The Open Court

A MONTHLY MAGAZINE

Devoted to the Science of Religion, the Religion of Science, and the Extension of the Religious Parliament Idea

Editor: Dr. Paul Carus.
Assistant Editor: T. J. McCormack.

Associates: E. C. Hegeler. Mary Carus.

VOL. XVI. (no. 9) September, 1902. NO. 556

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One of the Great Assembly Rooms in the Buddhist Cave Temples at Ajantâ.

Frontispiece to The Open Court.

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THE FOUNDATIONS OF GEOMETRY.[1]

BY DR. GEORGE BRUCE HALSTED.

ON the 13th of July, 1733, received the *imprimatur* of the Inquisition a book entitled *Euclid Vindicated from Every Fleck*, by the Jesuit Saccheri. In this book is given an entirely new turn to a question of centuries, the deduction of Euclid's celebrated parallel-postulate from his remaining assumptions. Here begins for the first time in the world a procedure whose latest brilliant flowering is seen in Hilbert's *Festschrift*, just now appearing in English.

If the postulate in question is no consequence of the others, a geometry may be exhibited in which they hold, but it does not.

Both the very recent books, Manning's *Non-Euclidean Geometry*, 1901, and Barbarin's *La Géométrie non-euclidienne*, 1902, adopt Saccheri's presentation, starting from an isosceles birectangular quadrilateral $ACDB$ in which the angles at A and B are right, and the sides AC and BD perpendicular to AB are equal, and considering the hypothesis, taken as equivalent to Euclid's parallel-postulate, that the equal angles at C and D are right, and the two hypotheses contradictory to this, namely that the two are obtuse and that the two are acute. And the Italian, the American, and the Frenchman exhibit the geometries corresponding to these two new hypotheses.

But Saccheri erecting his imposing structures with marvellous genius and elegance, and with a perfection which, as Staeckel says, represents the work of a life-time, professes only to build them that his destruction of them may prove the parallel-postulate a consequence of Euclid's other assumptions.

[1] *The Foundations of Geometry.* By David Hilbert, Ph. D., Professor of Mathematics, University of Göttingen. Authorised translation by E. J. Townsend, Ph. D., University of Illinois, Chicago: The Open Court Publishing Company. 1902. 8vo. Pages, vii, 132.

My friend the erudite Father Hagen of the Society of Jesus has written me, and gives me permission to use his opinion, the weightiest on this point of any living man's, that Saccheri, contrary to what Staeckel supposes, not only doubted the necessity of Euclid's postulate but knew that the slow, gentle, and feeble blows which he delivered nominally to demolish his structures really left them unscathed. In no other form, says Father Hagen, would the publication of such revolutionary ideas have then been permitted by the Provincial of the Jesuits, whose official authorisation was necessary, and was granted August 16, 1733. However, the book was so completely lost that Staeckel in 1895 speaks of its discovery by Beltrami in 1889 as creating a sensation, and my copy, from which I made the first translation into any modern tongue, is still, so far as I know, the only one on this continent.

Nearly a century later, in 1812, the German Schweikart at the Russian University Charkov invented what he called his Astral Geometry, the very system which bulks most largely in Saccheri. Returned to Germany, Schweikart sends in 1818 through Gerling to Gauss a *résumé* of his creation, which may fairly be considered the first *published* (not printed) treatise on non-Euclidean geometry. This, the non-Euclidean geometry of 1812 by Schweikart, I have given in *Science*, 1900, pp. 842–846.

Again in 1823 John Bolyai, a young Magyar, at Temesvár, as he writes, "from nothing created another wholly new world." This very year is his centenary, and Hungary will honor herself in honoring this truest genius, her son. He never published anything but, in a book by his father, one brief appendix, which he had the courage to call *The Science Absolute of Space*, and which remains the most extraordinary two dozen pages in all the history of human thought.

It is usual to date Lobachévski's discovery of this non-Euclidean geometry from 1826. In 1836 in his *Introduction to New Elements of Geometry*, of which I was the first to publish a translation (Vol. V., *Neomonic Series*, 1897), he says: "Believing myself to have completely solved the difficult question, I wrote a paper on it in the year 1826, *Exposition succincte des principes de la Géométrie*, AVEC UNE DÉMONSTRATION RIGOUREUSE DU THÉORÈME DES PARALLÈLES, read February 12, 1826, in the *séance* of the Physico-Mathematic Faculty of the University of Kazan, but never printed." No part of this French manuscript has ever been found. The latter half of the title is ominous. For centuries the world had been deluged with rigorous (!) demonstrations of the theorem of parallels. We know that three years later Lobachévski himself realised its absolute indemonstrability. Yet the paper said to contain material to stop forever this twenty-centuries-old striving still was headed *démonstration rigoureuse*, just as Saccheri's book of 1733 containing a coherent treatise on non-Euclidean geometry ended by one more pitiful proof of the parallel-postulate.

If Saccheri, like Lobachévski, had lived three years longer (he died Oct. 25, 1733), and had realised (as Father Hagen says he

NICOLÁI IVÁNOVICH LOBACHÉVSKI.
Portrait from the memorial circular on the centenary of his birth.

did) the pearl in his net, with the new meaning, he could have retained his old title: *Euclides ab omni naevo vindicatus*, since the non-Euclidean geometry is a perfect vindication and explanation of Euclid.

But Lobachévski's title is made wholly indefensible. A new geometry, founded on the contradictory opposite of the theorem of parallels, and so proving every demonstration of that theorem fallacious, could not very well pose under Lobachévski's old title.

He himself never tells what he meant by it, never tries to explain it. When citing the title in 1829, he substitutes *etc.* for the eight tell-tale words. When at last conscious of the new geometric science, the name he gave it, *Imaginary Geometry*, was a personal calamity.

But time has at length relented, and the world will always know this marvellous creation henceforth as the Bolyai-Lobachévski geometry, as it is now called by Hilbert, whom it justifies in making of Euclid's Axiom of Parallels a whole group, "die Axiomgruppe III."

In 1847, in the quaint and ancient Nuremberg of Albrecht Duehrer was published by the Erlangen professor von Staudt his *Geometrie der Lage*, an epoch-making work which leads to the cutting apart of Hilbert's "Axiomgruppe IV: Axiome der Congruenz." On the title page of this extraordinary book, now very rare, his name stands as Dr. Georg Karl Christian v. Staudt, but history and Max Noether, who should know, reverse the order of the first two names.

Georg von Staudt, born on the 24th of January, 1798, at Rothenburg ob der Tauber, was an aristocrat, issue of the union of two of the few *regierenden* families of the then still free Reichsstadt, which four years later closed the 630 years of its renowned existence as an independent republic. But his creation of a geometry of position disembarrassed of all quantity, wholly non-metric, neither positively nor negatively quantitative, was the outcome of a creation due to a French boy of low birth, born in 1746 at Beaune.

The construction of a plan of Beaune won this boy, Gaspard Monge, admission to the college of engineers at Mézières. From the consideration of certain problems in fortification he was led to generalise all the isolated methods hitherto employed, not merely in fortification, but in perspective, dialling, stone-cutting, etc., and to create a code theoretical and practical, which he termed *La Géométrie Descriptive*, which supplied the means of preparing on uniform principles the working drawings necessary in the various arts, and also of graphically solving problems in solid geometry, by general methods, capable of the most extensive practical application. Henceforth the name of Monge was inseparably associated with the development of French technical education.

Monge's pupil at the Paris Polytechnic School, Jean Victor Poncelet, in Napoleon's Russian campaign was abandoned as dead on the bloody field of Krasnoi and taken prisoner to Saratoff.

There in 1812 and 1813 he based chiefly on what he had studied with Monge the first draught of his famous treatise published in 1822. It gave us projective geometry by the simple substitution of central for Monge's parallel projection.

In the hands of the great Swiss, Jacob Steiner, this modern synthetic geometry develops with mighty power. But still its theory was almost an *argumentum in circulo.* Steiner still based it on magnitude-assumptions. Georg von Staudt it was whose creative genius availed to build the projective geometry without magnitude or congruence assumptions, and without motion.

Hilbert calls his Axiomgruppe IV, Axiome der Congruenz (oder der Bewegung), and Schur calls Hilbert's first two groups "the projective axioms."

In 1854 Riemann pronounced his astonishing discourse *On the hypotheses which lie at the basis of geometry*, containing the epoch-making idea that though space be unbounded, it is not therefore infinitely great. From the unboundedness of space its infinity in no way follows. Thus it may be that the whole universe could contain only a certain finite number of common building brick, so that then there would not be room for one more brick in the universe. From this it follows that even Euclid's very first proposition: "To describe an equilateral triangle on a given sect (on a given finite straight line)" involved a set of assumptions sufficient to make the straight line infinite, open, not finite and closed. In elliptic space it is not always possible to construct an equilateral triangle on a given base. Yet Euclid deduces even parallels before using his parallel postulate.

A brilliant filling in of these gaps by creating a set of "betweenness" assumptions was accomplished by Dr. M. Pasch of Giessen in 1882 in his book *Vorlesungen über neuere Geometrie*, to which Hilbert credits his second group of axioms. In the April number, 1902, of the *American Mathematical Monthly* in an article entitled "The Betweenness Assumptions," Hilberts's II. 4 has been shown to be an unnecessary redundancy. Thus the betweenness assumptions have been reduced from five to four.

The remaining axioms are Hilbert's group I., Axioms of Association. But again that very same Prop. I. of Euclid requires the assumption: If A and B be any two given points, there is at least one point C whose sects from A and B are both congruent to AB. This can only be covered by an axiom of continuity. Such is Hilbert's group V. (Archimedes's Axiom).

This historical investigation of the different colored threads which are to enter into the warp and the woof of Hilbert's weaving, brings out a surprising increase of penetration and clearness regarding fundamental assumptions for thousands of years subconscious.

It is due primarily to Baltzer and Hoüel that, beginning only from 1866, the world of science became conscious of the profound penetrations into, and remakings of, the foundations of geometry, which since then have been the key in the study and mastering of the fundamental concepts of all science.

Hilbert's *Festschrift* is still the most brilliant example of efforts to find for a special branch of science a sufficient and closed system of mutually independent first principles, assumptions; though America has bettered it by the annihilation of II. 4, its most troublesome and undesirable member.

In an article on like efforts for Mechanics, Dr. E. B. Wilson, of Yale, writes (*Bulletin Amer. Math. Soc.*, 1902, p. 342):

"This lack of satisfaction is but one of the many similar manifestations of the present state of mathematical instruction and mathematical science. We are no longer content to bear with superficially clear statements which seldom if ever lead into actual error,—nor does it suffice to start with inaccurate statements and, as we advance, to modify them so as to bring them into accord with our wider vision and our more stringent requirements. No. We must from the beginning bring up ourselves and our pupils on not only the truth but the whole truth.

"How soon the recent researches of Hilbert and others on the foundations of geometry must take their place in elementary textbooks on plane and solid geometry cannot be said. But that is purely a matter of time."

To have made these inspiring researches accessible in English is a weighty addition to the debt we already owe the Open Court Publishing Company.

Unconsciously from the time of Descartes and before, consciously, openly from the time of Newton, there has been in progress a procedure which may be called the arithmetisation of geometry. This brilliant *Festschrift* of Hilbert's may be most deeply characterised as a reversal of that procedure. It is a return to Euclid and the spirit of Euclid. It is anti-French, for in France elementary geometry has never recovered from Clairaut and Legendre. Even the latest and best French geometry, that of Hadamard, published under the editorship of the great and lovable Gaston Darboux, never presents nor consciously considers the question of its own foundations. It seems childishly unconscious of the great and final question for a scientific geometry, namely: What are the necessary and sufficient and independent conditions which must be fulfilled by a system of things in order that every property of these things may correspond to a geometric fact, and inversely; and that so these things may be a complete and simple picture of geometric reality.

Again, of the order of his propositions Hilbert said in his lectures: "The order of propositions is important. Mine differs strongly from that usual in text-books of elementary geometry; on the other hand, it greatly agrees with Euclid's order. So these wholly modern investigations lead us rightly to appreciate and in the highest degree to wonder at the penetrating wisdom of this ancient geometer."

Again he says of Euclid's parallel postulate:

"What penetration the setting-up of this axiom required, we recognise best if we cast a glance over the history of the axiom of parallels. As for Euclid himself (*circa* 300 B. C.), he, for example, proves the theorem of the exterior angle before introducing the parallel axiom, a sign, how deeply he had penetrated into the interdependence of the geometric theorems."

But also in two other exceedingly important respects Hilbert is more of a return to Euclid than he himself seems to know. Hilbert discards proof by superposition. Motion itself needs a geometric foundation, and so cannot be a foundation for geometry. So Hilbert in his lectures assumed Euclid I., 4, and even in the *Festschrift* he still assumes two-thirds of it as "IV. 6. If, in the two triangles ABC and $A'B'C'$, the congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle BAC \equiv B'A'C'$ hold, then the congruences $\angle ABC \equiv A'B'C'$ and $\angle ACB \equiv A'C'B'$ also hold."

But in all the most ancient manuscripts of Euclid the so-called proof of I. 4 by superposition is evidently corrupt, and in general Euclid's avoidance of direct superposition has always been noted, for example in I. 5, I. 6, I. 26, III. 26, III. 27.

Bertrand Russell says of the corrupt proof of I. 4: "The fourth proposition is a tissue of nonsense. Superposition is a logically worthless device."

But in another still more subtle respect Hilbert is a return to the real Euclid. In his lectures, as lithographed, Hilbert makes a serious blunder in regard to Euclid's treatment of proportion. Hilbert says: "The fundamental importance of the just-proven theorem lies therein, that it puts us in condition to found the theory of proportion without any new axiom. We see, therefore, that here also Euclid is finally justified: he also introduces the theory of proportion without a new axiom. However we must add: Die Art dieser Einführung bei Euclid ist gänzlich verfehlt. Euclid bases, namely, the theory of proportion on the following two theorems: (1) If in a triangle ABC we draw the parallel $A'B'$ to AB, then is $AC:BC::A'C:B'C$. (2) The inverse: If in a triangle $AC:BC::A'C:B'C$, then is AB parallel to $A'B'$.

"The proofs of these theorems in Euclid are rigorous throughout, where AC and BC both result from repeated laying off of one and the same sect. But now Euclid refers to general magnitude-relations while he takes the above proportion as a numeric equation, and concludes so, that the theorem remains valid for any position of A and A'.

"Against this is to be objected: (1) It is a new axiom, that we may always take a proportion between sects as a number-relation. (2) Even if we have introduced this new axiom, we must expressly prove, that the thereby newly introduced numbers follow the same algorithmic laws as those already known."

Here we see that Hilbert wholly misunderstands Euclid's treatment of proportion. Hilbert's misconception comes from the modern attempts at the "arithmetisation" of the subject, and his objections hold good against those who define a ratio as a quotient or a number, and a proportion as an equality between two ratios. This is equivalent to the introduction of irrational numbers, which must then certainly be proved to obey the ordinary laws of operation.

But on the other hand, it was Isaac Newton, not Euclid, who first identified number and ratio. Euclid never thought of or treated a ratio as a number, or a proportion as an equality between numbers. In Euclid's time irrational numbers had not been created. They did not exist. Euclid gave of proportion a treatment which may be applied to sects, and hence to all geometry, in as purely geometric a way as Hilbert's own. Euclid uses V, the Archimedes assumption. On the other hand his treatment is simpler than Hilbert's, in that it only needs the addition of sects and not their product.

Hilbert's supposition that in Euclid's treatment sects need to be represented by numbers in terms of some common unit sect, and consequently that Euclid's proofs are only rigorous for commensurable sects, shows an entire misconception of Euclid and the fifth book. Euclid's treatment is admirable for the same reason as is that of the *Festschrift*, namely that it had no need to represent sects by numbers, and consequently has no need of irrational numbers. Thus we see that here also Hilbert might have taken as his battle-cry, "Back to Euclid!"

Though the fundamental theorems above mentioned can be simply proven without the assumption of magnitudes other than Hilbert's sects (see Halsted's *Elements*, pp. 183–184), yet it is true that Euclid in his proof (Eu. VI. 2) uses the *content* of a triangle, which content he has, in I. 39, assumed to be a magnitude. That this assumption is unnecessary and redundant was shown by Schur, using the axiom of Archimedes.

A more elegant demonstration, without the Archimedes postulate, constitutes §§ 18–21 of the *Festschrift*.

Still another point in which Hilbert returns to Euclid is in regard to that fundamental geometric entity, the angle. In Euclid the two sides of an angle "are not in the same straight line." The moderns attempting to remove this supposed restriction introduced the flat or straight angle, and convex or re-entrant angles. I myself introduced from the rare *Pelicotetics* the word perigon, which other writers of geometries, even Italian, adopted from my book, as Beman and Smith found by correspondence when discussing their adoption of my phrase "partition of a perigon" and the theorems and corollaries under that heading. But in Hilbert an angle is defined as a bi-ray whose two rays are co-initial but not co-straight.

Thus, as in Euclid, there are no angles greater than two right angles. The angle is unambiguous.

Throughout there is successful revolt against arithmetisation. As Hilbert said at Paris: "I oppose the opinion that only the concepts of analysis, or even those of arithmetic alone, are susceptible of a fully rigorous treatment." And well he may, who has so established geometry upon a simple and complete system of assumptions, that the exactness of the geometric ideas and their applicability to deduction is in no respect inferior to those of the old arithmetical concepts.

Said Hilbert: "The most suggestive and notable achievement of the last century is the discovery of non-Euclidean geometry." May I add: Its most fascinating outcome is Hilbert's *Festschrift*.