Fall 11-6-2012

Primitive Matrices with Combinatorial Properties

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PRIMITIVE MATRICES WITH COMBINATORIAL PROPERTIES

by

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B.S., Kuwait University, 2008

A Research Paper
Submitted in Algebra of the Requirements for the
Master of Science Degree

Department of Mathematics
in the Graduate School
Southern Illinois University Carbondale
December, 2012

RESEARCH PAPER APPROVAL

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A Research Paper Submitted in Partial

Fulfillment of the Requirements

for the Degree of

Master of Science

in the field of Mathematics

Approved by:

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Graduate School
Southern Illinois University Carbondale
November 1, 2012

AN ABSTRACT OF THE RESEARCH PAPER OF

ABDULKAREM ALHURAIJI, for the master degree in ALGEBRA, presented on NOVEMBER 1, 2012, at Southern Illinois University Carbondale.

TITLE: Primitive Matrices with Combinatorial Properties

MAJOR PROFESSOR: Dr. R. Fitzgerald

I present a proof of Wielandt's theorem by using combinatorial properties of directed graph.

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INTRODUCTION

This paper presents some theorems in linear algebra and graph theory and this paper is suitable to read for the students, who had courses in graph theory and linear algebra, which I highly recommended.

A matrix called primitive if there exists an integer power which makes all the entries positive. It is difficult to show a matrix is not primitive using this definition one most think infintely many powers of the matrix. Wielandt's theorem solves this: just test $A^{(n-1)^2+1}$ is positive. If so A is primitive, otherwise not primitive.

Primitive matrices are useful because of Frobenius' Theorem which has applications to business and biology.

In my research paper, we concern about the proof of Wielandts theorem and then some theorems that extend Wielandts theorem.

The proofs in this paper provide some lemmas, definitions, and notations in graph theory. Furthermore, examples are helpful to understand the definitions or the notations, so I consist to put these examples to help the reader to get the right point.

Chapter 1 introduces some basic linear algebra and definitions with some examples in graph theory.

Chapter 2 concerns about Wielandts theorem and some lemmas that help us to prove theorem.

Chapter 3 deals with some definitions and notations in graph theory to help us to prove some theorems in linear algebra. We present an extension of Wielandt's theorem due to Dulmage and Mendelsohn.

Chapter 4 has the story of Helmut Wielandt's proof of his theorem, and how Hans Schneider got to know about this proof.

CHAPTER 1

NON-NEGATIVE MATRICES

1.1 STRUCTURE OF A GENERAL NON-NEGATIVE MATRIX

In this section we shall introduce some definitions about irreducible matrices and some notations for directed graphs to understand what we will do on the next chapter. This material is from [1] and [3].

Definition. A *permutation matrix* is a square matrix and its entries are 0 and 1 which has for each row and for each column 1 and elsewhere are zeros.

Definition. An $n \times n$ matrix T is *reducible* if there exists a permutation matrix P such that $P^{-1}TP = \left(\begin{smallmatrix} B & 0 \\ D & C \end{smallmatrix}\right)$ where B and C are square matrices and 0 is a zero matrix. Otherwise, T is *irreducible*.

Definition. A matrix T is *positive* if every entry is positive, we write $T > 0$ is $non-negative$ and if every entry is zero or positive, we write $T \geq 0$.

Definition. A square non-negative matrix T is said to be *primitive* if there exists a positive integer k such that $T^k > 0$.

Theorem 1.1.1. *Every primitive matrix is irreducible matrix.*

Proof. Let us prove Not irreducible matrix implies Not primitive matrix.
Let T is reducible matrix that means $P^{-1}TP = \left(\begin{smallmatrix} B & 0 \\ D & C \end{smallmatrix}\right)$. So, $T^n = P\left(\begin{smallmatrix} B^n & 0 \\ D' & C^n \end{smallmatrix}\right)P^{-1}$, for all $n \geq 1$. So, T^n is not positive for all $n \geq 1$. Hence, T is not primitive.
Thus, primitive matrix implies irreducible matrix.

□

Definition. A sequence $(i, i_1, i_2, .., i_{t-1}, j)$, for $t \geq 1$, (where $i_0 = i$), from the index set $\{1, 2, ..., n\}$ of a non-negative matrix T is said to form a *chain* of length t between the ordered pair (i, j) if

$$t_{ii_1}, t_{i_1 i_2}, ..., t_{i_{t-1} j} > 0.$$

In this definition clearly, we may without loss of generality impose the restriction that, for fixed (i, j), the indices $i, i_1, i_2, ..., i_{t-1}, j$ are distinct to obtain a minimal length chain or cycle, from given one.

I want to explain a little bit about the indices and how we can deal with them. Let i, j, k be arbitrary indices from the index set $1, 2, ..., n$ of the matrix T. We say that i leads to j , and we write $i \to j$, if there exists an integer $m \geq 1$ such that $t_{ij}^{(m)} > 0$. Here $t^{(m)}$ denotes the i, j-entry of T^m. If i does not lead to j, we write $i \nrightarrow j$. We can clearly say if $i \to j$ and $j \to k$, then from the rule of multiplication $i \to k$, let say $t_{ij}^{(m)} > 0$ and $t_{jk}^{(l)} > 0$ then $(T^{m+l})_{ik} = \sum (T^m)_{is}(T^l)_{sk} \geq (T^m)_{ij}(T^l)_{jk} > 0$. Moreover, we say that i and j communicate if $i \to j$ and $j \to i$, so in this case we write $i \leftrightarrow j$.

Now, we will use an alternative formulation for the study of the power structure of non-negative matrix is in graph theoretic terms. We shall study the mapping, F, induced by a matrix T, of its index set $\{1, 2, ..., n\}$ into itself. We shall study combinatorial properties of non-negative matrices.

Now denote by S the set of indices $\{1, 2, ..., n\}$ of an irreducible matrix T, and let $L \subset S$. Furthermore, for integer $h \geq 0$, let $F^h(L)$ be the set of indices $j \in S$ such that

$$t_{ij}^{(h)} > 0, i \in L, \ \ for \ \ all \ \ i \in L$$

Thus $F^h(i)$ is the set of $j \in S$ such that $t_{ij}^{(h)} > 0$. Also $F^0(L) = L$ by convention. Further, we notice the following consequences of these definitions:

(i) If $A \subset B \subset S$, then $F(A) \subset F(B)$.

(ii) If $A \subset S, B \subset S$, then $F(A \cup B) = F(A) \cup F(B)$.

(iii) For integer $h \geq 0$, and $L \subset S$,

$$F^{h+1}(L) = F(F^h(L)) = F^h(F(L)).$$

(iv) The mapping F^h may be interpreted as the F-mapping associated with the non-negative matrix T^h.

Example 1.1.1. $n = 5$, $S = \{1, 2, 3, 4, 5\}$

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad T^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

In this example I want to make it clear for the notation $F(i)$, let try to find $F(1), F(5), F^2(1), F^3(3), F^4(5)$.

Let start with $F(1) = \{2\}$ because the first row in the matrix T, the second column is positive. $F(5) = \{1, 2\}$ as you see in the matrix T in the fifth row, the first and the second column are positive. $F^2(1) = \{3\}$ because in the matrix T^2 in the first row, the third column is positive. $F^3(3) = \{1, 2\}$ if you see the matrix T^3 in the third row it has the first and the second column positive. $F^4(5) = \{4, 5\}$ in the matrix T^4 and in the fifth row there are two columns positive which they are the fourth and the fifth.

A directed graph consists of a vertex set $V = (1, 2, 3, ..., n)$ and a set of edges each of which is an ordered pair (i, j) of vertices. An edge (i, j) may also be called a path of length 1 from vertex i to vertex j. If vertices $k_1, k_2, ..., k_{t-1}$ exist such that $(i, k_1), (k_1, k_2), ..., (k_{t-1}, j)$ are edges of the graph, then i is said to be connected to j by a path of length t. A directed graph is said to be strongly connected, if for any two vertices i, j of the vertex set with $i \neq j$, there is a path of some length connecting i to j.

Definition. A *cycle* is a path which begins and ends with the same vertex.

Example 1.1.2. A cycle is $i \to k \to l \to .... \to j \to i$ as you see, it begins and ends with vertex i.

In such a path an edge may appear more than once.

Example 1.1.3. i is a cycle with repeat edges $i \to k \to l \to i \to k \to l \to .... \to j \to i$ as you see here the path from i to k to l repeated two times.

Definition. A *circuit* is a cycle of which no proper subgraph is a cycle.

Example 1.1.4. A *circuit* like $i \to k \to l \to m \to j \to i$ as you see, it begins and ends with the same vertex i and with no proper subgraph.

If the pair (i, i) is an edge, then this is circuit of length 1 is called a loop and i is called a loop vertex. The greatest common divisor of the lengths of all cycles is equal to the greatest common divisor of the lengths of all circuits i.e. $g.c.d(\ the\ lengths\ of\ all\ cycles\ ) = g.c.d(\ the\ lengths\ of\ all\ circuits)$. A strongly connected directed graph D in which the greatest common divisor of all cycles and the greatest common divisor of all circuits is 1 will be called *primitive*. The exponent $\gamma(D)$ of a primitive graph D is the least integer with the property that for every $t \geq \gamma(D)$ and every ordered pair of vertices, there is a directed path from the first vertex to the second of length t.

The directed graph D_A of an $n \times n$ matrix $A = (a_{ij})$ has vertex set $V = (1, 2, 3, ..., n)$ and the ordered pair (i, j) is an edge of D_A if and only if $a_{ij} \neq 0$. It is well known for a non-negative matrix A, that A is primitive if and only if the graph D_A is primitive as well. Furthermore, the exponent $\gamma(A)$ is equal to the exponent $\gamma(D_A)$.

CHAPTER 2

IRREDUCIBLE MATRICES

2.1 IRREDUCIBLE MATRICES

We shall assume that $n \geq 2$, to avoid trivialities.

Lemma 2.1.1. *(a) $F(S) = S$.*

(b) If L is a proper subset of S and $L \neq \phi$ then $F(L)$ contains some index not in L.

Proof. (a) We have to show that $F(S) \subset S$ and $S \subset F(S)$

It is obvious that $F(S) \subset S$ because S consists of all indices, so we need to prove $S \subset F(S)$

Pick $j \in S$, and we want to show $j \in F(S)$,

which means $t_{ij} \neq 0$ for some $i \in S$ by the definition of F.

Instead, suppose that $t_{ij} = 0$ for all $i \in S$, (Note: we say $t_{ij} = 0$, not $t_{ij} \leq 0$. Because we deal with non-negative matrices.) This implies there is a zero j^{th} column $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Now, we will pick a permutation matrix P to make the last column of $P^{-1}TP$ is zero. $P^{-1}TP = \left(\begin{smallmatrix} B & 0 \\ D & 0 \end{smallmatrix}\right)$, where B is a square matrix and $C = 0$ is the 1×1 zero matrix. So, T is an reducible matrix, which is contradiction.

Thus, $F(S) = S$.

(b) We want to prove it by contradiction. Suppose that $F(L) \subset L$

Claim 1: $F^k(L) \subset L$ for all $k \geq 1$.

We prove this by contradiction, take case $k = 1$ bring our supposition. Property (i) page 3 in chapter 1 : $A \subset B$ that implies

$$F(A) \subset F(B)$$

Assume $F^n(L) \subset L$ we have

$$F^{n+1}(L) = F[F^n(L)] \subset F(L) \subset L$$

Now, $\phi \neq L \subset S$. We pick $i \in L$ and $j \notin L$

Claim 2: There is no path from i to j.

Suppose $i \to k_1 \to k_2 \to k_3 \to ... \to k_n \to j$ is a path. There is an edge $i \to j$ iff $j \in F(i)$.

So,

$$k_1 \in F(i) \in F(L) \subset L$$

$$k_2 \in F(k_1) \in F^2(L) \subset L$$

$$.....$$

$$k_n \in F(k_{n-1}) \in F^n(L) \subset L$$

$$j \in F(k_n) \in F^{n+1}(L) \subset L$$

This is contradiction, so $F(L)$ contains some index not in L. $\square$

Lemma 2.1.2. *For* $0 \leq h \leq n-1$, $\{i\} \cup F(i) \cup .... \cup F^h(i)$ *contains at least* $h+1$ *indices.*

Proof. The proposition is evidently true for $h = 0$. Assume it is true for some h, $0 \leq h < n-1$; then

$$L = \{i\} \cup F(i) \cup .... \cup F^h(i)$$

contains at least $h+1$ indices, and one of two situations occurs: (a) $L = S$, in which case

$$\{i\} \cup F(i) \cup .... \cup F^{h+1}(i) = S$$

also, containing $n > h+1$ elements, so that $n \geq h+2$, and the hypothesis is verified; or (b) L is a proper non-empty subset of S in which case

$$F(L) = F(i) \cup .... \cup F^{h+1}(i)$$

contains at least one index not in L (by Lemma 2.1.1), and since $i \in L$

$$\{i\} \cup F(L) = \{i\} \cup F(i) \cup .... \cup F^{h+1}(i) = L \cup F^{h+1}(i)$$

contains all the indices of the L and at least one not in L, thus containing at least $h+2$ elements.

$\square$

Corollary 2.1.3. *(a) If* $\{i\} \cup F(i) \cup .... \cup F^{h-1}(i)$ *is a proper subset of* S*, then, with the union of* $F^h(i)$*, at least one new element is added. Thus,(b) if*

$$\{i\} \cup F(i) \cup .... \cup F^h(i), \ \ h \leq n-2,$$

contains precisely $h+1$ *elements, then the union with each successive* $F^r(i)$*, where* $r = 1, 2, ..., n-2$ *adds precisely one new element.*

Proof. Let prove part (a) let say $L = \{i\} \cup F(i) \cup F^2(i) \cup ... \cup F^{h-1}(i) \subset S$
and $F(L) = F(i) \cup F^2(i) \cup ... \cup F^{h-1}(i) \cup F^h(i)$. By Lemma 2.1.1 (b). $F(L)$ contains some index not in L. So, pick $j \in F(L)$, where $j \notin L$. Now $j \in F(L)$ implies $j \in \{i\} \cup F(L) = L \cup F^h(i)$, so $j \in F^h(i)$, $j \notin L$. So, with $F^h(i)$, at least one element is added.
Now we are in part (b), note $\{i\}$ has one element by definition. Let $r = 1$, $\{i\} \cup F(i)$ has $1 + 1 = 2$ elements by the assumption
$\{i\}$ has one element, so $F(i)$ adds precisely one new element.
We want to prove (b) by induction. Assume $F^{r-1}(i)$ adds precisely one new element, we want to show $F^r(i)$ adds precisely one new element, $r = 1, 2, ..., n-1$
$L = \{i\} \cup F(i) \cup ... \cup F^{r-1}(i)$ this set has r elements by the assumption. If we take F of both sides we will get $F(L) = F(i) \cup F^2(i) \cup ... \cup F^{r-1}(i) \cup F^r(i)$.
And if we also add $\{i\}$ to both sides we will get

$$\{i\} \cup F(L) = \{i\} \cup F(i) \cup F^2(i) \cup ... \cup F^{r-1}(i) \cup F^r(i)$$

this set has $r+1$ elements by the assumption and we know before that the set $\{i\} \cup F(i) \cup ... \cup F^{r-1}(i)$ has r elements, so that implies $F^r(i)$ adds precisely one new element as desire.

$\square$

Corollary 2.1.4. *For any* $i \in S$, $\{i\} \cup F(i) \cup .... \cup F^{n-1}(i) = S$.

Proof. By Lemma 2.1.2 $\{i\} \cup F(i) \cup F^2(i) \cup ... \cup F^{n-1}(i)$ has at least n elements. As the union is in S which has n elements $\{i\} \cup F(i) \cup F^2(i) \cup ... \cup F^{n-1}(i) = S$.

□

Let us take an example to understand this Corollary.

Example 2.1.1. $n = 5$, so $S = \{1, 2, 3, 4, 5\}$, to see the relationship for each i, $\{i\} \cup F(i) \cup F^2(i) \cup F^3(i) \cup F^4(i) = S$

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad T^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Let take $i = \{1\}$, $F(1) = \{2\}$, $F^2(1) = \{3\}$, $F^3(1) = \{4\}$, $F^4(1) = \{5\}$

So

$$\{1\} \cup F(1) \cup F^2(1) \cup F^3(1) \cup F^4(1) = S$$

this satisfies the corollary for $i = 1$ also it is true for all i. I will let the reader check that out.

Lemma 2.1.5. *For $i \in S$, $F^h(i)$ contains at least two indices for some $h, 1 \leq h \leq n$ unless T is a permutation matrix.*

Proof. Since $\{i\} \cup F(i) \cup .... \cup F^{n-1}(i) = S$, two cases are possible for this proof:

(a) Suppose this $F^h(i)$ has only one index. Pick $j \in S$. As $\{i\} \cup ... \cup F^{h-1}(i) = S$, there exist h with $j \in F^h(i)$. As $F^h(i)$ has size 1, $F^h(i) = \{j\}$. Then $F(j) = F^{h+1}(i)$ has only one index. Thus, T is permutation matrix.
(b) Some $F^h(i)$, $1 \leq h \leq n-1$ contains at least two indices.

$\square$

We now pass on to a study of a general upper bound for $\gamma(T)$ depending only on the dimension n (≥ 2 by earlier assumption) for a primitive matrix T. For subclasses of $(n \times n)$ primitive matrices T satisfying additional structural conditions, stronger results are possible. Wielandt's theorem is important because it gives a simple way to test if a matrix is primitive or not.

Theorem 2.1.6. *(Wielandt's Theorem [4]) For a primitive* $(n \times n)$ *matrix* T, $\gamma(T) \leq n^2 - 2n + 2$.

Proof. According to Corollary 2.1.4, for arbitrary fixed $i \in S$,

$$\{i\} \cup F(i) \cup .... \cup F^{n-1}(i) = S,$$

and either (a) $F^h(i)$, $h = 0, ..., n-1$ all contain precisely one index, and by Lemma 2.1.5 in which case $F^n(i)$ contains at least two indices and the permutation case is excluded by the assumption of primitivity of T (i.e. any power of a permutation matrix is again a permutation); or (b) one of the $F^h(i)$, $1 \leq h \leq n-1$ contains at least two indices.

Right now let's prove the first case.

(a) Since $\{i\} \cup F(i) \cup .... \cup F^{n-1}(i) = S$, it follows $F(i) \cup .... \cup F^n(i) = F(S) = S$, in which case $F^n(i)$ must contains i. Namely, $i \in F^h(i)$ for some $1 \leq h \leq n$. If $h < n$ then $F^h(i) = \{i\}$ (By assumption that $F^h(i)$ has only one element). But then i appears twice in $\{i\} \cup F(i) \cup .... \cup F^{n-1}(i)$. And so, this contains at most $n-1$ elements, contrary to Lemma 2.1.2.

We know from Corollary 2.1.4

$$\{i\} \cup F(i) \cup .... \cup F^{n-1}(i) = S$$

since $j \in S$, now $j \in F^m(i)$ where $1 \leq m \leq n-1$ ($j \neq i$ so $j \notin F^0(i)$).

Thus $F^m(i) = j$ since we are in case (a) and by Lemma 2.1.5 that implies $F^m(i) \subseteq F^n(i)$

Hence for some integer $m \equiv m(i)$, $1 \leq m < n$,

$$F^m(i) \subseteq F^n(i) = F^{m+(n-m)}(i)$$

Apply $F^{n-m}(i)$:

$$F^{n-m}[F^m(i)] \subseteq F^{n-m}[F^{m+(n-m)}(i)]$$

$$F^m(i) \subseteq F^n(i) \subseteq F^{m+2(n-m)}(i)$$

So we repeated $n-1$ times to get:

$$F^m(i) \subseteq F^{m+(n-m)}(i) \subseteq F^{m+2(n-m)}(i) \subseteq ... \subseteq F^{m+(n-1)(n-m)}(i)$$

Now, let $B = A^{n-m}$. Let $F_A,\ \ F_B$ be the F-function for A and B respectively. Note that $F_B^h(L) = F_A^{h(n-m)}(L)$ since $B^h = A^{h(n-m)}$. By Corollary 2.1.4; applied to B and j:

$$\{j\} \cup F_B(j) \cup F_B^2(j) \cup ... \cup F_B^{n-1}(j) = S.$$

$$\{j\} \cup F_A^{(n-m)}(j) \cup F_A^{2(n-m)}(j) \cup ... \cup F_A^{(n-1)(n-m)}(j) = S.$$

As $j = F_A^m(i)$:

$$F_A^m(i) \cup F_A^{m+(n-m)}(i) \cup F_A^{m+2(n-m)}(i) \cup ... \cup F_A^{m+(n-1)(n-m)}(i) = S.$$

But each of these terms is contained in the last one, so

$$F_A^{m+(n-1)(n-m)}(i) = S.$$

Since $m + (n-1)(n-m) \leq n^2 - 2n + 2$:

$$F^{n^2-2n+2}(i) = S.$$

(b) If one of the $F^h(i)$, $h = 1, ..., n-1$ contains at least two indices, we further differentiate between two cases:

(b.1) $\{i\} \cup F(i) \cup .... \cup F^{n-2}(i) \neq S$. Then by Corollary 2.1.3 of Lemma 2.1.2, each of $F^h(i)$, $h = 0, ..., n-2$ adds precisely one new element, and by Corollary 2.1.4, $F^{n-1}(i)$ contributes the last element required to make up S. Let $p \equiv p(i)$, $1 \leq p \leq n-1$, be the smallest positive integer such that $F^p(i)$ contains at least two elements. Then there exists an integer m, $0 \leq m < p$, where $F^m(i)$ contains only one element because $m < p$ and p is the smallest one to get two elements and by Corollary 2.1.3 $F^h(i)$ adds precisely one new element that implies $F^m(i) \subseteq F^p(i)$.

Proceeding as in (a),

$$F^m(i) \subseteq F^p(i) = F^{m+(p-m)}(i)$$

so that operating repeatedly with F^{p-m}:

We have

$$F^m(i) \subseteq F^p(i) = F^{m+(p-m)}(i)$$

Apply $F^{p-m}(i)$:

$$F^{p-m}[F^m(i)] \subseteq F^{p-m}[F^{m+(p-m)}(i)]$$

$$F^m(i) \subseteq F^{m+(p-m)}(i) \subseteq F^{m+2(p-m)}(i)$$

So we repeated $n-1$ times to get:

$$F^m(i) \subseteq F^{m+(p-m)}(i) \subseteq F^{m+2(p-m)}(i) \subseteq ... \subseteq F^{m+(n-2)(p-m)}(i) \subseteq F^{m+(n-1)(p-m)}(i)$$

as in part (a) if we let $B = A^{p-m}$ will get by Corollary 2.1.4

$$F^m(i) \cup F^{m+(p-m)}(i) \cup F^{m+2(p-m)}(i) \cup ... \cup F^{m+(n-2)(p-m)}(i) \cup F^{m+(n-1)(p-m)}(i) = S$$

and we show that $F^m(i), F^{m+(p-m)}(i), F^{m+2(p-m)}(i)$,..., and $F^{m+(n-2)(p-m)}(i)$ subset of $F^{m+(n-1)(p-m)}(i)$

Thus

$$F^{m+(n-1)(p-m)}(i) = S$$

and by using basic algebra

$m+(n-1)(p-m) = m+np-nm-p+m = m(2-n)+np-p = -m(n-2)+np-p+[p-p] = -m(n-2)+p(n-2)+p = p+(n-2)(p-m) \leq (n-1)+(n-2)(n-1) = (n-1)^2 < n^2-2n+2$.

So in this case for all $i \in S$ we have:

$$F^{n^2-2n+2}(i) = S.$$

(b.2) $\{i\} \cup F(i) \cup .... \cup F^{n-2}(i) = S$. Then by Lemma 2.1.1

$$S = F(S) = F(i) \cup .... \cup F^{n-1}(i)$$

as before, by Corollary 2.1.3 of Lemma 2.1.2 for some p, $1 \leq p \leq n-1$, $\{i\} = F^0(i) \subseteq F^p(i)$. Proceeding as before ,

$$F^p[F^0(i)] \subseteq F^p[F^p(i)]$$

$$F^{0+p}(i) \subseteq F^{2p}(i)$$

$$F^p[F^{0+p}(i)] \subseteq F^p[F^{2p}(i)]$$

$$F^{0+2p}(i) \subseteq F^{3p}(i)$$

Which implies:

$$F^0(i) \subseteq F^{0+p}(i) \subseteq F^{0+2p}(i) \subseteq ... \subseteq F^{0+(n-1)p}(i)$$

also as in part (a) if $B = A^p$ and by Corollary 2.1.4:

$$F^0(i) \cup F^{0+p}(i) \cup F^{0+2p}(i) \cup ... \cup F^{0+(n-1)p}(i) = S$$

and we show $F^0(i), F^{0+p}(i), F^{0+2p}(i)$,..., and $F^{0+(n-2)p}(i)$ subset of $F^{0+(n-1)p}(i)$
Thus

$$F^{0+(n-1)p}(i) = S$$

with

$$(n-1)p \leq (n-1)^2 < n^2 - 2n + 2$$

So in this case also we have:

$$F^{n^2-2n+2}(i) = S$$

Thus combining (a) and (b), we have that for each $i \in S$,

$$F^{n^2-2n+2}(i) = S$$

which proves the theorem.

$\square$

CHAPTER 3

GAPS IN THE SET OF EXPONENTS OF PRIMITIVE MATRICES

3.1 INTRODUCTION AND DEFINITIONS

Wielandt's theorem states that $\gamma(A) \leq (n-1)^2 + 1$ and we proved it earlier. Let S be the set of all exponents of $n \times n$ primitive matrices. The main result of this chapter concerns about gaps in this exponent set S. Explicitly, if n is even, there is no primitive matrix A for which

$$n^2 - 4n + 6 < \gamma(A) < (n-1)^2.$$

If n is odd, there is no primitive matrix A for which

$$n^2 - 3n + 4 < \gamma(A) < (n-1)^2,$$

or

$$n^2 - 4n + 6 < \gamma(A) < n^2 - 3n + 2.$$

Two directed graphs are isomorphic if there is a one to one correspondence between vertices which preserves edges. For two $n \times n$ matrices A and B, there exists a permutation matrix P such that A and $P^{-1}BP$ have the same zero entries if and only if the graphs D_A and D_B are isomorphic.

If D is a directed graph with vertex set $V = (1, 2, 3, ..., n)$ then the t^{th} power of D, denoted by D^t is the directed graph with the same vertex set V, such that the ordered pair (i, j) is an edge of D^t if and only if there is a path in D from vertex i to vertex j of length t. Thus $\gamma(D)$ is the smallest power of D which is a complete graph with n loops.

3.2 THEOREMS ON THE EXPONENT OF A PRIMITIVE GRAPH

In this section, the theorems on the gaps in the exponent set of $n \times n$ primitive matrices are established. This material is taken from [1].

We will give another proof of Wielandt's theorem in the next theorem which depends on the length of the circuit in the graph. And, we will base a few preliminary Lemmas that help us with proofs next theorems.

Lemma 3.2.1. *If D is a primitive graph, then D^t is primitive for all $t > 0$.*

Proof. D is primitive that implies there exists an integer $s > 0$ such that $D^s > 0$ and that implies $D^{st} > 0$ for some $t > 0$ so we can write it like $[D^t]^s > 0$ that implies D^t is primitive.

□

Lemma 3.2.2. *If D is a primitive graph, then for every vertex i there exists an integer h with the property that for every vertex j there is a path from i to j of length h.*

Proof. D is primitive, so there exists an integer $h \geq 1$ such that $D^h > 0$, which means there exists a path of length h from i to j.

□

Before we go to the next Lemma the least such integer h of 3.2.2 is called the *reach* of vertex i.

Lemma 3.2.3. *Let D be a primitive graph and let h_i be the reach of vertex i. If $p \geq h_i$ then there exists a path from i to any vertex j of length p.*

Proof. Since D is strongly connected so we know that for any i, there exist a path from i to j of length h_i. We want to prove the Lemma by induction.

Write $p = h_i + q$, where $q \geq 0$ [we want to prove: Given j, there exist path from i to j of length $h_i + q$].

If $q = 0$: it is true by definition of h_i. For $q > 0$: Given j and we pick k such that $k \to j$. By induction: there exist path from i to k of length $h_i + (q - 1)$. To make it more clear, it looks like $i \to ... \to k \to j$. Thus there is a path of length $h_i + 1$ from i to j for every j.

□

Lemma 3.2.4. *If D is primitive graph then $\gamma(D) = Max[h_1, h_2, ..., h_n]$.*

Proof. We need to show for any i, any j there exist path from i to j of length $m = Max[h_1, h_2, ..., h_n]$ and we want to show m is the smallest such integer.

First part: Given i and j, there exists a path from i to j of length h_i and so one of length $m \geq h_i$ by Lemma 3.2.3.

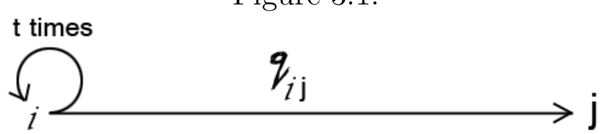
Second part: $m = h_i$ for some i . By definition of h_i, there exist j such that there is no path from i to j of length less than $h_i = m$.

So m is smallest number such that there exists a path from i to j of length m, for all i, j. □

Lemma 3.2.5. *If D strongly connected and i is a loop vertex then $h_i \leq n - 1$.*

Proof. There is a path from i to j of length $q_{ij} \leq n - 1$ by Corollary 2.1.4.

Figure 3.1.



Combining this with $n - 1 - q_{ij}$ loops, we have a path from i to j of length $n - 1$.

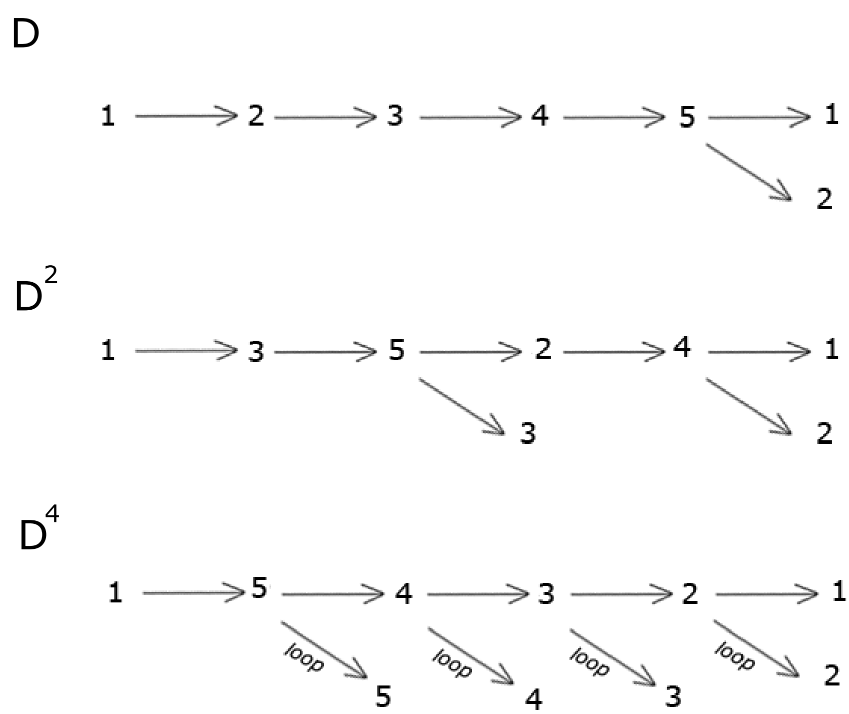
□

Before we go to the next theorem I want to put this example for making proof the next theorem more clear. For example, if D has a circuit of length s, then D^s has at least s loop vertices.

Example 3.2.1.

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 3.2.



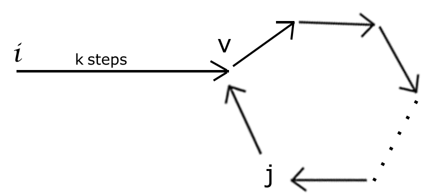
Let us write T in D graph:

We can see ,in the figure 3.2, here $n = 5$,and $s = 4$ which is the length of the shortest circuit of D. So, $n - s = 1$ there is a path of length 1 from any i to some vertex k is a loop in D^4.

Theorem 3.2.6. *If D is a primitive graph and if s is the length of the shortest circuit in D then $\gamma(D) \leq n + s(n-2)$. In other words, if A is a primitive matrix, and if s is the length of the shortest circuit in the directed graph D_A then $\gamma(A) \leq n + s(n-2)$.*

Proof. Since D is primitive, D^s is primitive by Lemma 3.2.1. Since D has a circuit of length s, D^s has at least s loop vertices. In Figure 3.3, the length from i to v is k steps which is the shortest length, and the length from v to j is $s-1$ because s is the length of the circuit. So, the length from i to j is $k+s-1$ and we know that by Corollary 2.1.4 $k+s-1 \leq n-1$. which implies $k \leq n-s$.

Figure 3.3.



Thus for any vertex i of D, there is a path in D of length $p_i \leq n-s$ from i to some vertex v of D which is a loop vertex in D^s.

Since v is a loop vertex in D^s, there exists, by Lemma 3.2.5., for any vertex j, a path of D^s from v to j of length exactly $n-1$. Thus there is, for any vertex j, a path in D from v to j of length $(n-1)s$. Combining these paths we have a path from i to any vertex j of length exactly $p_i + (n-1)s$. It follows that $h_i \leq p_i + (n-1)s$. Thus

$$\gamma(D) = Max[h_1, h_2, ..., h_n] \leq n - s + (n-1)s = n + s(n-2).$$

□

We can use 3.2.1 to give another proof of Wielandt's Theorem in next remark.

Remark. Since the greatest common divisor of the lengths of the circuits in a primitive graph is 1, it follows that $s \leq n-1$. Thus

$$\gamma(D) \leq (n-1)^2 + 1.$$

Proof. Say $s = n$. Then the length of any circuit $s \geq n$. But there are only n vertices, so the length of any circuit is less than n. Thus every circuit has length n. But D is primitive so the greatest common divisor of circuit length is 1. So, $s \leq n-1$. And, by previous theorem $\gamma(D) \leq n + s(n-2) \leq n + (n-1)(n-2) = (n-1)^2 + 1$

□

We will now use F differently from Chapter 1. Let $p_1, p_2, ..., p_u$ be relatively prime and let $F(p_1, p_2, ..., p_u)$ denote the largest integer which is not expressible in the form $a_1p_1 + a_2p_2 + ... + a_up_u$ where a_r is non-negative integer for $r = 1, 2, ..., u$. It is well known, if m and n are relatively prime, that $F(m, n) = mn - m - n$. And, if $a_j = a_0 + jd, j = 0, 1, ..., s, a_0 \leq 2$, then

$$F(a_0, a_1, ..., a_s) = ([\frac{a_0 - 2}{s}] + 1)a_0 + (d-1)(a_0 - 1) - 1.$$

Example 3.2.2. Let take $F(3, 7)$ and how we calculate it.

$$X = 3a_1 + 7a_2$$

$$3 = 3(1) + 7(0)$$

$$4 \neq 3a_1 + 7a_2$$

$$5 \neq 3a_1 + 7a_2$$

$$6 = 3(2) + 7(0)$$

$$7 = 3(0) + 7(1)$$

$$8 \neq 3a_1 + 7a_2$$

$$9 = 3(3) + 7(0)$$

$$10 = 3(1) + 7(1)$$

$$11 \neq 3a_1 + 7a_2$$

$$12 = 3(4) + 7(0)$$

$$13 = 3(2) + 7(1)$$

and all the numbers after 11 there is a solution so $F(3,7) = 11$ because 11 is the largest number which is not expressible in the form $3a_1 + 7a_2$.

Here we will define a new length which called r_{ij}.

Definition. Let D be a primitive graph in which every circuit is of length $p_1, p_2, ...,$ or p_u. For any ordered pair (i,j) of vertices, a non-negative integer r_{ij} is defined as follows. If $i = j$ and if for $s = 1, 2, ..., u$ there is a circuit through vertex i of length p_s then $r_{ij} = 0$; otherwise r_{ij} is the length of the shortest path from i to j which has at least one vertex on some circuit of length p_s for $s = 1, 2, ..., u$.

Let say $r = Max(r_{ij})$ taken over all ordered pairs (i,j).

Theorem 3.2.7. *If D is a primitive graph then*

$$\gamma(D) \leq F(p_1, p_2, ..., p_u) + 1 + r.$$

Proof. For any set of non-negative integers $a_1, a_2, ..., a_u$ and any ordered pair (i,j) of vertices, if $i \neq j$ there is a path from vertex i to vertex j of length r_{ij} that contains a vertex on a circuit of length p_u for each u. So that p plus $u's$ many trips around the circuit of length p_u. See Figure 3.4 and 3.5.

$$r_{ij} + a_1p_1 + a_2p_2 + ... + a_up_u$$

Figure 3.4.

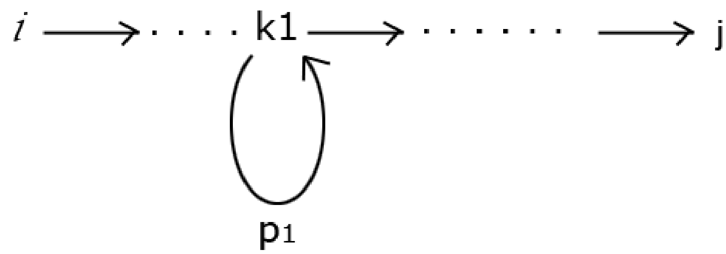
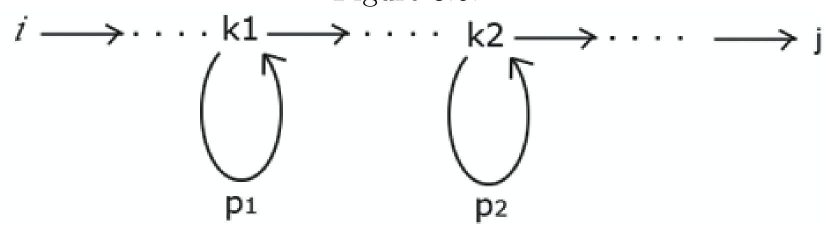


Figure 3.5.



Thus there is a path from vertex i to vertex j of length

$$F(p_1, p_2, ..., p_u) + r_{ij} + N$$

for every $N \geq 1$, by definition if $N \geq 1$, $F(p_1, p_2, ..., p_u) + N$ can be expressed as $\sum a_i p_i$ and so there is a path of length $F(p_1, p_2, ..., p_u) + r_{ij} + N$. Choosing $N = 1 + r - r_{ij}$, we have a path from vertex i to vertex j of length

$$F(p_1, p_2, ..., p_u) + 1 + r,$$

so that

$$h_i \leq F(p_1, p_2, ..., p_u) + 1 + r.$$

Thus by Lemma 3.2.4

$$\gamma(D) = Max\{h_i\} \leq F(p_1, p_2, ..., p_u) + 1 + r.$$

□

Definition. An ordered pair (k, l) of vertices in a primitive graph D is said to have *the unique path length property* if every path from vertex k to vertex l which has length $\geq r_{kl}$ consists of some path α of length r_{kl} augmented by a number of circuits each of which has a vertex in common with α.

Theorem 3.2.8. *If D is a primitive graph in which the ordered pair of vertices (k, l) has the unique path length property, then*

$$F(p_1, p_2, ..., p_u) + 1 + r_{kl} \leq \gamma(D).$$

Proof. There is no path from vertex k to vertex l of length

$$w = F(p_1, p_2, ..., p_u) + r_{kl}$$

for such a path would imply the existence of non-negative $a_1, a_2, ..., a_u$ with

$$F(p_1, p_2, ..., p_u) = a_1 p_1 + a_2 p_2 + ... + a_u p_u.$$

Thus by Lemma 3.2.3., we have $F(p_1, p_2, ..., p_u) + r_{kl} < h_k$. Since $h_k \leq Max h_i = \gamma(D)$, the result follows.

□

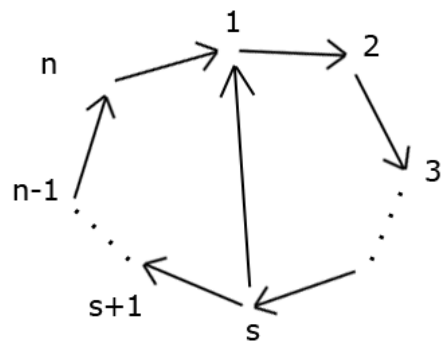
Corollary 3.2.9. *If in theorem 3.2.8, $r_{kl} = r$ then*

$$h_k = \gamma(D) = F(p_1, p_2, ..., p_u) + 1 + r.$$

Theorem 3.2.10. *If s and n are relatively prime $(s < n)$, there exists a primitive graph D with n vertices and $n+1$ edges for which $\gamma(D) = n + s(n-2)$.*

Proof. The graph D with the $n+1$ edges $(1, 2), (2, 3), ...(n-1, n), (n, 1)$ and $(s, 1)$ has a circuit of length s, and a circuit of length n as you see in Figure 3.6. Since s and n are relatively prime, D is primitive. The ordered pair $(s+1, n)$ has the unique path length property, so we want all paths $s+1$ to n that have at least one vertex on a circuit called p_1

Figure 3.6.



and p_2 of length $p_1 = n$ and $p_2 = s$ i.e. we pick $s+1$ not $s+2$ because the unique length path property for $s+1$ to n is longer.

Here we have two circuit:

$$1 \to ... \to n \to 1$$

and

$$1 \to ... \to s \to 1$$

$p_1 = n$ and $p_2 = s$

Let check the path from $s+1$ to n:

$$s+1 \to s+2 \to ... \to n$$

this does not have vertex in common in p_2, so the unique path length property is:

$$s+1 \to s+2 \to ... \to n \to 1 \to 2 \to ... \to n$$

the length from $s+1$ to 1 is $n-s$ and the length from 1 to n is $n-1$, if you add them you will have $r_{s+1,n} = 2n - s - 1$. Moreover, $r_{s+1,n} = r$. By Corollary 3.2.9.,

$$\gamma(D) = F(n,s) + 1 + r = ns - s - n + 1 + 2n - s - 1 = n + s(n-2)$$

$\square$

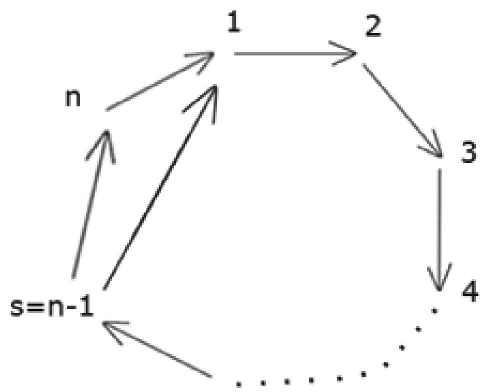
Theorem 3.2.11. *Apart from isomorphism, there is exactly one primitive graph D on n vertices for which $\gamma(D) = (n-1)^2 + 1$, and exactly one for which $\gamma(D) = (n-1)^2$. These are the only graphs for which the length of the shortest circuit is $n-1$.*

Proof. If $s < n-1$, then from Theorem 3.2.6,

$$\gamma(D) \leq n + (n-2)^2 = n^2 - 3n + 4$$

If $s = n-1$, then since the greatest common divisor of the lengths of the circuits is 1,

Figure 3.7.



the graph must have a circuit of length $n-1$ and another of length n. Thus the graph D must have as a subgraph, a graph which is isomorphic to the graph of Theorem 3.2.10 with $s = n-1$ see Figure 3.7. Denote this graph by E. The graph E is isomorphic to the graph of the matrix used by Wielandt theorem to show that his result was best possible. There are two cases to consider.

Case (i). $D = E$. From Theorem 3.2.10, we have

$$\gamma(E) = n + (n-1)(n-2) = (n-1)^2 + 1.$$

Case (ii). E is a proper subgraph of D. The only edge which can be added to E without introducing a circuit of length less than $n-1$ is the edge $(n, 2)$ or the edge $(n-2, n)$. Since the resulting graphs are isomorphic, it is sufficient to consider the first case.

The ordered pair $(1, n)$ has the unique path length property with $r_{1,n} = n-1$. Moreover,

$r_{1,n} = r$. By Corollary 3.2.9.,

$$\gamma(D) = F(n, n-1) + 1 + r = n(n-1) - n - (n-1) + 1 + n - 1 = (n-1)^2$$

$\square$

Theorem 3.2.12. *If n is even ($n > 4$), then there is no primitive graph D such that*

$$n^2 - 4n + 6 < \gamma(D) < (n-1)^2$$

Proof. From Theorem 3.2.11, if $\gamma(D) < (n-1)^2$, we have $s \leq n-2$. For $s \leq n-3$, by Theorem 3.2.6.,

$$\gamma(D) \leq n + (n-3)(n-2) = n^2 - 4n + 6.$$

If $s = n-2$, since n and $n-2$ are not relatively prime, a primitive graph D must have circuits of length $n-2$ and $n-1$. Beginning with a circuit of length $n-2$, a circuit of length $n-1$ must either involve both of the remaining vertices or one of the remaining vertices. It follows, that D must have as subgraph, a graph which is isomorphic to one of the graphs F in Figure 3.8 or G in Figure 3.8. There are two cases here.

Case (i). F is a subgraph of D. In F the ordered pair $(n-1, n)$ has the unique path length property and $r_{n-1,n} = n = r$. By Corollary 3.2.9.,

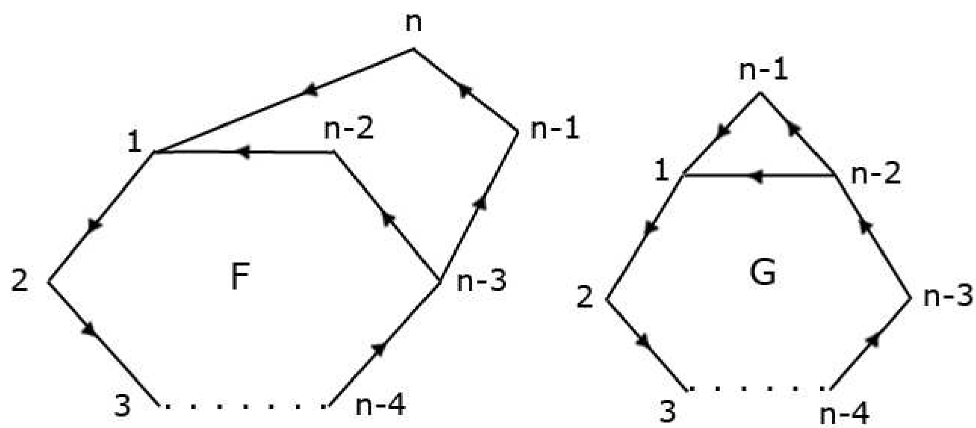
$$\gamma(D) \leq \gamma(F) = F(n-1, n-2) + 1 + r = n^2 - 4n + 6$$

Case(ii). G is a subgraph of D. G is a primitive graph with $n-1$ vertices of the same type as the graph E of theorem 3.2.11. Thus

$$\gamma(G) = (n-2)^2 + 1 = n^2 - 4n + 6.$$

There must be at least one edge (n, i_1) of D, $i_1 \neq n$, and at least one edge (i_2, n) of D, $i_2 \neq n$. We may assume $i_1 \neq i_2$, since otherwise we have a circuit of length 2 which is less than s for $n > 4$. Let E be the subgraph of D consisting of G together with the two edges

Figure 3.8.



(n, i_1) and (i_2, n). We show that $\gamma(H) \leq n^2 - 4n + 6$. If $j \neq n$, there is a path from i_1 to j in G of length exactly $n^2 - 4n + 5$ and adjoining (n, i_1) we have a path from n to j of length exactly $n^2 - 4n + 6$ in H. If $i \neq n$, there is a path from i to i_2 in G of length exactly $n^2 - 4n + 5$ and adjoining (i_2, n) there is a path of length exactly $n^2 - 4n + 6$ from i to n in H. For a path from n to n we use the fact that at least one of i_2 and $i_2 \neq n - 1$. If $i_1 \neq n - 1$, there exists a vertex n_1 of G such that (n_1, i_1) is the only edge in G out of n_1. there is a path from n_1 to i_2 in G of length exactly $n^2 - 4n + 5$. Replacing (n_1, i_1) by (n, i_1) and adjoining (i_2, n) yield a path from n to n in H of length $n^2 - 4n + 6$. Similarly, if $i_2 \neq n - 1$, there exists a vertex n_2 of G such that (i_2, n_2) is the only edge into n_2. there is a path from i_1 to n_2 in G of length exactly $n^2 - 4n + 5$. Adjoining (n, i_1) and replacing (i_2, n_2) by (i_2, n)

$$\gamma(D) \leq \gamma(H) \leq n^2 - 4n + 6.$$

If $i_1 \neq n - 1$ and $i_2 \neq n - 1$ it is easy to see, using the replacement edges (n_1, i_1) and (i_2, n_2) that $\gamma(D) \leq \gamma(H) \leq n^2 - 4n + 5$.

$\square$

Theorem 3.2.13. *If n is odd ($n > 3$), then there is no primitive graph D such that*

$$n^2 - 3n + 4 < \gamma(D) < (n-1)^2$$

or

$$n^2 - 4n + 6 < \gamma(D) < n^2 - 3n + 2.$$

Proof. If $\gamma(D) < (n-1)^2$, then $s \leq n-2$ and hence, by Theorem 3.2.6, $\gamma(D) \leq n^2-3n+4$. If $s \leq n-2$, and if the graph has a circuit of length $n-1$, then $\gamma(D) \leq n^2-4n+6$. This follows , because the proof of this result given in Theorem 3.2.12 for n even holds also for n odd.

□

CHAPTER 4

HISTORY OF WIELANDT'S PROOF

There are many things happened in our lives which we write them just to remember the past and the old friends. It says in wisdom every picture has a story and I want to say each mathematics proof has a story behind it. Wielandt's Proof has a different story as Hans Schneider tells us.

Wielandt published his theorem in 1950 [4] but did not give a proof. The first published proof was given in 1957 by Rosenblatt.

Hans Schneider published a paper [2] which had Wielandt's Proof and its story . In 1950s, there were two mathematicians named Helmut Wielandt and Hans Schneider. One day, Helmut and Hans were having a conversation where Helmut told Hans that keeping mathematical notes and thoughts in notebooks he called "Tagebcher" (diaries) was his habit. After the death of Helmut, Heinrich Wefelscheid and Hans visited Helmut's wife and son at Bavaria (an area southeast of Germany), and with their help, they were able to get a hold of Helmut's diaries. After Hans analyzed Helmut's work, he discovred Wielandt's proof of his theorem. No one knows why Wielandt never published it.

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