^+) : 1 \leq i \leq n, 1 \leq j \leq m\}$ to define a finite Markov partition for the semiflow.

The restriction of the semiflow to this partition is called a *Sub- $\mathcal{L}(n, m)$ template*.

We took this theorem to a higher level, by trying to get the relationship between the zeta function of a star-product of two $\mathcal{L}(n, m)$ admissible pairs where n and m are nonnegative even numbers and their zeta functions.

This is what we got:

Theorem 2.3.5. *For a reducible kneading pair $(X, Y)*(S, W)$ with both (X, Y) and (S, W) , admissible finite $\mathcal{L}(n, m)$ pairs where n and m are nonnegative even numbers, we have that*

$$\zeta_{T_{(X,Y)*(S,W)}}^S(t^{2+n}, t^{2+m}) = \zeta_{T_{(X,Y)}}^S(t^{2+n}, t^{2+m}) \zeta_{T_{(S,W)}}^S(t^{[(2+n)X_L+(2+m)X_R]}, t^{[(2+n)Y_L+(2+m)Y_R]}).$$

Where

$X_L = \#$ of L symbols in the sequence X .

$X_R = \#$ of R symbols in the sequence X .

$Y_L = \#$ of L symbols in the sequence Y .

$Y_R = \#$ of R symbols in the sequence Y .

Proof. The Markov partition P associated to $T_{(X,Y)*(S,W)}$ can be split as $P = RB \cup P \setminus RB$ in such a way that all the iterates of each periodic orbit are exclusively contained in RB

or in $P \setminus RB$. So we can split the sum;

$$\sum_{q=2}^{\infty} \tau_q \frac{t^q}{q} = \sum_{q=2}^{\infty} \tau_q(RB) \frac{t^q}{q} + \sum_{q=2}^{\infty} \tau_q(P \setminus RB) \frac{t^q}{q}$$

where $\tau_q(RB)$ (resp. $\tau_q(P \setminus RB)$) means simply that we are counting orbits contained in RB (resp. $P \setminus RB$).

Now lets calculate zeta function of each group:

1.

$$\exp\left(\sum_{q=2}^{\infty} \tau_q(P \setminus RB) \frac{t^q}{q}\right) = \zeta_{T_{(X,Y)}^S}(t^{2+n}, t^{2+m})$$

since it is the same as the original.

2. Each ribbon leaving a cell I_k makes $((2+n)/2)X_L + ((2+m)/2)X_R$ curls without splitting if I_k is on the left of 0 and makes $((2+n)/2)Y_L + ((2+m)/2)Y_R$ curls without splitting if I_k is on the right of 0, before reenter in $I_{X|X|-m(X,Y)-1}$. Moreover $\sigma^{|X|}(I_k) \cap I_{k'}$ (resp. $\sigma^{|Y|}(I_k) \cap I_{k'}$) is not empty if and only if $\sigma(P_k) \cap \sigma(P_{k'}) \neq \emptyset$, so

$$\exp\left(\sum_{q=2}^{\infty} \tau_q(RB) \frac{t^q}{q}\right) = \zeta_{T_{(S,W)}^S}(t^{[(2+n)X_L+(2+m)X_R]}, t^{[(2+n)Y_L+(2+m)Y_R]}).$$

and the result follows. □

Example 2.3.3. Let $(X, Y) = (LRRL0, RLLR0)$ and $(S, W) = (LRR0, RL0)$ two admissible pairs in $\mathcal{L}(4, 6)$. Then,

$$I - A_{T(X,Y)}(t^4, t^6) = \begin{bmatrix} 1 & 0 & -t^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t^4 & -t^4 & 0 \\ 0 & 0 & 1 & 0 & 0 & -t^4 \\ -t^6 & 0 & 0 & 1 & 0 & 0 \\ 0 & -t^6 & -t^6 & 0 & 1 & 0 \\ 0 & 0 & 0 & -t^6 & 0 & 1 \end{bmatrix}$$

$$\det(I - A_{T(X,Y)}(t^4, t^6)) = 1 + t^{30} - t^{20} - t^{10}$$

$$I - A_{T(S,W)}(t^{20}, t^{20}) = \begin{bmatrix} 1 & -t^{20} & -t^{20} \\ -t^{20} & 1 & 0 \\ 0 & -t^{20} & 1 \end{bmatrix}$$

$$\det(I - A_{T(S,W)}(t^{20}, t^{20})) = 1 - t^{40} - t^{60}$$

Now the matrix of $(I - A_{T(X,Y)*(S,W)})$ is:

$$\det(I - A_{T_{(X,Y) * (S,W)}}) = \det(I - A_{T_{(X,Y)}}(t^4, t^6)) * \det(I - A_{T_{(S,W)}}(t^{20}, t^{20}))$$

Which give us;

$$\zeta_{T_{(X,Y) * (S,W)}}^S(t^2, t^2) = \zeta_{T_{(X,Y)}}^S(t^2, t^2) \zeta_{T_{(S,W)}}^S(t^{[(2+n)X_L + (2+m)X_R]}, t^{[(2+n)Y_L + (2+m)Y_R]}).$$

2.3.3 the Sub-horseshoe template

Definition. If we have a horseshoe map f that has a double saddle connection, we can use the points $\{f^i(0^-), f^j(0^+) : 1 \leq i \leq n, 1 \leq j \leq m\}$ to define a finite Markov partition for the semiflow.

The restriction of the semiflow to this partition is called *a Sub-horseshoe template*.

Example 2.3.4. For $(X, Y) = (LRLLO, RRLLR0)$, we construct the sub-horseshoe template $T(X, Y)$ following the procedure of Figure 2.6 where;

$$\sigma(X) = RLL0 \quad (8) \quad \sigma(Y) = RLLR0 \quad (7)$$

$$\sigma^2(X) = LL0 \quad (1) \quad \sigma^2(Y) = LLR0 \quad (2)$$

$$\sigma^3(X) = L0 \quad (3) \quad \sigma^3(Y) = LR0 \quad (4)$$

$$\sigma^4(X) = 0 \quad (5) \quad \sigma^4(Y) = R0 \quad (6)$$

$$\sigma^5(Y) = 0 \quad (5)$$

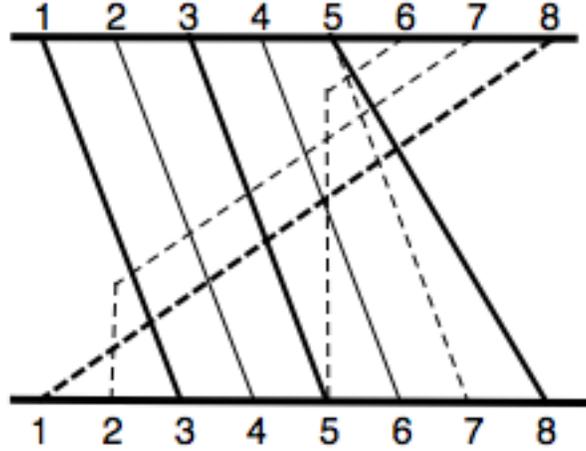


Figure 2.6. Sub-horseshoe template $T_{(LRLLO,RRLLR0)}$.

We associate to the template the transition matrix $A_{T(X,Y)}(t^2, t^3)$;

$$A_{T(X,Y)}(t^2, t^3) = \begin{bmatrix} 0 & 0 & t^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t^2 & t^2 \\ 0 & 0 & 0 & 0 & t^3 & t^3 & 0 \\ 0 & t^3 & t^3 & t^3 & 0 & 0 & 0 \\ t^3 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 2.3.5. By using $(X, Y) = (LRLLO, RRLLR0)$ as in the previous example and $(S, W) = (LRL0, RRLR0)$.

$(X, Y) * (S, W) = (LRLLLRLLRLLRLL0, RRLLLRRLLRLLRLLRLLR0)$;

see Figure 2.7 for the template of $T_{(LRL0,RRLR0)}$.

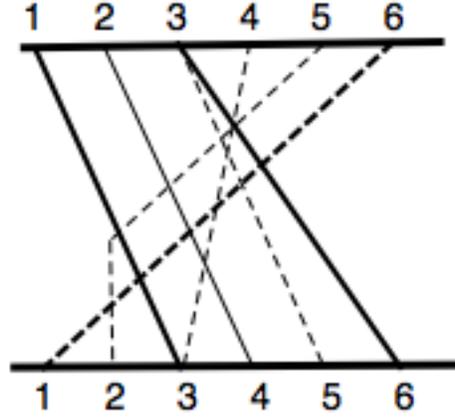


Figure 2.7. Sub-horseshoe template $T_{(LRR0,RL0)}$.

We can see that,

$$I - A_{T_{(X,Y)}}(t^2, t^3) = \begin{bmatrix} 1 & 0 & -t^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -t^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -t^2 & -t^2 \\ 0 & 0 & 0 & 0 & 1 - t^3 & -t^3 & 0 \\ 0 & -t^3 & -t^3 & -t^3 & 0 & 1 & 0 \\ -t^3 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(I - A_{T_{(X,Y)}}(t^2, t^3)) = 1 - t^3 - t^5 - t^7 + t^{10} - t^{15} - t^{17}.$$

And

$$I - A_{T(S,W)}(t^2, t^3) = \begin{bmatrix} 1 & 0 & -t^2 & 0 & 0 \\ 0 & 1 & 0 & -t^2 & -t^2 \\ 0 & 0 & 1 - t^3 & -t^3 & 0 \\ 0 & -t^3 & 0 & 1 & 0 \\ -t^3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(I - A_{T(S,W)}(t^2, t^3)) = 1 - t^3 - t^5 - t^8 - t^{13}.$$

For the template of $(X, Y) * (S, W)$ see Figure 2.8. Since the matrix for their $*$ -product

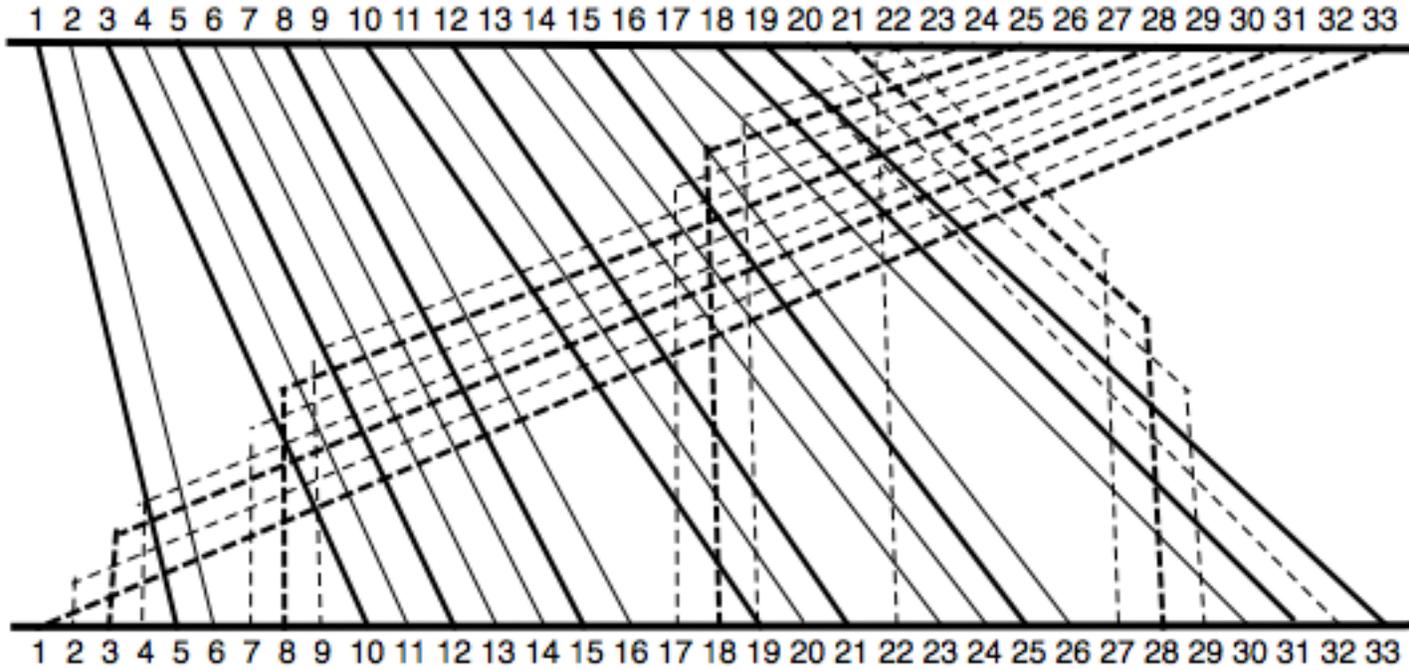


Figure 2.8. Sub-horseshoe template $T_{(X,Y)*(S,W)}$.

is too big (32×32), we will explain where every interval in its Markov partition goes:

1 \longrightarrow 5	19 \longrightarrow 29, 30, 31
2 \longrightarrow 6, 7, 8, 9	20 \longrightarrow 28
3 \longrightarrow 10	21 \longrightarrow 27
4 \longrightarrow 11	22 \longrightarrow 22, 23, 24, 25, 26
5 \longrightarrow 12	23 \longrightarrow 19, 20, 21
6 \longrightarrow 13	24 \longrightarrow 18
7 \longrightarrow 14	25 \longrightarrow 17
8 \longrightarrow 15	26 \longrightarrow 9, 10, 11, 12, 13, 14, 15, 16
9 \longrightarrow 16, 17, 18	27 \longrightarrow 8
10 \longrightarrow 19	28 \longrightarrow 7
11 \longrightarrow 20	29 \longrightarrow 4, 5, 6
12 \longrightarrow 21, 22	30 \longrightarrow 3
13 \longrightarrow 23	31 \longrightarrow 2
14 \longrightarrow 24	32 \longrightarrow 1,
15 \longrightarrow 25	
16 \longrightarrow 26, 27, 28, 29	
17 \longrightarrow 30	
18 \longrightarrow 31	

where every arrow in the left makes two half twists (t^2) and on the right makes three half twists (t^3).

$$\det(I - A_{T_{(X,Y)*(S,W)}}) = 1 - t^9 + t^{12} + t^{78} - t^{15} - t^{17} + t^{18} - t^{19} + t^{20} + t^{22} + t^{24} - t^{25} - t^{27} - t^{29} + t^{30} - t^{31} + t^{32} + t^{34} - t^{37} - t^{39} + t^{40} - t^{41} + t^{42} - t^{43} + t^{44} + t^{46} - t^{47} - t^{49} + t^{52} + t^{54} + t^{56} - t^{61} + t^{64} + t^{66} + t^{68} - t^{71} + t^{76} - t^3 - t^5 - t^7.$$

The question we have here is whether there a relationship between the zeta function of a $*$ -product of two horseshoe admissible pairs and the zeta function of these two admissible pairs?

And if this relationship exists, can we take it to a higher level?

2.4 FUTURE WORK

In general the zeta function of the sub-horseshoe template of the $*$ -product of two horseshoe admissible pairs cannot be factorized into two other zeta functions, so we will look for a different relationship between the zeta function of two horseshoe admissible pairs and their $*$ -product using the sub-horseshoe templat of the $*$ -product and the sub-horseshoe templates of the elements of the product. For example, if we have the two horseshoe admissible pairs (X, Y) and (S, W) , the elements of their product are: (X, Y) , (X_L, Y_L) , and (S, W) .

Then we will try to take that to a higher level by finding the relationship between the zeta function of the $*$ -product of two $\mathcal{L}(n, m)$ admissible pairs and the zeta function of the originals, where n and m are odd.

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Earned the membership of Golden Key International Honor Society.

Earned the membership of The Honor Society of Phi Kappa Phi.

Earned the membership of The Honor Society.

Dissertation Title:

RENORMALIZATION AND *-PRODUCT OF THE HORSESHOE MAPS

Major Professor: Prof. Michael Sullivan