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Error Classification and Reduction in Solid State Qubits

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Abstract

Quantum computers have enormous advantages over classical computers. A quantum computer can be used to calculate the factors of a number, which is sometimes impossible during one's lifetime with a classical computer. Quantum information processing techniques can also be used for encryption, which makes eavesdropping impossible. Noise from the environment is a great challenge in building a reliable quantum computer. To build a reliable quantum computer, one has to protect the information content of the system from the environment. Otherwise, the information associated with the system will decay as the system interacts with the environment. The noise resulting from the system-bath interaction can be removed by using quantum error correcting codes or decoupling pulses. In addition, one could also use DFS encoding to make the information content of the system immune to the noise. In this paper, the various types of errors that could arise in a specific DFS encoding of a 3 spin qubit have been classified according to their effect on the state of the qubit. Also, the application of a specific decoupling pulse on the system, which is coupled to the environment through hyperfine Hamiltonian, has been analyzed.

1 Introduction

A quantum dot is a physical entity that is used to confine a finite number of electrons or holes in a specific region. The states associated with the spins of these electrons can be used to represent information. The idea of a quantum computer is to use the states associated with some quantum mechanical property, which can be spin, to store information and perform calculations faster than a classical computer using superposition states and entangled states. More specifically, a classical computer uses a high and a low voltage to represent 0's and 1's. Using this representation, a classical computer stores information and does all the required processing. A quantum computer, on the other hand, can use the z-component of the spin angular momentum to represent 0's and 1's. Since the z-component of the spin can also be in a superposition state and could also be entangled with the z-component of other spin, these states could be used for storing information and performing faster calculations. Therefore, one could use a single quantum dot with a single electron in it as a system and represent the information of the system by the superposition state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $|0\rangle \equiv |S_z = \frac{1}{2}\rangle$ and $|1\rangle \equiv |S_z = -\frac{1}{2}\rangle$. However, using a 3 or higher dimensional spin system allows one to use a noise avoiding method called DFS encoding, which is described in more detail in section 3. Hence, this paper is focused towards a triple quantum dot system where a single electron is present in each of the quantum dots, which is coupled to the electrons present in the other two quantum dots as in [1]. This system allows one to take advantage of the DFS encoding. However, this DFS encoding is immune to only certain types of noise acting on the system. The remaining noise, at least a part of it, can be eliminated by using quantum error correcting codes (QECC) and decoupling pulses. In fact, in this paper, I will present a combination scheme of DFS and decoupling pulses to eliminate as much noise as possible that results from the noise Hamiltonian. Even though similar work has been done before in [2], it has been done using a different DFS encoding and it did not utilize the noise obtained from the experimental data as in this paper.

If one could completely eliminate noise resulting from the coupling of the spin states of the electrons in the quantum dots with the environment, and if the spin states of these electrons can be reliably manipulated, one has a reliable quantum device. This paper has been written with an intention to help achieve this purpose. The remaining part of the paper is organized as follows. First, a brief discussion on the spin angular momentum states is presented in section 2 to clarify the discussion on the DFS encoding mentioned in section 3. In section 4, the noise is classified in the DFS subspace. Then, in section 5, a brief discussion of the exchange pulses is presented in order to help the reader understand the decoupling theory mentioned in section 6. Followed by it, the application of a pulse sequence on a specific Hamiltonian has been analyzed in section 7. Finally, all the results of this project have been summarized in the conclusion along with the future direction of this project.

2 Addition of Angular Momentum

Since this paper is based on the spin system of a linear triple quantum dot, I will present the eigen-states of S^2 of a triple spin system in this section. If there are two particles present in a system with the corresponding spin quantum numbers, S_1 and S_3, then the total spin quantum number of the system can have one of the following values: $|S_1 + S_3|, |S_1 + S_3 - 1|, ..., |S_1 - S_3|$. In a triple quantum dot, if the spins of the electrons in dot 1 and dot 3 are added, S_{13} can be either 0 or 1. If S_{13} is 0, S_{123} can only be 1/2. On the other hand, if S_{13} is 1, S_{123} can be either 3/2 or 1/2. Since the operators S_Z, S_{13}, and S_{123} form a complete set of commuting observables, the system is in one of the linear combination of the following eigen-states:

$$
\begin{aligned}
|1\rangle = & \quad |\frac{1}{2}, 1, \frac{1}{2}\rangle & = \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle - 2|\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) \\
|2\rangle = & \quad |\frac{1}{2}, 0, \frac{1}{2}\rangle & = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle) \\
|3\rangle = & \quad |\frac{1}{2}, 1, \frac{-1}{2}\rangle & = \frac{1}{\sqrt{6}}(|\uparrow\downarrow\downarrow\rangle - 2|\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) \\
|4\rangle = & \quad |\frac{1}{2}, 0, \frac{-1}{2}\rangle & = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle) \\
|5\rangle = & \quad |\frac{3}{2}, 1, \frac{3}{2}\rangle & = (|\uparrow\uparrow\uparrow\rangle) \\
|6\rangle = & \quad |\frac{3}{2}, 1, \frac{1}{2}\rangle & = \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle - 2|\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) \\
|7\rangle = & \quad |\frac{3}{2}, 1, \frac{-1}{2}\rangle & = \frac{1}{\sqrt{6}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) \\
|8\rangle = & \quad |\frac{3}{2}, 1, \frac{-3}{2}\rangle & = (|\downarrow\downarrow\downarrow\rangle)
\end{aligned}
$$

where eigen-states are in the form $|S, S_{13}, S_Z\rangle$

3 Decoherence Free Subsystems

As mentioned in the introduction, the information associated with the system is chosen to be represented by the eigen-states of the system rather than the physical states, which are measured experimentally. More specifically, the physical states of the electron spins in the quantum dots can be represented by the following eight states: $|000\rangle$, $|001\rangle$, $|010\rangle$, $|011\rangle$, $|100\rangle$, $|101\rangle$, $|110\rangle$, $|111\rangle$ where $|0\rangle$ represents $S_z = +\frac{1}{2}$ state and $|1\rangle$ represents S_z = -$\frac{1}{2}$ state. However, the information associated with the system used in my project is rather represented by $|\psi\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$ where $|0_L\rangle$ and $|1_L\rangle$ are the first two eigen-states of the system mentioned in section 2. Information is represented this way because any arbitrary linear combination of $|0_L\rangle$ and $|1_L\rangle$ is invariant under the action of noise

Hamiltonian generated by the uniform external magnetic field [1]. Hence, the subspace spanned by $|0_L\rangle$ and $|1_L\rangle$ is called the DFS for my system against the uniform external magnetic field. Also, these logical qubits allow the Heisenberg exchange interaction alone to be used as a universal gate for this system, which is discussed in more detail in section 5. The transformation, U_{dfs}, that generates logical states or encoded states from physical states, as shown below, is called the DFS transformation matrix for the system.

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |\uparrow\uparrow\uparrow\rangle \\ |\uparrow\uparrow\downarrow\rangle \\ |\uparrow\downarrow\uparrow\rangle \\ |\uparrow\downarrow\downarrow\rangle \\ |\downarrow\uparrow\uparrow\rangle \\ |\downarrow\uparrow\downarrow\rangle \\ |\downarrow\downarrow\uparrow\rangle \\ |\downarrow\downarrow\downarrow\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle - 2|\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) \\ \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle) \\ \frac{1}{\sqrt{6}}(|\uparrow\downarrow\downarrow\rangle - 2|\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle) \\ (|\uparrow\uparrow\uparrow\rangle) \\ \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle - 2|\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) \\ \frac{1}{\sqrt{6}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) \\ (|\downarrow\downarrow\downarrow\rangle) \end{pmatrix}$$

Hence, U_{dfs} is given by

$$U_{dfs} = \begin{pmatrix} 0 & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

One should also note that the transformation, $U_{dfs}.A_\alpha.U_{dfs}^\dagger$, will block diagonalize the operators against which the system is immune. Hence, the logical states will have the same form when a noise operator acts on the system. The physical qubits or the noise operators transformed by the DFS transformation are said to be in the DFS basis. A more detailed review of DFS is provided in [3],[4] and [5].

4 Classification of noise

The error algebra, $\mathcal{A}$, which consists of operators that can be used to frame all the types of noises acting on the system, is generated by the set $\{H_S, S_\alpha\}$ [2]. The operators H_S and S_α are discussed in section 6. The error algebra for a 3 qubit system consists of 64 operators, which is shown explicitly in the appendix. One should note that these error operators are denoted by $\widetilde{a}_i$ in the DFS basis as opposed to a_i in the physical basis. The set $\{\widetilde{a}_i\}$ can be classified into three types of operators: logical operators, leakage operators and orthogonal operators.

4.1 Logical Operators

Logical operators are the operators that act only on the logical qubits (or logical subspace). These operators contain elements in either/all of the positions labeled by a_{11}, a_{12}, a_{21} and a_{22} as shown in the matrix below.

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The action of these logical operators on the state of qubit is shown below:

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle - 2|\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) \\ \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle) \\ \frac{1}{\sqrt{6}}(|\uparrow\downarrow\downarrow\rangle - 2|\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle) \\ (|\uparrow\uparrow\uparrow\rangle) \\ \frac{1}{\sqrt{6}}(|\downarrow\uparrow\uparrow\rangle - 2|\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) \\ \frac{1}{\sqrt{6}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) \\ (|\downarrow\downarrow\downarrow\rangle) \end{pmatrix} = \begin{pmatrix} a_{11}|0_L\rangle + a_{12}|1_L\rangle \\ a_{21}|0_L\rangle + a_{22}|1_L\rangle \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

There are four such logical operators in the error algebra, $\widetilde{a}_1$ through $\widetilde{a}_4$. These four elements can be written as $(1+\widetilde{\sigma}_z)\otimes(1+\widetilde{\sigma}_z)\otimes\widetilde{\sigma}_i$ where $\widetilde{\sigma}_z$ is a Pauli matrix and $\widetilde{\sigma}_i$ can be any of the Pauli matrices or a 2 by 2 identity operator. One should note that there is a tilde over the Pauli matrices just to emphasize that $\widetilde{a}_i$'s are in the DFS basis.

4.2 Leakage Operators

The leakage operators move elements of the qubit from the logical subspace to the orthogonal subspace. These operators are of the form shown below by the matrix L.

$$L = \begin{pmatrix} 0 & 0 & l_{13} & l_{14} & l_{15} & l_{16} & l_{17} & l_{18} \\ 0 & 0 & l_{23} & l_{24} & l_{25} & l_{26} & l_{27} & l_{28} \\ l_{31} & l_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{41} & l_{42} & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{51} & l_{52} & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{61} & l_{62} & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{71} & l_{72} & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{81} & l_{82} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The action of these leakage operators on the qubit state is shown below:

$$\begin{pmatrix} 0 & 0 & l_{13} & l_{14} & l_{15} & l_{16} & l_{17} & l_{18} \\ 0 & 0 & l_{23} & l_{24} & l_{25} & l_{26} & l_{27} & l_{28} \\ l_{31} & l_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{41} & l_{42} & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{51} & l_{52} & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{61} & l_{62} & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{71} & l_{72} & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{81} & l_{82} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} |0_L\rangle \\ |1_L\rangle \\ |3\rangle \\ |4\rangle \\ |5\rangle \\ |6\rangle \\ |7\rangle \\ |8\rangle \end{pmatrix} = \begin{pmatrix} l_{13}|3\rangle + ... + l_{18}|8\rangle \\ l_{23}|3\rangle + ... + l_{28}|8\rangle \\ l_{31}|0_L\rangle + l_{32}|1_L\rangle \\ l_{41}|0_L\rangle + l_{42}|1_L\rangle \\ l_{51}|0_L\rangle + l_{52}|1_L\rangle \\ l_{61}|0_L\rangle + l_{62}|1_L\rangle \\ l_{71}|0_L\rangle + l_{72}|1_L\rangle \\ l_{81}|0_L\rangle + l_{82}|1_L\rangle \end{pmatrix}$$

where the sates $|0_L\rangle$, $|1_L\rangle$, $|3\rangle$, ... , $|8\rangle$ are the same S^2 eigen-states mentioned in section 2. There are 24 such leakage operators in the error algebra. These leakage elements are of the form $((\widetilde{\sigma}_x \otimes \widetilde{\sigma}_y + \widetilde{\sigma}_y \otimes \widetilde{\sigma}_x) \otimes \widetilde{\sigma}_i)$, $((\widetilde{\sigma}_x \otimes \widetilde{\sigma}_x - \widetilde{\sigma}_y \otimes \widetilde{\sigma}_y) \otimes \widetilde{\sigma}_i)$, $((\widetilde{\sigma}_x + \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_x \otimes \widetilde{\sigma}_i)$, $((\widetilde{\sigma}_x + \sigma_z) \otimes \widetilde{\sigma}_y \otimes \widetilde{\sigma}_i)$, $(\widetilde{\sigma}_x \otimes (\widetilde{\sigma}_x + \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_i)$ and $(\widetilde{\sigma}_y \otimes (\widetilde{\sigma}_x + \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_i)$.

4.3 Orthogonal Operators

The orthogonal operators act only on the orthogonal subspace of the qubit state. These operators have the form shown by the matrix O below.

$$O = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & o_{33} & o_{34} & o_{35} & o_{36} & o_{37} & o_{38} \\ 0 & 0 & o_{43} & o_{44} & o_{45} & o_{46} & o_{47} & o_{48} \\ 0 & 0 & o_{53} & o_{54} & o_{55} & o_{56} & o_{57} & o_{58} \\ 0 & 0 & o_{63} & o_{64} & o_{65} & o_{66} & o_{67} & o_{68} \\ 0 & 0 & o_{73} & o_{74} & o_{75} & o_{76} & o_{77} & o_{78} \\ 0 & 0 & o_{83} & o_{84} & o_{85} & o_{86} & o_{87} & o_{88} \end{pmatrix}$$

The action of these orthogonal operators on the qubit state is shown below:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & o_{33} & o_{34} & o_{35} & o_{36} & o_{37} & o_{38} \\ 0 & 0 & o_{43} & o_{44} & o_{45} & o_{46} & o_{47} & o_{48} \\ 0 & 0 & o_{53} & o_{54} & o_{55} & o_{56} & o_{57} & o_{58} \\ 0 & 0 & o_{63} & o_{64} & o_{65} & o_{66} & o_{67} & o_{68} \\ 0 & 0 & o_{73} & o_{74} & o_{75} & o_{76} & o_{77} & o_{78} \\ 0 & 0 & o_{83} & o_{84} & o_{85} & o_{86} & o_{87} & o_{88} \end{pmatrix} \begin{pmatrix} |0_L\rangle \\ |1_L\rangle \\ |3\rangle \\ |4\rangle \\ |5\rangle \\ |6\rangle \\ |7\rangle \\ |8\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ o_{33}|3\rangle + ... + o_{38}|8\rangle \\ o_{43}|3\rangle + ... + o_{48}|8\rangle \\ o_{53}|3\rangle + ... + o_{58}|8\rangle \\ o_{63}|3\rangle + ... + o_{68}|8\rangle \\ o_{73}|3\rangle + ... + o_{78}|8\rangle \\ o_{83}|3\rangle + ... + o_{88}|8\rangle \end{pmatrix}$$

Note that these operators do not affect either the $|0_L\rangle$ or the $|1_L\rangle$. Therefore, if an error Hamiltonian can be decomposed into orthogonal operator(s), the orthogonal operator(s) will not affect the information stored in the logical subspace of the qubit state. The error algebra contains 36 of these orthogonal operators, all of which can be condensed into one of the following forms: $((1+\widetilde{\sigma}_z)\otimes\widetilde{\sigma}_i\otimes\widetilde{\sigma}_i)$, $((1+\widetilde{\sigma}_z)\otimes(1-\widetilde{\sigma}_z)\otimes\widetilde{\sigma}_i)$, $(\widetilde{\sigma}_x\otimes(1-\widetilde{\sigma}_z)\otimes\widetilde{\sigma}_i)$, $(\widetilde{\sigma}_y\otimes(1-\widetilde{\sigma}_z)\otimes\widetilde{\sigma}_i)$, $((\widetilde{\sigma}_x\otimes\widetilde{\sigma}_x+\widetilde{\sigma}_y\otimes\widetilde{\sigma}_y)\otimes\widetilde{\sigma}_i)$ and $((\widetilde{\sigma}_x\otimes\widetilde{\sigma}_y-\widetilde{\sigma}_y\otimes\widetilde{\sigma}_x)\otimes\widetilde{\sigma}_i)$.

5 Exchange Pulses

The states of the electron spins in the quantum dots are generally manipulated by either performing a single qubit operation, which is achieved by manipulating the local magnetic field in the dot, or by performing a two qubit operation with the help of an exchange interaction. This paper is focused on the systems that uses exchange interactions, which are two orders of magnitude faster to perform than the single qubit operations [6][7]. Since the Heisenberg exchange Hamiltonian commutes with S^z and S^2, it can only manipulate states to other states having same S^z and S^2. Hence, our encoded states were chosen to have same S^z and S^2, so that the Heisenberg exchange interaction alone would suffice as a universal quantum gate [6]. The Heisenberg exchange Hamiltonian, H_{ex}, for a 3 spin system is given by

$$H_{ex} = J_{12}\vec{S}_1 \cdot \vec{S}_2 + J_{23}\vec{S}_2 \cdot \vec{S}_3 + J_{31}\vec{S}_3 \cdot \vec{S}_1 \tag{1}$$

where S_1, S_2, and S_3 are the total spin operators acting on the first, second and the third electron correspondingly, and J_{12}, J_{23} and J_{31} are the exchange coupling parameters. We also set $J_{13} = 0$ so as to be compatible with the experimental implementation of the system discussed in [1]. Setting $J_{13} = 0$ and expanding spin terms in equation 1, we obtain

$$H_{ex} = J_{12}(\sigma_x \otimes \sigma_x \otimes \mathbb{1} + \sigma_y \otimes \sigma_y \otimes \mathbb{1} + \sigma_z \otimes \sigma_z \otimes \mathbb{1}) + J_{23}(\mathbb{1} \otimes \sigma_x \otimes \sigma_x + \mathbb{1} \otimes \sigma_y \otimes \sigma_y + \mathbb{1} \otimes \sigma_z \otimes \sigma_z) \tag{2}$$

$$H_{ex} = \begin{pmatrix} J_{12}+J_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & J_{12} & J_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & J_{23} & 0 & 0 & J_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{23} & 0 & J_{12} & 0 & 0 \\ 0 & 0 & J_{12} & 0 & J_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{12} & 0 & 0 & J_{23} & 0 \\ 0 & 0 & 0 & 0 & J_{23} & J_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{12}+J_{23} \end{pmatrix}$$

Transforming this exchange Hamiltonian into DFS basis through the transformation $U_{dfs}.H_{ex}.U_{dfs}^{\dagger}$, we obtain

$$H_{ex} = \begin{pmatrix} -\frac{1}{2}(J_{12}+J_{23}) & -\frac{\sqrt{3}}{2}(-J_{12}+J_{23}) & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2}(-J_{12}+J_{23}) & \frac{1}{2}(J_{12}+J_{23}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(J_{12}+J_{23}) & -\frac{\sqrt{3}}{2}(-J_{12}+J_{23}) & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2}(-J_{12}+J_{23}) & \frac{1}{2}(J_{12}+J_{23}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{12}+J_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{12}+J_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_{12}+J_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{12}+J_{23} \end{pmatrix}$$

Hence, the exchange Hamiltonian in the logical qubit space is given by

$$H_{ex} = \begin{pmatrix} -\frac{1}{2}(J_{12}+J_{23}) & -\frac{\sqrt{3}}{2}(-J_{12}+J_{23}) \\ -\frac{\sqrt{3}}{2}(-J_{12}+J_{23}) & \frac{1}{2}(J_{12}+J_{23}) \end{pmatrix}$$

$$H_{ex} = -\frac{\sqrt{3}(-J_{12}+J_{23})}{2}\sigma_x - \frac{(J_{12}+J_{23})}{2}\sigma_z \tag{3}$$

6 Decoupling Theory

Given a Hamiltonian under which the system will evolve, decoupling theory provides a means to alter the evolution of the state of the system such that, over an average time interval, the evolution operator becomes equivalent to an identity operator. Therefore, over an average time interval, the state of the system remains unchanged. In order to use the decoupling theory, we first need to solve for the unitary associated with the Hamiltonian acting on the system. The Hamiltonian acting on a system can be always be written as

$$H = H_S \otimes I_B + I_S \otimes H_B + H_{SB} \tag{4}$$

The first term of the Hamiltonian acts only on the system, and the second term only on the bath. However, the third term, which acts both on the system and the bath, couples the system to the bath. The third term, H_{SB}, can be expanded as follows:

$$H_{SB} = \Sigma_{\alpha=X,Y,Z} \;\; S_\alpha \otimes B_\alpha \tag{5}$$

Given this Hamiltonian and the initial state of the system, the state of the system can be solved by using the Schrodinger's equation expressed below:

$$\frac{\partial \Psi(t)}{\partial t} = -iH(t)\Psi(t) \tag{6}$$

The above problem is equivalent to

$$\frac{dU(t)}{dt} = -iH(t)U(t) \quad \& \quad U(0) = \mathbb{1} \tag{7}$$

The solution for U(t) can be obtained using the Magnus expansion [8]. According to Magnus expansion, U(t) is given by the following expression:

$$U(t) = \exp(\sum_{i=0}^{\infty} \Omega(t)).U(0) \tag{8}$$

where

$$\Omega_1(t) = \int_0^t -iH(t_1)dt_1$$

$$\Omega_2(t) = \frac{1}{2}\int_0^t dt_1 \int_0^{t_1} dt_2[-iH(t_1), -iH(t_2)]$$

$$\Omega_3(t) = \frac{1}{6}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3([-iH(t_1), [-iH(t_2), -iH(t_3)]] + [-iH(t_3), [-iH(t_2), -iH(t_1)]])$$

and so on. Therefore, through this method, one could obtain unitary, U(t), associated with the state of the system upto any order of interest. Having obtained U(t), the goal is to implement N pulses periodically with the system undergoing free evolution for time period Δt between those pulses such that the $U_{eff}(T)$ becomes equivalent to an identity operator. Also, if the external pulses are strong enough then the Hamiltonian, H_{SB}, can be neglected when the external pulses are applied. The effective unitary, $U_{eff}(T)$, for T = nΔt can then be expressed mathematically as

$$U_{eff}(T) = \prod_{n=0}^{N-1} U_i[U(\Delta t)]U_i^{-1} \tag{9}$$

where U_i is the unitary associated with the external pulse that will be acting on the system. Since the effective unitary, $U_{eff}(T)$, will become equivalent to an identity operator after T, the idea is to repeat such pulse sequence after every time interval T. However, for our project, U(t) that contains only terms upto first order in time will be used. This simplifies the problem to a great extent. Using this approximation, H_{eff} is solved upto first order in time in the following subsection.

6.1 First Order Theory

In order to use the first order theory, one has to make the assumption that the time interval over which the free evolution occurs is small enough that one can neglect second or higher order terms in Δt. In other words, this assumption allows us to express U(Δt) as U(Δt) $\approx \mathbb{1}$ - $iH_{SB}\Delta t$. Substituting this expression into equation 9 and expanding it, we obtain the following expression:

$$U_{eff}(T) \approx [U_0^{-1}(\mathbb{1} - iH_{SB}(\Delta t))U_0][U_1^{-1}(\mathbb{1} - iH_{SB}(\Delta t))U_1] \;\; ... \;\; [U_{N-1}^{-1}(\mathbb{1} - iH_{SB}(\Delta t))U_{N-1}] \tag{10}$$

Neglecting higher order terms in Δt, we obtain

$$U_{eff}(T) \approx \mathbb{1} - i\Delta t U_0^{-1} H_{SB} U_0 - i\Delta t U_1^{-1} H_{SB} U_1 \;\; ... \;\; - i\Delta t U_{N-1}^{-1} H_{SB} U_{N-1} \tag{11}$$

Defining $U_{eff}(T) = exp(-iH_{eff} n\Delta t)$, which we can then be approximated as $U_{eff}(T) \approx \mathbb{1} - iH_{eff} n\Delta t$. Substituting this expression in equation 7, we obtain

$$H_{eff} \approx \frac{1}{N} \sum_{n=0}^{N-1} U_n^{-1} H_{SB} U_n \tag{12}$$

Having obtained an expression for H_{eff}, the goal is to modify H_{eff} by manipulating the unitaries associated with the external pulses so that the H_{eff} will be zero. If the H_{eff} is made zero, the U_{eff} will be equivalent to an identity operator. Therefore, the state of the system in the presence of noise Hamiltonian will not be altered after a time period T. In general, however, H_{eff} cannot be made zero. In that case, the goal is to eliminate as many coefficients, which are associated with the decomposition of H_{SB} in terms of the elements of the error algebra, as possible. In the following section, a pulse sequence has been suggested that will eliminate most of the noise associated with a general noise Hamiltonian. For more information on decoupling pulses refer [8]. Also, a different method of decoupling the system from the noise has been provided in [7].

6.2 Elimination of errors

The unitary associated with an arbitrary exchange pulse, as mentioned in equation 2, is given by

$$U = \exp\left[i\frac{t}{\hbar}(J_{12}\vec{S}_1\cdot\vec{S}_2 + J_{23}\vec{S}_2\cdot\vec{S}_3)\right]$$

With this unitary various values for $\frac{t}{\hbar}J_{12}$ and $\frac{t}{\hbar}J_{23}$ have been tried to eliminate as many error coefficients as possible in the decomposition of H_{SB}. It was found that the following sequence could eliminate 2 errors coefficients associated with the logical operators and 20 error coefficients associated with the leakage operators:

$$U_{eff}(T=4t) = [P_3 exp(\frac{-it}{\hbar}H_{SB})P_3^\dagger][P_2 exp(\frac{-it}{\hbar}H_{SB})P_2^\dagger][P_1 exp(\frac{-it}{\hbar}H_{SB})P_1^\dagger]exp(\frac{-it}{\hbar}H_{SB}) \tag{13}$$

where $P_1 = exp\left[i\pi(\vec{S}_1\cdot\vec{S}_2+\vec{S}_2\cdot\vec{S}_3)\right]$, $P_2 = exp\left[i\frac{\pi}{2}(\vec{S}_1\cdot\vec{S}_2+\vec{S}_2\cdot\vec{S}_3)\right]$ and $P_3 = exp\left[i\frac{3\pi}{2}(\vec{S}_1\cdot\vec{S}_2+\vec{S}_2\cdot\vec{S}_3)\right]$ In particular, the effective Hamiltonian was obtained using first order theory (refer to equation 8), which is expressed below.

$$\widetilde{H}_{eff} = \frac{1}{4}[H_{SB} + P_1^\dagger H_{SB}P_1 + P_2^\dagger H_{SB}P_2 + P_3^\dagger H_{SB}P_3] \tag{14}$$

Then, the $\widetilde{H}_{eff}$ was decomposed in terms of error operators ($\widetilde{H}_{eff} = \Sigma_{i=1}^{64} c_i * \widetilde{a}_i$). The decomposition was done utilizing the fact that the set $\{\widetilde{a}_i\}$ has been constructed to be orthonormal, so the coefficient, c_i, can be obtained by multiplying the decomposition form of the Hamiltonian with some basis element $\widetilde{a}_j$ on both sides as shown below.

$$\widetilde{H}_{eff} = \Sigma_{i=1}^{64} c_i.\widetilde{a}_i$$

$$\widetilde{H}_{eff}.\widetilde{a}_j = (\Sigma_{i=1}^{64} c_i.\widetilde{a}_i).\widetilde{a}_j$$

$$\widetilde{H}_{eff}.\widetilde{a}_j = c_j.\mathbb{1}$$

$$\frac{1}{8} * Trace(c_j.\mathbb{1}) = c_j = \frac{1}{8} * Trace(\widetilde{H}_{eff}.\widetilde{a}_j)$$

$$c_j = \frac{1}{8} * Trace(\widetilde{H}_{eff}.\widetilde{a}_j) \tag{15}$$

Using this expression, all the coefficients, c_i, of the effective Hamiltonian in the basis $\{a_i\}$ has been obtained with the help of MATHEMATICA. It has been found that $c_i = 0$ for i = {2,3,5, ..., 20, 22, 23, 26, 27}. Also, for i from 29 through 64, the basis elements are orthogonal operators. Hence, one need not be concerned about the presence of the nonzero coefficients associated with these operators. Therefore, in total, there are only 5 nonzero coefficients that affect the system.

7 Application of our pulse to a specific system

In this section, the above mentioned pulse sequence will be applied to a specific Hamiltonian, which has been obtained experimentally. More specifically, our pulse sequence will be applied to the hyperfine Hamiltonian in a uniformly coupled triple dot given in [1].

This hyperfine Hamiltonian is shown below.

$$\widetilde{H}_{HF} = \begin{pmatrix}
\frac{B^z_{2\bar{1}2}}{6} & \frac{-(B^x_{2\bar{1}2} - iB^y_{\bar{2}1\bar{2}})}{6} & \frac{-B^z_{10\bar{1}}}{2\sqrt{3}} & \frac{(B^x_{10\bar{1}} - iB^y_{10\bar{1}})}{2\sqrt{3}} & \frac{(B^x_{1\bar{2}1} + iB^y_{\bar{1}2\bar{1}})}{2\sqrt{6}} & \frac{-B^z_{1\bar{2}1}}{3\sqrt{2}} & \frac{-(B^x_{1\bar{2}1} - iB^y_{1\bar{2}1})}{6\sqrt{2}} & 0 \\
\frac{-B^z_{10\bar{1}}}{2\sqrt{3}} & \frac{(B^x_{10\bar{1}} - iB^y_{10\bar{1}})}{2\sqrt{3}} & \frac{B^z_{010}}{2} & \frac{-(B^x_{010} - iB^y_{010})}{2} & \frac{(B^x_{10\bar{1}} + iB^y_{10\bar{1}})}{2\sqrt{2}} & \frac{-B^z_{10\bar{1}}}{\sqrt{6}} & \frac{-(B^x_{10\bar{1}} - iB^y_{10\bar{1}})}{2\sqrt{6}} & 0 \\
\frac{-(B^x_{2\bar{1}2} + iB^y_{2\bar{1}2})}{6} & \frac{-B^z_{\bar{2}1\bar{2}}}{6} & \frac{(B^x_{10\bar{1}} + iB^y_{10\bar{1}})}{2\sqrt{3}} & \frac{B^z_{10\bar{1}}}{2\sqrt{3}} & 0 & \frac{-(B^x_{1\bar{2}1} + iB^y_{1\bar{2}1})}{6\sqrt{2}} & \frac{B^z_{1\bar{2}1}}{3\sqrt{2}} & \frac{(B^x_{1\bar{2}1} - iB^y_{1\bar{2}1})}{2\sqrt{6}} \\
\frac{(B^x_{10\bar{1}} + iB^y_{10\bar{1}})}{2\sqrt{3}} & \frac{B^z_{10\bar{1}}}{2\sqrt{3}} & \frac{-(B^x_{0\bar{1}0} + iB^y_{0\bar{1}0})}{2} & \frac{-B^z_{010}}{2} & 0 & \frac{-(B^x_{10\bar{1}} - iB^y_{10\bar{1}})}{2\sqrt{6}} & \frac{B^z_{\bar{1}01}}{\sqrt{6}} & \frac{(B^x_{10\bar{1}} + iB^y_{10\bar{1}})}{2\sqrt{2}} \\
\frac{(B^x_{1\bar{2}1} - iB^y_{1\bar{2}1})}{2\sqrt{6}} & 0 & \frac{(B^x_{10\bar{1}} - iB^y_{10\bar{1}})}{2\sqrt{2}} & 0 & \frac{B^z_{111}}{2} & \frac{(B^x_{111} - iB^y_{111})}{2\sqrt{3}} & 0 & 0 \\
\frac{-B^z_{1\bar{2}1}}{3\sqrt{2}} & \frac{-1(B^x_{1\bar{2}1} - iB^y_{1\bar{2}1})}{6\sqrt{2}} & \frac{-B^z_{10\bar{1}}}{\sqrt{6}} & \frac{-(B^x_{10\bar{1}} - iB^y_{10\bar{1}})}{2\sqrt{6}} & \frac{(B^x_{111} - iB^y_{111})}{2\sqrt{3}} & \frac{B^z_{111}}{6} & \frac{(B^x_{111} - iB^y_{111})}{3} & 0 \\
\frac{-(B^x_{1\bar{2}1} + iB^y_{1\bar{2}1})}{6\sqrt{2}} & \frac{B^z_{1\bar{2}1}}{3\sqrt{2}} & \frac{-(B^x_{10\bar{1}} + iB^y_{10\bar{1}})}{2\sqrt{6}} & \frac{B^z_{10\bar{1}}}{\sqrt{6}} & 0 & \frac{(B^x_{111} + iB^y_{111})}{3} & \frac{-B^z_{111}}{6} & \frac{(B^x_{111} - iB^y_{111})}{2\sqrt{3}} \\
0 & \frac{(B^x_{1\bar{2}1} + iB^y_{1\bar{2}1})}{2\sqrt{6}} & 0 & \frac{(B^x_{10\bar{1}} + iB^y_{10\bar{1}})}{2\sqrt{2}} & 0 & 0 & \frac{(B^x_{111} + iB^y_{111})}{2\sqrt{3}} & \frac{-B^z_{111}}{2}
\end{pmatrix}$$

where $B^p_{l,m,n} = lB^p_{N1} + mB^p_{N2} + nB^p_{N3}$, in which l, m, and n are real numbers and p is either x, y or z.

Note that the 2^{nd} row and 3^{rd} row of the Hamiltonian, H_{HF}, are interchanged in comparison to the Hamiltonian mentioned in [1]. These rows have been interchanged because the DFS transformation mentioned in this paper is not equivalent to the DFS transformation in [1] rather the transformation mentioned in this paper interchanges the 2^{nd} and 3^{rd} eigen-states of the qubit mentioned in [1]. The application of the above mentioned pulse sequence resulted in the Hamiltonian (in the DFS basis), which has been decomposed in terms of basis elements, $\{a_i\}$, as shown below.

$$\begin{aligned}
\widetilde{H}_{HF,eff} = \frac{(B^x_{-4+\sqrt{6},\bar{4},-(4+\sqrt{6})} + B^z_{-2\sqrt{6},0,2\sqrt{6}})}{72}\widetilde{a}_1 + \frac{B^x_{-4+\sqrt{6},\bar{4},-(4+\sqrt{6})} - B^z_{3+4\sqrt{6},6,3-4\sqrt{6}}}{72}\widetilde{a}_4 \\
+\frac{B^x_{(1+\sqrt{3}-\sqrt{6}),(-2+3\sqrt{2}-\sqrt{6}),(1-\sqrt{3}-\sqrt{6})} + iB^y_{\sqrt{6},\sqrt{6},\sqrt{6}} + B^z_{(4-4\sqrt{3}-\sqrt{6}),-8,(4+4\sqrt{3}+\sqrt{6})}}{72}\widetilde{a}_{21} \\
+\frac{B^x_{(1+\sqrt{3}-\sqrt{6}),(-2+3\sqrt{2}-\sqrt{6}),(1-3\sqrt{3}-\sqrt{6})} + iB^y_{\sqrt{6},\sqrt{6},\sqrt{6}} + 2B^z_{(-1-2\sqrt{3}+\sqrt{6}),2,(-1+2\sqrt{3}-\sqrt{6})}}{72}\widetilde{a}_{24} \\
+\frac{B^y_{(-3+\sqrt{3}),-\sqrt{3}(2+3\sqrt{2}),(3+\sqrt{3})}}{72\sqrt{3}}\widetilde{a}_{25} + \frac{B^y_{(-3+\sqrt{3}),-\sqrt{3}(2+3\sqrt{2}),(3+\sqrt{3})}}{72\sqrt{3}}\widetilde{a}_{28} + \sum_{i=29}^{64} c_i^{\perp}\widetilde{a}_i
\end{aligned} \tag{16}$$

8 Conclusion

To summarize this paper, first, a specific DFS encoding was utilized, which is compatible with [1], to encode the information. This DFS encoding also allows the qubit states to be immune from the noise resulting from the uniform external magnetic field [1]. Next, the errors that could arise in such a DFS encoding were classified as either logical errors, leakage errors or orthogonal errors. Then, a decoupling pulse sequence, which consisted of a π pulse, a $\frac{\pi}{2}$ pulse and a $\frac{3\pi}{2}$ pulse, was utilized to eliminate errors. This pulse sequence is found to eliminate 2 logical errors and 20 leakage errors. However, the system is still vulnerable to 1 logical error and 4 leakage errors. Finally, the application of the same pulse sequence on a specific noise Hamiltonian mentioned in [1] has been analyzed so as to provide useful information to the experimental physicists working on this system to eliminate noise.

In future, we hope to perform similar analysis on the data associated with noise Hamiltonian obtained from the hybrid qubit mentioned in [9].

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9 Appendix

Let A be any generic 8 by 8 matrix, whose elements $A_{1,1}$ through $A_{4,4}$ are 2 by 2 block matrices.

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix}$$

Then, the elements of the error algebra, which also span 8 by 8 matrices are expressed below.

1)Logical operators:

Basis elements $\widetilde{a}_1$ through $\widetilde{a}_4$ span block $A_{1,1}$ of a 8 by 8 matrix.

$$\widetilde{a}_1 = \frac{1}{2}((1+\widetilde{\sigma}_z) \otimes (1+\sigma_z) \otimes 1)$$
$$\widetilde{a}_2 = \frac{1}{2}((1+\widetilde{\sigma}_z) \otimes (1+\widetilde{\sigma}_z) \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_3 = \frac{1}{2}((1+\widetilde{\sigma}_z) \otimes (1+\widetilde{\sigma}_z) \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_4 = \frac{1}{2}((1+\widetilde{\sigma}_z) \otimes (1+\widetilde{\sigma}_z) \otimes \widetilde{\sigma}_z)$$

2) Leakage operators:

Basis elements $\widetilde{a}_5$ through $\widetilde{a}_{12}$ span $A_{1,4}$ and $A_{4,1}$ of a 8 by 8 matrix.

$$\widetilde{a}_5 = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x \otimes \widetilde{\sigma}_y) + (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_x) \otimes 1)$$
$$\widetilde{a}_6 = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x + \widetilde{\sigma}_y) + (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_x) \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_7 = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x + \widetilde{\sigma}_y) + (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_x) \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_8 = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x + \widetilde{\sigma}_y) + (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_x) \otimes \widetilde{\sigma}_z)$$

$$\widetilde{a}_9 = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x + \widetilde{\sigma}_x) - (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_y) \otimes 1)$$
$$\widetilde{a}_{10} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x + \widetilde{\sigma}_x) - (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_y) \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{11} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x + \widetilde{\sigma}_x) - (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_y) \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{12} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x + \widetilde{\sigma}_x) - (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_y) \otimes \widetilde{\sigma}_z)$$

Basis elements $\widetilde{a}_{13}$ through $\widetilde{a}_{20}$ span blocks $A_{1,3}$ and $A_{3,1}$ of a 8 by 8 matrix.

$$\widetilde{a}_{13} = \frac{1}{\sqrt{2}}(\widetilde{\sigma}_x \otimes (1 + \widetilde{\sigma}_y)) \otimes 1)$$
$$\widetilde{a}_{14} = \frac{1}{\sqrt{2}}(\widetilde{\sigma}_x \otimes (1 + \widetilde{\sigma}_y)) \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{15} = \frac{1}{\sqrt{2}}(\widetilde{\sigma}_x \otimes (1 + \widetilde{\sigma}_y)) \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{16} = \frac{1}{\sqrt{2}}(\widetilde{\sigma}_x \otimes (1 + \widetilde{\sigma}_y)) \otimes \widetilde{\sigma}_z)$$

$$\widetilde{a}_{17} = \frac{1}{\sqrt{2}}(\widetilde{\sigma}_y \otimes (1+\widetilde{\sigma}_z)) \otimes 1)$$
$$\widetilde{a}_{18} = \frac{1}{\sqrt{2}}(\widetilde{\sigma}_y \otimes (1+\widetilde{\sigma}_z)) \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{19} = \frac{1}{\sqrt{2}}(\widetilde{\sigma}_y \otimes (1+\widetilde{\sigma}_z)) \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{20} = \frac{1}{\sqrt{2}}(\widetilde{\sigma}_y \otimes (1+\widetilde{\sigma}_z)) \otimes \widetilde{\sigma}_z)$$

Basis elements $\widetilde{a}_{21}$ through $\widetilde{a}_{28}$ span $A_{1,2}$ and $A_{2,1}$ of a 8 by 8 matrix.

$$\widetilde{a}_{21} = \frac{1}{\sqrt{2}}((1+\widetilde{\sigma}_z) \otimes \widetilde{\sigma}_x \otimes 1)$$
$$\widetilde{a}_{22} = \frac{1}{\sqrt{2}}((1+\widetilde{\sigma}_z) \otimes \widetilde{\sigma}_x \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{23} = \frac{1}{\sqrt{2}}((1+\widetilde{\sigma}_z) \otimes \widetilde{\sigma}_x \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{24} = \frac{1}{\sqrt{2}}((1+\widetilde{\sigma}_z) \otimes \widetilde{\sigma}_x \otimes \widetilde{\sigma}_z)$$

$$\widetilde{a}_{25} = \frac{1}{\sqrt{2}}((1+\widetilde{\sigma}_z) \otimes \widetilde{\sigma}_y \otimes 1)$$
$$\widetilde{a}_{26} = \frac{1}{\sqrt{2}}((1+\widetilde{\sigma}_z) \otimes \widetilde{\sigma}_y \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{27} = \frac{1}{\sqrt{2}}((1+\widetilde{\sigma}_z) \otimes \widetilde{\sigma}_y \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{28} = \frac{1}{\sqrt{2}}((1+\widetilde{\sigma}_z) \otimes \widetilde{\sigma}_y \otimes \widetilde{\sigma}_z)$$

3) Orthogonal operators:

Basis elements $\widetilde{a}_{29}$ through $\widetilde{a}_{44}$ span blocks $A_{3,3}$, $A_{3,4}$, $A_{4,3}$, and $A_{4,4}$ of a 8 by 8 matrix.

$$\widetilde{a}_{29} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes 1 \otimes 1)$$
$$\widetilde{a}_{30} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes 1 \otimes \widetilde{\sigma}_x)$$
$$\widetilde{e}_{31} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes 1 \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{32} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes 1 \otimes \widetilde{\sigma}_z)$$

$$\widetilde{a}_{33} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_x \otimes 1)$$
$$\widetilde{a}_{34} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_x \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{35} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_x \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{36} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_x \otimes \widetilde{\sigma}_z)$$

$$\widetilde{a}_{37} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_y \otimes 1)$$
$$\widetilde{a}_{38} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_y \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{39} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_y \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{40} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_y \otimes \widetilde{\sigma}_z)$$

$$\widetilde{a}_{41} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_z \otimes 1)$$
$$\widetilde{a}_{42} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_z \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{43} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_z \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{44} = \frac{1}{\sqrt{2}}((1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_z \otimes \widetilde{\sigma}_z)$$

Basis elements $\widetilde{a}_{45}$ through $\widetilde{a}_{48}$ span block $A_{2,2}$ of a 8 by 8 matrix.

$$\widetilde{a}_{45} = \frac{1}{2}((1 + \widetilde{\sigma}_z) \otimes (1 - \widetilde{\sigma}_z) \otimes 1)$$
$$\widetilde{a}_{46} = \frac{1}{2}((1 + \widetilde{\sigma}_z) \otimes (1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{47} = \frac{1}{2}((1 + \widetilde{\sigma}_z) \otimes (1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{48} = \frac{1}{2}((1 + \widetilde{\sigma}_z) \otimes (1 - \widetilde{\sigma}_z) \otimes \widetilde{\sigma}_z)$$

Basis elements $\widetilde{a}_{49}$ through $\widetilde{a}_{56}$ span blocks $A_{3,2}$ and $A_{2,3}$ of a 8 by 8 matrix.

$$\widetilde{a}_{49} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x \otimes \widetilde{\sigma}_x) \otimes (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_y) \otimes 1)$$
$$\widetilde{a}_{50} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x \otimes \widetilde{\sigma}_x) \otimes (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_y) \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{51} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x \otimes \widetilde{\sigma}_x) \otimes (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_y) \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{52} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x \otimes \widetilde{\sigma}_x) \otimes (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_y) \otimes \widetilde{\sigma}_z)$$

$$\widetilde{a}_{53} = \frac{1}{\sqrt{2}}(((\widetilde{\sigma}_x \otimes \widetilde{\sigma}_y) - (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_x)) \otimes 1)$$
$$\widetilde{a}_{54} = \frac{1}{\sqrt{2}}(((\widetilde{\sigma}_x \otimes \widetilde{\sigma}_y) - (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_x)) \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{55} = \frac{1}{\sqrt{2}}(((\widetilde{\sigma}_x \otimes \widetilde{\sigma}_y) - (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_x)) \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{56} = \frac{1}{\sqrt{2}}(((\widetilde{\sigma}_x \otimes \widetilde{\sigma}_y) - (\widetilde{\sigma}_y \otimes \widetilde{\sigma}_x)) \otimes \widetilde{\sigma}_z)$$

Basis elements $\widetilde{a}_{57}$ through $\widetilde{a}_{64}$ span $A_{2,4}$ and $A_{4,2}$ blocks of a 8 by 8 matrix.

$$\widetilde{a}_{57} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x \otimes (1 - \widetilde{\sigma}_z)) \otimes 1)$$
$$\widetilde{a}_{58} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x \otimes (1 - \widetilde{\sigma}_z)) \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{59} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x \otimes (1 - \widetilde{\sigma}_z)) \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{60} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_x \otimes (1 - \widetilde{\sigma}_z)) \otimes \widetilde{\sigma}_z)$$

$$\widetilde{a}_{61} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_y \otimes (1 - \widetilde{\sigma}_z)) \otimes 1)$$
$$\widetilde{a}_{62} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_y \otimes (1 - \widetilde{\sigma}_z)) \otimes \widetilde{\sigma}_x)$$
$$\widetilde{a}_{63} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_y \otimes (1 - \widetilde{\sigma}_z)) \otimes \widetilde{\sigma}_y)$$
$$\widetilde{a}_{64} = \frac{1}{\sqrt{2}}((\widetilde{\sigma}_y \otimes (1 - \widetilde{\sigma}_z)) \otimes \widetilde{\sigma}_z)$$