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Problem Solving Theory and Practice

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Recent studies have shown that on all grade levels, U. S. children are at a very low end of the scale compared to children in other countries concerning the subject of mathematics. It seems as if children are not learning, or do not want to learn "mathematics". A lot of fault lies in how and what "mathematics" is taught. There needs to be a move to put pizzaz back into mathematics and work toward students having successful experiences to boot. I believe that this can be started by using problem solving in the curriculum. This paper will look at some of the problems concerning mathematics, how problem solving can help, and conclude with some problem solving strategies and ideas.

The mathematics yield, the sum total of mathematics learned by all students, of U. S. schools is substantially less than that of other industrialized nations (Steen, 1987). Since our society claims to be so advanced, the previous statement in itself proves that there is something wrong with the way that the children are "learning" mathematics It seems that mathematics has reduced to following rules. A recent NAEP mathematics assessment found that students seem to be able to follow the rules and do the computation, but "fail Miserably" when asked to reason. (Burns, 1986) Teachers have not helped students develop higher-level skills and understandings that go beyond rote, step-by-step, learning. Children become prisoners to blindly following rules and not even thinking about what or why they are doing what they are doing.

Today, mathematics classes turn out answer - centered persons who get the answer out of blind memory, instead of problem - centered persons who get the problem - solving process out of the problem itself, not out of memory.

Getting the correct answers is so deeply implanted as important into these young mathematics students, they cannot function any other way. In contrast, there are a few "thinkers", students who try to thind about the meaning, the reality, of whatever it is he/she is working on, who become very discouraged in this answer - centered atmosphere. This ridiculous right / wrong merry - go - round brings about all kinds of defensive strategies in children (discussed in Holt's How Children Fail). "What hampers children's thinking, what drives them into these narrow and defensive strategies, is a feeling that they must please the grownups at all costs. The really able thinkers in our class turn out to be, without exception, children who don't feel so strongly the need to please grownups." (Holt) This supports the important need for children to have s successes.

A good place to start to turn the mathematics curriculum around is with problem - solving. Using problem - solving in the classroom could help change the trend of answer - centered persons, producers, into thinkers. Problem iolving shifts the focus of importance from the answer to the plan used to get the answer. Using problem solving makes the students use their brain! It makes them draw upon their knowledge and combine what they know instead of blindly pulling from their memory some procedure. It also provides many and varied opportunities for successes. These successes could be the beginning of breaking down the built up defensive strategies.

Problem solving can be used in more than one way. One of the many ways it can be used is as a "Friday" activity. The teacher presents a problem to the students on Monday. Then on Friday, they go through the problem solving model together as a whole, in groups, or on an individual basis (this depending on many factors, for example: level of students, time of year, type of problem, etc.) The problems used on Friday are problems that the students have not done before; ones that do not fit into a "formula". When Friday problems are used, one day a week is cut from the "routine" curriculum. P. R. Halmos summed that problem up by this idea about covering forty topics: "Would it be better to give twenty topics a ten minute mention and to treat the other twenty in depth by student solved problems, student - constructed counter-examples, and student - discovered applications? Some of the material doesn't get covered, but a lot of material gets discovered." (Halmos, 1980)

That leads to another way to use problem solving; incorporating it into lessons that will work effectively with guided discovery. As one becomes a more experienced teacher, knowing which lessons work best with a guided discovery, problem solving, approach becomes more evident. Eventhough that is true, a lesson that works well in the problem solving mode one year, may be the worst thing to do the next year because of discipline, variety of students, etc. In order to effectively use problem solving in the class as a lesson, other factors need to be buckled down first. Many of these factors, such as time management, effective discipline, and routine, are built through experience. It doesn't matter how good the lesson is, it will fail to be effective if there is not adequate tome to teach or if the students are not paying attention. (Leinhardt, 1986)

In planning a discovery lesson, the teacher has several important roles: (Simon, 1986)

1. Identify and prioritize what needs to be learned (go through the textbook)
2. Distinguish between facts, procedures, and concepts.

3. Organize concepts hierarchically.

4. Divide what is to be learned into appropriate increments.

5. Create or adapt activities that stimulate the development of the desired concept.

Doing mathematics is figuring out what to do when you don't know what to do. When teachers use problem solving, students who are exploring and discovering new concepts for themselves have the opportunity actually to do mathematics rather than passively learn about it. This challenges the students to think more deeply about it and to connect it with prior experience in a personally meaningful way. These students tend to retain their understanding of the concept longer than students who have only the teacher's or textbook's explanation. (Simon, 1986)

When going about solving problems, the teacher wants to guide the students into a problem solving mode with a problem solving strategy. The problem solving strategy that follows is in accordance to Polya's How To Solve It (except for step 4 which was added by Dr. Katherine Pedersen, Southern Illinois University):

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Check to see if the solution is reasonable.
5. Generalize the solution; the problem.

In this strategy most of the time should be spent at step 1, understanding the problem. At this step the students should determine, and write down: everything the problem tells them, everything they know, all their feelings toward the problem, all of their ideas, what they are looking for, what would be a reasonable answer, etc., anything that relates to the problem at hand. After that step has been throughly exhausted (which is usually never, but one must quit sometime!), the next step is to devise a plan. This is done from the information in step 1. Possible strategies are picked over until the one that seems most suitable is found. Next, carry out the plan; solve the problem. Step 4 has the student(s) checking the result of step 3. Is it a reasonable answer to the question being asked? If not, the student(s) should "flow" back up to step 1 and begin again. The last step is to examine the solution. Following are some of the questions that the student(s) need to ask themselves at this step: Could the problem be extended? Can the problem be solved another way? If so, do it. Can the problem/solution be generalized? Is there a similar problem that could be asked? Is thare another problem that could be solved the same way? etc.

The remainder of this paper gives a few suggestions for problem solving activities. The first is to take advantage of computers and calculators. Computers and calculators change both what is feasible and what is important in the mathematics curriculum. Widely available computer packages, and calculators, can carry out almost every mathematical technique taught through the sophomore year in college both in the purely symbolic form that mathematicians are fond of and in the graphical and numerical forms that are needed by scientists.(Steen, 1987)

Many problem solving activities are made especially for calculators. Using calculator problems for Friday problems not only helps the students be in a problem solving mode, but also increases the students' awareness of the usefulness of calculators as well as familiarizes the students with their many functions. Another advantage is that most fields in the "real" world that the students will someday be entering, take advantage of calculators and computers in their day to day activities.

Another type of problem solving activity involves Direct Quote Word Problems (DQWP). (Pinker, 1979) Direct Quote Word Problems are based on a direct quotation from a disciplinary or general information publication. A DQWP consists of three parts:

1. a direct quotation from some publication
2. the identification of this publication
3. a question or questions that must be answered

Some DQWP's require additional parts:

1. a short introduction giving the background for the question or explaining its significance
2. a glossary of terms appearing in the quotation that may be unknown to the reader
3. the identification of the subject areas in which the problem would be of special interest

These DQWP's seem to be a good idea mainly because a majority of high school students have some academic goals in mind and these problems link mathematics to other fields and create a motivational vehicle for learning mathematics. DQWP's can easily be made by the teacher from articles out of magazines and newspapers. The teacher can vary the difficulty of the problems to suit the varying classes that will use them.

A way to incorporate data collection and analysis in the problem solving routine is through "human variability". (Pagni, 1979) People "own" the special traits that determine their personality, appearance, and daily life - style. Because students are interested in sharing information concerning themselves, they can make useful applications fron these traits in the classroom. In the 1979 Yearbook of the NCTM, Chapter Five, David L. Pagin lists many human characteristics that can be measured on a continuous basis. A few examples of human characteristics that can be represented by discrete variables are: eye color, freckles, sex, tongue type, etc. A few examples of human characteristics that can be measured and recorded by continuous variables are: length of arm, jumping distance, optical illusion, sensitivity to touch, etc. The students will have a good time performing the experiments and gathering the data (they may even have so much fun doing this that you will have to remind them that they are doing math!). After gathering the data, these techniques require further application of mathematics in the form of graphing, reading graphs, estimation, computing, and prediction. Another extension of these problems is to collect data from cooperating classes or on a random basis, then further predict about larger populations from the results obtained. High ability students could go as far as creating confidence intervals for the data.

As discussed in this paper, problem-solving should be an integral and important part of all mathematics curriculum. There are many ways to incorporate problem solving in the classroom, but no matter how it is used, it must be there to further students' mathematical thinking. Mathematics needs to return to the "thinking" stages instead of the role following directions that many classes are in now. This will bring students back to "using their head" and therefore schools will begin to turn out more mathematicians that are able to solve problems that the "real" world and their prospective jobs deal out. The future U. S. mathematicians need to once again be able to compete with their competitive nations.

The following pages are some examples of problem solving activities. The pages numbered 19 and 63 (Pederson, 1987) are examples of things to use for every-day. These problems are an extension of something that the students have done before. The pages numbered 20, 66, 16, 40, and 64 are examples of weekly problems that should be introduced on Monday, then worked on in Friday's problem solving mode (groups, entire class, etc.). The remaining five pages are examples of problem solving activities that can be used in the intermediate grades.(Cox, 1986).

MONDAY

On a stereo purchase you are offered a 20% discount and a 10% discount to be taken in either order. Which do you ask for first in order to reach a lower price?

TUESDAY

What is the least positive integer by which 180 should be multiplied, to give a product that is (a) a perfect square; (b) a perfect cube?

WEDNESDAY

Find a value for the sum of the first n terms of the following series:

$$+1 - 1 + 1 - 1 + 1 - 1 + 1 \ldots$$

THURSDAY

The arithmetic mean of five numbers is 2. If the smallest of the five numbers is deleted from the set, the average of the remaining numbers is 4. What is the smallest number in the original set?

FRIDAY

In which quadrant(s) are points that satisfy $x \cdot y < 0$?

MONDAY

What are the dimensions of a square inscribed in a circle of radius 1?

TUESDAY

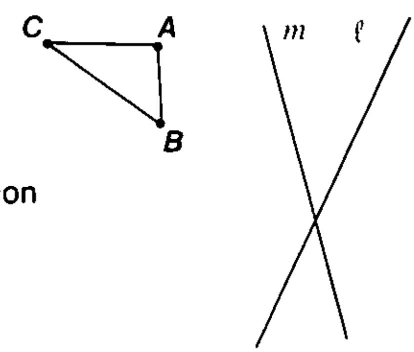
Rearrange the letters from items 1–5 to form a mathematical word.

1. The letter describing $\sqrt{-1}$.
2. The initial letter of the name for a line which touches a circle at exactly one point.
3. The letter which looks like the answer to $0 \cdot \pi \cdot r$.
4. The initial letter of the word for a parallelogram with equal sides whose angles may not be 90°.
5. The initial letter of the word for the measure of a plane region in terms of square units.

WEDNESDAY

What are the coordinates of the points that divide the line segment from (1, 1) to (5, 3) into four equal parts?

THURSDAY



What is the image of $\triangle ABC$ after a reflection in line ℓ followed by a reflection in line m?

FRIDAY

What are the coordinates of the midpoint of the segment that joins (0, 5) and (6, 1)?

Fill the Boxes!

You are to place one of the digits 0 through 9 in each of the boxes below. You may use a digit more than once; you need not use every digit.

Follow this rule:

The digit you place in the first box (under the 0) is to indicate the total number of zeros in all the boxes; the digit you place in the second box (under the 1) indicates the total number of 1s in all the boxes; and so on, to the last box (under the 9), so that the digit placed there indicates the total number of 9s in all the boxes.

Can you fill in the boxes? It is interesting to note that there is only one correct answer!

0	1	2	3	4	5	6	7	8	9

A Colorful Puzzle

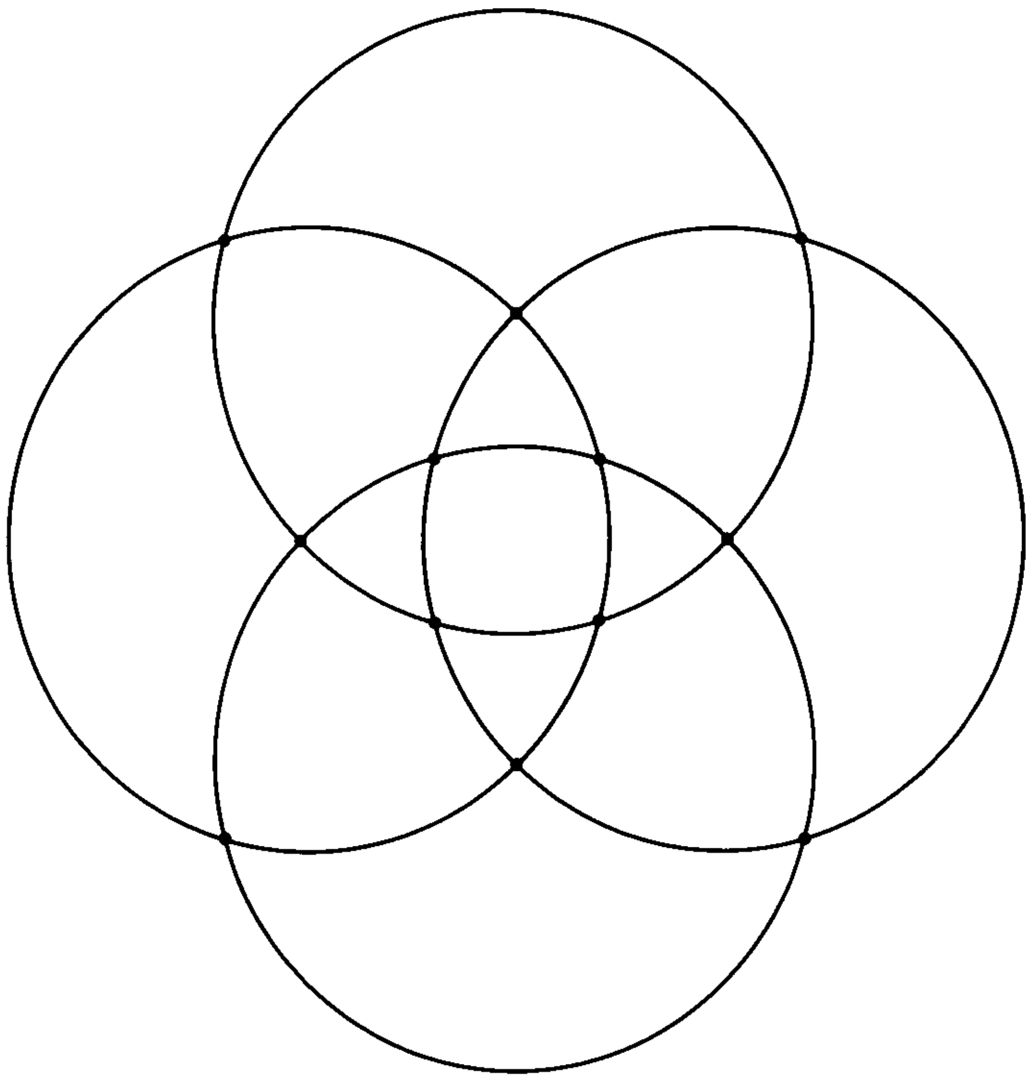
You are given 12 wooden cubes, all different in size and in color. The largest cube is 12 cm on a side, the next is 11 cm on a side and so on down to the smallest cube, which is 1 cm on a side. The cubes are colored red, blue, orange, yellow, green, purple, silver, white, violet, tangerine, crimson, and mauve. The following relationships hold among the cubes:

1. The sum of the volumes of the yellow and blue cubes equals the sum of the volumes of the green and mauve cubes.
2. The volume of the purple cube alone equals the sum of the volumes of the red, violet, and silver cubes.
3. The sum of the volumes of the blue, orange, purple, and violet cubes equals the sum of the volumes of the mauve, green, tangerine, silver, white, and yellow cubes.
4. The volume of the green cube alone equals the sum of the volumes of the tangerine, purple, and yellow cubes.

Match the size of each of the 12 cubes with its proper color.

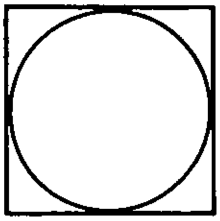
Magic Circles! Because *you* make them so

Arrange the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 at the points of intersection of the circles below so that the sum of the numbers lying on any given circle is equal to the sum of the numbers lying on any other one.

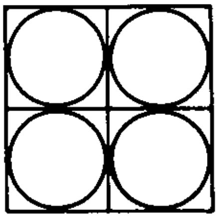


All My Circles!

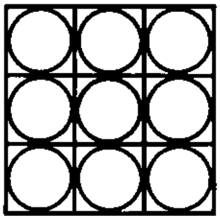
1. Inscribe a circle in a square of side 12. What is the area of the circle?



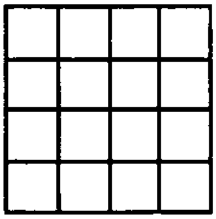
2. Divide the original square into 4 smaller squares, as shown. Inscribe a circle in each of the smaller squares. What is the sum of the areas of the circles?



3. Divide the original square into 9 smaller squares, as shown. Inscribe a circle in each of the smaller squares. What is the sum of the areas of the circles?



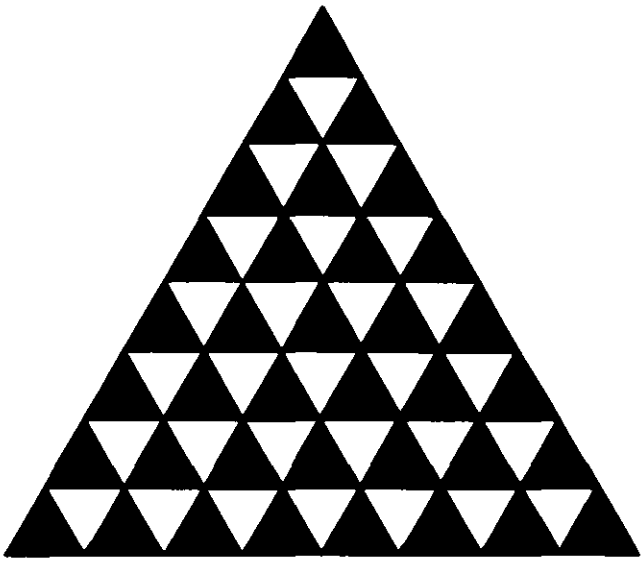
4. Divide the original square into 16 smaller squares as indicated. Inscribe a circle in each of the smaller squares. What is the sum of the areas of the circles?



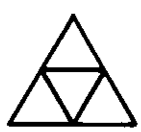
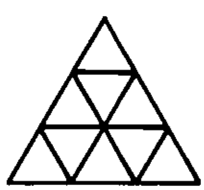
5. Repeat the process eight more times. What is the sum of the areas of the inscribed circles the last time this process is carried out?

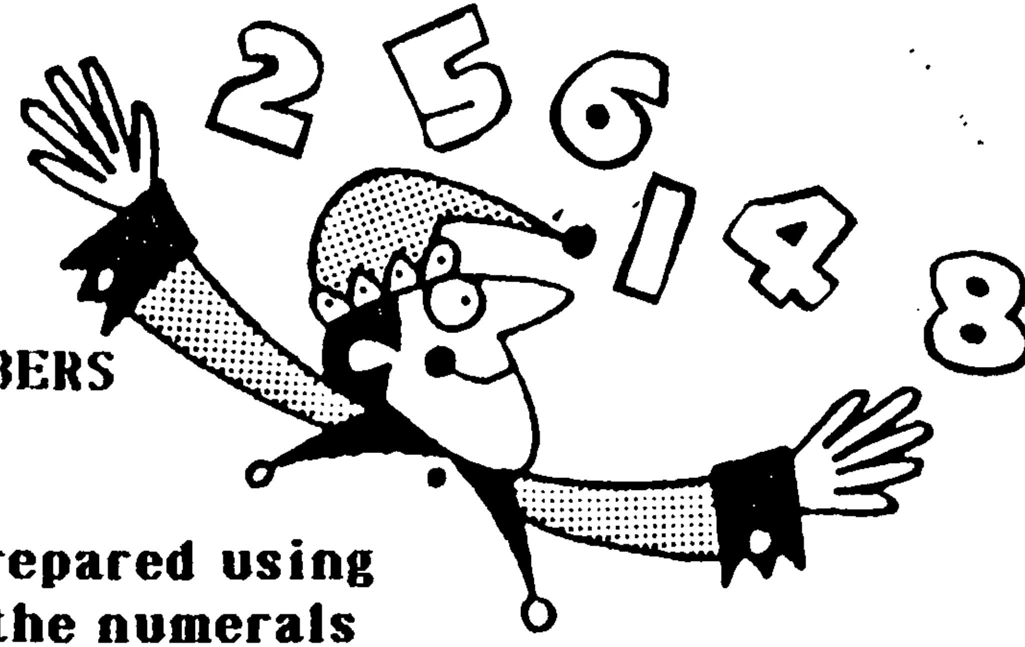
How Many Triangles?

How many triangles? Count only *upward-pointing* triangles in the figure below.



Complete the table below to find a pattern.

Size	Triangle	Number of Size				Total
		1×1×1	2×2×2	3×3×3	4×4×4	
1 × 1 × 1		1				1
2 × 2 × 2		3	1			4
3 × 3 × 3		6	3	1		10
4 × 4 × 4						
⋮						



LETTERS FOR NUMBERS

Some simple problems were prepared using the digits 0 – 9. Then, all of the numerals were changed to letters. You have to discover the code for each problem. Within each problem, the letter stands for the same number. But, the letter "value" changes in each problem.

```
  X X X X
  Y Y Y Y
+ Z Z Z Z
---------
Y X X X Z
```

```
    P N X
  x   N X
  -------
    R N X
  N X S
  -------
  Z P N X
```

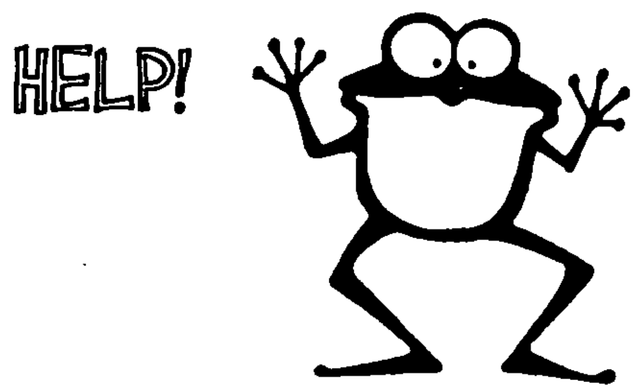
```
        H I L
      -------
I L | P H I L
      I L
      ---
        T I
        L S
        ---
        H I L
        H I L
        -----
```

TEACHING READING

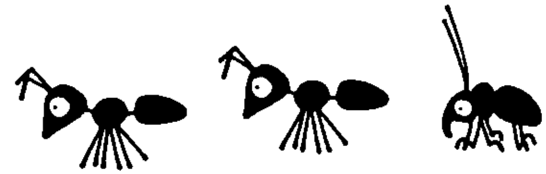
In the newly developing country of Xamba, teaching reading is the main focus of the new government program. The plan is for each individual who can read to spend one year teaching two others to read. That individual is then finished, but the two new readers must each spend one year teaching two others to read. If 1,000 people in Xamba can now read, how many people will be able to read in ten years?

FROGGIE

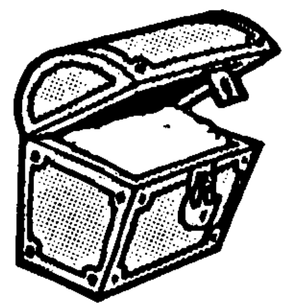
A frog is at the bottom of a ten foot deep well. Each day, the frog climbs up five feet but, at night, it slides back four feet when it sleeps. At this rate, how many days will it take for the frog to get out of the well?

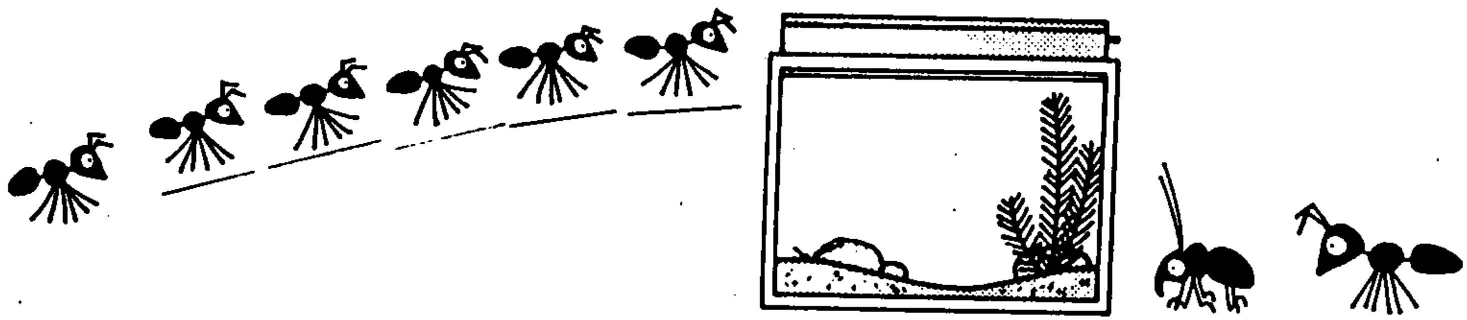


ANTS IN THE ATTIC



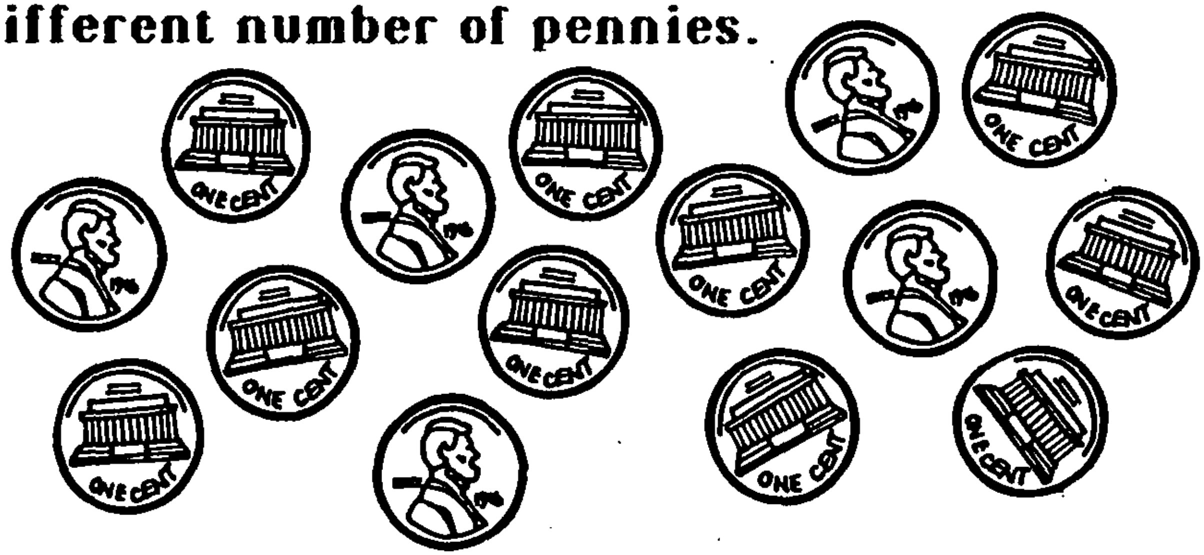
In a corner of the attic, Julie found an old rectangular fish tank frame. All of the glass had been removed. An ant decided to crawl to the opposite corner. How many different ways could the ant get to the opposite corner by walking along exactly three edges of the frame?





PILES OF PENNIES

Show all of the ways that 15 pennies can be put in 4 piles so that each pile has a different number of pennies.



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