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ADMISSIBLE JORDAN BLOCKS OF CERTAIN REPRESENTATIONS OF THE SYMPLECTIC GROUP WITH SPLIT-RANK 2

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ADMISSIBLE JORDAN BLOCKS OF CERTAIN REPRESENTATIONS OF THE SYMPLECTIC GROUP WITH SPLIT-RANK 2

MAJOR PROFESSOR: Dr. Dubravka Ban

The classification for irreducible square integrable representations of symplectic groups, as described in a joint paper by Mœglin and Tadić, gives a parameterization of irreducible tempered representations of these groups. The first parameter is given in terms of Jordan blocks which satisfy certain criteria. These are called admissible Jordan blocks. In this paper we will look at simple examples of the admissible Jordan blocks of irreducible tempered representations induced from irredible square integrable representations in the case of the symplectic group with split-rank 2.

DEDICATION

This paper is dedicated to my family and friends, and to pizza and the occasional steak.

ACKNOWLEDGMENTS

I would like to thank Dr. Ban for her invaluable assistance and dedication leading to the writing of this paper. I would also like to give a special thanks to Dr. Hundley for his assistance. My sincere thanks goes to all the members of my graduate committee for their patience and understanding during the years of education and effort that went into the production of this paper.

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PREFACE

The purpose of this study is to give some simple examples making use of the extensive machinery provided in the works of Jantzen, Mœglin, Tadić, and Sally, to name a few. Hopefully, by better understanding an example, a better understanding of the overarching theory may be reached.

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INTRODUCTION

This paper illustrates how to find the admissible Jordan blocks of certain representations induced from square integrable representations of the symplectic group with split-rank two over a p-adic field. We make use of a general classification provided by the work of Tadić. We also make extensive use of irreduciblity theorems provided in a paper by Jantzen. We only work within the confines of these irreducibility theorems, which means we exclude all of the irreducible square integrable representations with minimal parabolic support aside from the Steinberg representation.

The aim in writing this paper is to present the basic ideas of the theory being used via the examples given. The essence of the theory is the main focus. Formalism is secondary. Where possible, basic ideas are studied by means of the examples, which means that we ignore instances where we would be forced to utilize the ground theory of Zelevinsky segments.

The treatment of proofs varies. The proofs for the examples given are somewhat elementary in the context of all the machinery being used, and are presented in a style that overlooks some of the tedious computations. Again the examples are given mainly to outline some of the fundamental ideas. The proofs of the theorems in use are more difficult, but very valuable. They make use of a variety of different ideas ranging from Zelevinsky segments to Hopf algebras to analytical ideas about automorphic forms. These proofs are omitted completely, placing more emphasis on the theorems themselves.

1 BACKGROUND

Let G be a reductive group over a local non-archimedean field F (except where noted otherwise).

Remark. One important example is $G = GL(n, F)$, and $F = \mathbb{Q}_p$, i.e., the field of p-adic numbers.

1.1 Representations of p-adic groups

Definition. A **representation of a p-adic group** G is a homomorphism π of G into the group of linear automorphisms of a complex vector space V

$$\pi : G \to Aut(V)$$

(we shall consider only complex representations here; representations over other fields are also very important).

Quite often we shall think of the ordered pair (π, V) as the representation of G.

Example 1.1. Let $|\cdot|$ be the p-adic absolute value. Then $\nu = |\det|$ is a representation of $GL(n, F)$ on $\mathbb{C}$.

Definition. A subspace $U \leq V$ is said to be G**-invariant** (or G**-stable**) if

$$\pi(g)(u) \in U,$$

$\forall g \in G, \forall u \in U$.

Definition. The representation (π, V) is called **reducible** if there is a proper G-invariant subspace. Otherwise, we say that (π, V) is **irreducible**.

This naturally leads to the idea of a "sub"-representation, which will give us some idea of how representations are "built."

Definition. Suppose (π, V) is reducible and $U \leq V$ is a G-invariant subspace, $U \neq V$ or $\{0\}$. Define

$$\delta : G \to Aut(U)$$

by

$$\delta(g)(u) = \pi(g)(u),$$

$\forall g \in G, \forall u \in U$. Then (δ, U) is a representation of G. We call δ a **subrepresentation** of π.

Similar to how the notion of subspaces gave rise to the notion of subrepresentations, we have this idea of what a "quotient"-representation is based on the idea of a quotient space.

Definition. Suppose $U \leq V$ is a G-invariant subspace of V. We consider the exact sequence of spaces

$$0 \to U \to V \to V/U \to 0.$$

Define

$$\xi : G \to Aut(V/U)$$

by

$$\xi(g)(v + U) = \pi(g)(v) + U,$$

$\forall g \in G, \forall v \in V$. Then ξ is a representation of G on V/U. We say ξ is a **quotient of** π.

Building blocks

There is an important distinction to which we would like to draw the reader's attention. This idea of how representations may be "built" from "smaller" ones begs the question: What are our building blocks? Or more precisely, what representations are irreducible for a given V? It is important to note that when we think of "building" in this way, we are thinking of the ordered pair (π, V) as the representation of G (with more of an emphasis on the vector space V). Later, we will see that it is important to focus on how we "build" the group homomorphisms π, as well. This leads to the idea "induced"-representations or, in the context of $G = GL(n, F)$ or some subgroup of $GL(n, F)$, to the idea of "parabolically induced"-representations.

Recap: Let (π, V) be a representation of a group G.

- Attention on $V \rightsquigarrow$ reducibility-irreducibility.
- Attention on $\pi \rightsquigarrow$ induction (among other ideas).

However, these ideas are not unrelated in the case of p-adic groups. Here, induced representations play a very important role in the construction of irreducible representations. For example, one can get from one-dimensional representations of a subgroup, such as the trivial representation, interesting non-trivial representations of the group.

Smooth and admissible representations

Definition. Let (π, V) be a representation of G. A vector $v \in V$ is called a **smooth vector** if there exists an open subgroup $K \leq G$ such that

$$\pi(k)(v) = v,$$

$\forall k \in K$.

Definition. The set of all smooth vectors in V is denoted by V_∞ and called the **smooth part of** V. The smooth part of V is a G-invariant subspace of V.

Definition. Define (π, V) to be a **smooth representation of** G if $V_\infty = V$. A smooth one-dimensional representation χ is called a **character**.

Definition. Given a subgroup $K \leq G$, define

$$V^K = \{v \in V | \pi(k)v = v, \forall k \in K\}.$$

We say that (π, V) is an **admissible representation** if it is smooth and $\dim V^K < \infty$, for every open subgroup K of G.

1.2 Intertwining operators and Schurs lemma

Definition. An **isomorphism φ between two representations** (ρ_1, V_1) and (ρ_2, V_2) of G is a linear isomorphism $\varphi : V_1 \to V_2$ which intertwines with the action of G, i.e., satisfies

$$\varphi(\rho_1(g)(v)) = \rho_2(g)(\varphi(v)).$$

Definition. Note that the equality makes sense even if φ is not invertible, in which case it is just called an **intertwining operator** or **G-linear map**.

However, if φ is invertible, we can describe ρ_2 by the following conjugation:

$$\rho_2 = \varphi \circ \rho_1 \circ \varphi^{-1}.$$

Thus, we have an equality of linear maps after inserting any group element g.

Proposition 1.1. *Suppose (ρ_1, V_1) and (ρ_2, V_2) are irreducible representations of G. If $\varphi : V_1 \to V_2$ is an intertwining operator, then φ is a bijection or $\varphi = 0$.*

Lemma 1.2 (Schur's Lemma)**.** *Suppose (ρ, V) is an irreducible admissible representation of G. If $\varphi : V \to V$ is an intertwining operator, then φ is a scalar.*

2 PARABOLIC SUBGROUPS

2.1 Preliminary - Reductive Algebraic Groups

We will quickly recall some basic information concerning algebraic groups and, more specifically, reductive algebraic groups. For more information on algebraic groups we refer the reader to [Bor].

Let G be a connected reductive algebraic group.

- A **character** of an algebraic group G is any morphism of algebraic groups $\chi : G \to \mathbb{G}_m$, where $\mathbb{G}_m$ is the multiplicative group. If χ_1 and χ_2 are characters, we define their product $\chi_1\chi_2$ by

$$(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g).$$

  Notice that $\chi_1\chi_2$ is a character. We denote by $X(G)$ **the set of all characters of** G. $X(G)$ along with the given product forms an abelian group and in this context $X(G)$ is said to be **the abelian group of characters of** G.

- An algebraic group is called a **torus** if it is isomorphic to the group of all invertible diagonal $n \times n$ matrices, for some n.

- A **Borel Subgroup** of G is a maximal connected solvable subgroup of G.

- A closed subgroup P of G is called **parabolic** if G/P is a projective variety.

- **Levi Decomposition.** If P is a parabolic subgroup of G, then P has a decomposition

$$P = MU,$$

  where M is a reductive group and U is the unipotent radical of P. This decomposition is called the **Levi Decomposition** and M is called a **Levi factor** of P.

- **Standard Levis.** Once we have fixed a maximal torus T and Borel subgroup B containing the maximal torus, then we get a unique Levi decomposition $P = MU$, where M contains T. In this case, we call M the **standard Levi subgroup** of P.

2.2 Background on $Sp(2n, F)$

Let J_n be the $n \times n$ matrix with ones along the antidiagonal, i.e.

$$J_n = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{bmatrix}$$

and let J_+^- be the $2n \times 2n$ matrix

$$J_+^- = \left[\begin{array}{c|c} 0 & -J_n \\ \hline J_n & 0 \end{array}\right].$$

- The **symplectic group** $Sp(2n, F)$

$$Sp(2n, F) = \{g \in GL(2n, F) | {}^t g J_+^- g = J_+^-\},$$

where ${}^t g$ is the transpose of g.

- **Parabolic Subgroups of** $Sp(2n, F)$**.** In this paper, we will be working with symplectic representations induced from smaller symplectic representations. So we will be working over the standard parabolic subgroups of $Sp(2n, F)$. Here is a brief description of what these subgroups look like (it is important to remember that these are just the standard parabolic subgroups of $GL(2n, F)$, but then restricted to $Sp(2n, F)$):

Let S_n denote $Sp(2n, F)$. We take formally $Sp(0, F)$ to be the trivial group.

We fix in S_n the minimal parabolic subgroup $P_\emptyset^S$ which consists of all upper triangular matrices in the group. Let $M_\emptyset^S$ be the subgroup of all diagonal matrices in S_n. Then $M_\emptyset^S$ is a Levi factor of the standard minimal parabolic subgroup. It is also a maximal torus in S_n. In this setting, our parabolic subgroups are block upper triangular matrices of the form

$$P^S_{(m)} = \left\{ \begin{bmatrix} g & * & * \\ 0 & h & * \\ 0 & 0 & {}^\tau g^{-1} \end{bmatrix} \in S_n \,|\, g \in GL(m,F), h \in S_{n-m} \right\}.$$

Here ${}^\tau g$ denotes the transposed matrix of g with respect to the second diagonal.

2.3 Parabolic induction and Jacquet modules

We take a moment to recall a few general facts about parabolic induction (the reader can find a full treatment in the Casselman pre-print or the joint paper of Bernstein and Zelevinsky).

Fix a parabolic subgroup P of G and its Levi subgroup M, where $P = MU$. Then if we want to lift a smooth representation σ of M to a certain representation $Ind_M^G(\sigma)$ of G, we have a natural scheme for doing so by looking at the parabolic subgroups. For this reason we call $Ind_M^G(\sigma)$ the parabolically induced representation of σ.

- Denote by δ_P the modular character of P. Let V' be the set of all locally constant functions $f : G \to V$ satisfying

  $$f(mug) = \delta_P(m)^{1/2}\sigma(m)f(g),$$

  $\forall m \in M, \forall u \in U, \forall g \in G$. The group G acts by right translations:

  $$(R_g f)(x) = f(xg),$$

  $\forall g, x \in G, \forall f \in V'$. This defines a smooth representation $Ind_M^G(\sigma)$ of G, which is called the representation of G parabolically induced by σ from P.

- The requirement for this "normalizing" factor δ_P (the modular character of P) is so that parabolic induction carries unitarizable representations to unitarizable ones.

- Let $Ind_M^G(V) = V'$ and $Ind_M^G(\sigma)$ be the corresponding representation. This functor $Ind_M^G : AlgM \to AlgG$ is called **the functor of parabolic induction**.

- Parabolic induction commutes with contragredients:

$$\widetilde{Ind_M^G(\sigma)} \cong Ind_M^G(\tilde{\sigma})$$

Frobenius reciprocity for parabolically induced representations holds in the same way as for the finite groups. We first need to introduce another functor before we can make this idea more clear.

Let (π, V) be a smooth representation of a group G and have parabolic subgroup $P = MU$ of G. Let

$$V(U) = span_{\mathbb{C}}\{\pi(u)v - v : v \in V, u \in U\}.$$

Since M normalizes U, this space in M-invariant. Denote by

$$r_G^M(\pi)$$

the quotient representation of M on $V/V(U)$, twisted by $(\delta_P|_M)^{-1/2}$. That is,

$$r_G^M(\pi) = \delta_P^{-1/2}\pi_U,$$

where π_U is the natural quotient. We call $r_G^M(\pi)$ the Jacquet module of π with respect to the decompositon $P = MU$. The functor $r_G^M : AlgG \to AlgM$ is known as the Jacquet functor.

- The functors Ind_M^G and r_G^M are exact.

- (Frobenius Reciprocity) The functor r_G^M is left adjoint to Ind_M^G. In particular,

$$Hom_G(\pi, Ind_M^G(\sigma)) \cong Hom_M(r_G^M(\pi), \sigma)$$

for the representations π and σ of G and M respectively.

- We can induce in stages and we have transitivity of Jacquet modules: Let $P_1 = M_1U_1$ and $P_2 = M_2U_2$ such that $P_1 \leq P_2$. Then

$$Ind_{N_1}^{G} = Ind_{N_2}^{G} \circ Ind_{N_1}^{N_2}$$

and

$$r_G^{N_1} = r_G^{N_2} \circ r_{N_2}^{N_1}.$$

3 SQUARE INTEGRABLE REPRESENTATIONS

Definition. Let (π, V) be a smooth representation of G. On the dual space $V^* = \{f | f : V \to \mathbb{C}, \text{ linear}\}$ there exists a natural representation:

$$(\pi^*(g)v^*)(v) = v^*(\pi(g^{-1})v).$$

The set of all linear forms with open stabilizer is a (smooth) subrepresentation, which we denote by $(\tilde{\pi}, \tilde{V})$, and call the **contragredient** of (π, V).

Definition. Functions of the form

$$c_{v,\tilde{v}} : g \mapsto \tilde{v}(\pi(g)v),$$

$c_{v,\tilde{v}} : G \to \mathbb{C}$, for $v \in V, \tilde{v} \in \tilde{V}$, are called **matrix coefficients of the representation** (π, V).

Definition. Schur's lemma implies that for each smooth irreducible representation (π, V) there is a character ω_π of $Z(G)$ such that

$$\pi(z) = \omega_\pi(z) id_V$$

for all $z \in Z(G)$. The character ω_π is called the **central character of** π.

Definition. Suppose (π, V) is an admissible representation of G which has a unitary central character. Then π is said to be **square integrable** or a **discrete series representation** if the absolute values of all the matrix coefficients of π are square integrable functions on $G/Z(G)$. That is, the integral

$$\int_{Z\backslash G} |c_{v,\tilde{v}}(g)|^2 dg$$

is finite. If the center of G is compact, square integrable modulo center representations will be simply called square integrable.

Definition. Tempered representations are irreducible components of representations parabolically induced from discrete series (square integrable) representations.

4 NOTATION AND PRELIMINARY RESULTS

In this section, we introduce notation and recall some results that will be needed in the rest of the paper. Most of the results are from [J], [M-T], and [S-T].

Let F be a local non-archimedean field of characteristic zero. The topological modulus of F will be denoted by $|\cdot|_F$. As a homomorphism of $F^\times$, this character will be denoted by

$$\nu : F^\times \to \mathbb{R}^\times.$$

For two smooth representations π_1 of $GL(n_1, F)$ and π_2 of $GL(n_2, F)$, we denote by

$$\pi_1 \times \pi_2$$

the smooth representation of $GL(n_1 + n_2, F)$ parabolically induced by $\pi_1 \otimes \pi_2$ from the standard parabolic subgroup (with respect to the upper triangular matrices)

$$P_{(n_1,n_2)} = M_{(n_1,n_2)} N_{(n_1,n_2)}$$

whose Levi factor $M_{(n_1,n_2)}$ is naturally isomorphic to $GL(n_1, F) \times GL(n_2, F)$ (see [B-Z]).

For a smooth representation π of $GL(n, F)$ and σ of S_m we denote by

$$\pi \rtimes \sigma$$

the parabolically induced representation of S_{n+m} by $\pi \otimes \sigma$ from the parabolic subgroup

$$P_{(n)}^{S_m} = \left\{ \begin{bmatrix} g & * & * \\ 0 & h & * \\ 0 & 0 & {}^{\tau}g^{-1} \end{bmatrix} \in S_{n+m} \,|\, g \in GL(n,F), h \in S_m \right\}.$$

Here ${}^{\tau}g$ denotes the transposed matrix of g with respect to the second diagonal. To make the notation less cumbersome, we will say $P_{(n)}^{S}$ when $S = S_m$ is fixed.

4.1 Square integrable representations of general linear groups

The set of all equivalence classes or irreducible cuspidal representations of all $GL(n,F), n \geq 1$ will be denoted by C.

Definition. Let $\nu = |\det|_F$. Then ν is a character. For $\rho \in C, k \in \mathbb{N}$ the set

$$\Delta = [\rho,\, \nu^k\rho] = \{\rho,\, \nu\rho,\, \nu^2\rho, \ldots,\, \nu^k\rho\}$$

is called a **segment in** C.

The set of all such segments will be denoted S.

The induced representation

$$\nu^k\rho \times \nu^{k-1}\rho \times \cdots \times \rho$$

has a unique irreducible subrepresentation, which we denote by $\delta(\Delta)$ and a unique quotient, which we denote by $\zeta(\Delta)$.

Remark. For $\Delta \in S$, the representation $\delta(\Delta)$ is essentially square integrable, where

$$\delta(\Delta) \hookrightarrow \nu^k\rho \times \nu^{k-1}\rho \times \cdots \times \rho$$

Definition. A **balanced segment** is a segment of the form

$$[\nu^{-k}\rho,\, \nu^k\rho],$$

where $k \in \frac{1}{2}\mathbb{Z}$, $\rho \in C^u$ (C^u = unitary part of C).

Remark. Every square integrable representation π of $GL(n,F)$ is of the form $\delta(\Delta)$, for a balanced segment Δ.

4.2 Principal Series

Principal series representations of symplectic groups are representations of the form

$$\chi_1 \times \cdots \times \chi_k \rtimes 1_{S_0},$$

where χ_i are characters of $F^\times$. The representation is irreducible if and only if the following conditions hold

1. χ_i is not of order two $\forall i$
2. $\chi_i \neq \nu^{\pm 1}, \forall i$
3. $\chi_i \neq \nu^{\pm 1}{\chi_j}^{\pm 1}$, for $1 \leq i < j \leq k$.

4.3 $P_{(1)}$

For this parabolic subgroup, the cuspidal reducibilites were described by J.-L. Waldspurger in [W] (for $GSp(4)$) and later by F. Shahidi. We shall recall their description. Let $\sigma \in C^u(Sp(2))$ be irreducible and let $\chi = \nu^\alpha \chi_0$ be a character of $F^\times$, where $\alpha \in \mathbb{R}$ and χ_0 is a unitary character. The reducibility of $\chi \rtimes \sigma$ implies $\chi_0^2 = 1_{F^\times}$. Conversely, $\chi_0^2 = 1_{F^\times}$ implies reducibility for the same $\alpha \in \mathbb{R}$.

For $a \in F^\times$ consider the representation

$$\sigma_a : g \mapsto \sigma\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Denote

$$F_\sigma^\times = \{a \in F^\times, \sigma \cong \sigma_a\}$$

(each $\varphi \in (F^\times/F_\sigma^\times)^\wedge$ satisfies $\varphi^2 = 1_{F^\times}$). The list of all reducibility points of $\nu^\alpha\chi_0 \rtimes 1$ is:

1. $\chi_0 = 1_{F^\times}$ and $\alpha = 0$;

2. $\chi_0^2 = 1_{F^\times}, \chi_0 \notin (F^\times/F_\sigma^\times)^\wedge$ and $\alpha = 0$;

3. $\chi_0 \in (F^\times/F_\sigma^\times)^\wedge \backslash \{1_{F^\times}\}$ and $\alpha = \pm 1$.

4.4 $P_{(2)}$

Let $\rho = \nu^\alpha \rho_0$ be an irreducible cuspidal representation of $GL(2)$, with ρ_0 unitarizable and $\alpha \in \mathbb{R}$. To have reducibility ρ_0 must be selfdual, in the sense of being isomorphic to its contragredient. Conversely, $\rho_0 \cong \tilde{\rho_0}$ implies reducibility for some $\alpha \in \mathbb{R}$. If $\omega_{\rho_0} \neq 1_{F^\times}$, then the representation

$$\rho_0 \rtimes 1$$

of S_2 reduces and $\rho \rtimes 1 = \nu^\alpha \rho_0 \rtimes 1$ is irreducible for $\alpha \neq 0$. If $\omega_{\rho_0} = 1_{F^\times}$, then the representation

$$\nu^{\frac{1}{2}} \rho_0 \rtimes 1$$

of S_2 reduces and $\rho \rtimes 1 = \nu^\alpha \rho_0 \rtimes 1$ is irreducible for $\alpha \neq \pm\frac{1}{2}$.

4.5 Jantzen's Results

If ρ_0 is an irreducible unitarizable supercuspidal representation of $GL(p_0,\, F)$, then

$$\nu^{\frac{k-1}{2}} \rho_0 \times \nu^{\frac{k-1}{2}-1} \rho_0 \times \cdots \times \nu^{\frac{-k+1}{2}} \rho_0$$

has a unique irreducible subrepresentation $\delta(\rho_0,\, k)$.

Similarly, suppose that ρ is an irreducible unitarizable supercuspidal representation of $GL(p,\, F)$, and σ is an irreducible supercuspidal representation of S_m such that $\nu^{\pm\frac{1}{2}} \rho \rtimes \sigma$ (resp. $\nu^{\pm 1} \rho \rtimes \sigma$ is reducible and $\nu^\beta \rho \rtimes \sigma$ is irreducible $\forall \beta \in \mathbb{R}$ with $|\beta| \neq \frac{1}{2}$ (resp. $|\beta| \neq 1$). Then

$$\nu^{l-\frac{1}{2}} \rho \times \nu^{l-\frac{3}{2}} \rho \times \cdots \times \nu^{\frac{1}{2}} \rho \rtimes \sigma \ \ (\text{resp. } \nu^l \rho \times \nu^{l-1} \rho \times \cdots \times \nu^1 \rho \rtimes \sigma)$$

contains a unique irreducible subrepresentation which we denote $\delta(\rho,\, l;\, \sigma)$ (in either case).

In Jantzen's paper he looks at representations of the form

$$\nu^{\alpha}\delta(\rho_0, k) \rtimes \delta(\rho,\, l;\, \sigma),$$

$\alpha \in \mathbb{R}$. We adopt his notation and say that ρ satisfies (C0) (resp. (C1/2), (C1)) if ρ is an irreducible unitarizable supercuspidal representation of some $GL(p, F)$ satisfying

(C0) $\rho \rtimes \sigma$ is reducible and $\nu^{\beta}\rho \rtimes \sigma$ is irreducible $\forall \beta \in \mathbb{R}, \beta \neq 0$.

(C1/2) $\nu^{1/2}\rho \rtimes \sigma$ is reducible and $\nu^{\beta}\rho \rtimes \sigma$ is irreducible $\forall \beta \in \mathbb{R}, \beta \neq \pm\frac{1}{2}$.

(C1) $\nu\rho \rtimes \sigma$ is reducible and $\nu^{\beta}\rho \rtimes \sigma$ is irreducible $\forall \beta \in \mathbb{R}, \beta \neq \pm 1$.

Here are his reducibility theorems we will be needing:

Jantzen (C1/2) Theorem. Let ρ_0, ρ be irreducible unitarizable supercuspidal representations of $GL(p_0, F)$, $GL(p, F)$, resp.; σ an irreducible supercuspidal representation of S_m. Further, suppose that ρ satisfies (C1/2). Let $\pi = \nu^{\alpha}\delta(\rho_0, k) \rtimes \delta(\rho, l; \sigma), \alpha \in \mathbb{R}$.

1. Suppose $\rho_0 \cong \rho$. Then, π is reducible if and only if

$$\alpha \in \left\{-\frac{k}{2}, -\frac{k}{2}+1, ..., \frac{k}{2}\right\} \cup \left\{\pm(l+\frac{k}{2}), \pm(l+\frac{k}{2}-1), ..., \pm(l-\frac{k}{2}+1)\right\}$$

(noting that the sets are not necessarily disjoint) with the exception that if $k = 2l$ and $\alpha = 0$, there is irreducibility (i.e., $\delta(\rho, 2l) \rtimes \delta(\rho, l; \sigma)$ is irreducible).

2. Suppose $\rho_0 \ncong \rho$. Then, π is reducible if and only if $\nu^{\alpha}\delta(\rho_0, k) \rtimes \sigma$ is reducible.

Remark. For part two of the previously stated theorem:
If $\rho_0 \ncong \widetilde{\rho_0}$ then $\nu^{\alpha}\delta(\rho_0, k) \rtimes \sigma$ is irreducible.
Jantzen also considers the cases where ρ_0 satisfies (C1/2), (C1), or (C0), resp., and then gives the reducibility points of $\nu^{\alpha}\delta(\rho_0, k) \rtimes \sigma$ in his paper.

Here is the result we will need:

Proposition 4.1 (C1/2)**.** *Suppose σ is an irreducible supercuspidal representation of S_m and ρ is a representation of $GL(p,F)$ satisfying (C1/2). Let*

$$\pi = \delta(\rho, n) \rtimes \sigma, \quad n \geq 2.$$

Then π is irreducible if and only if $n \in 1 + 2\mathbb{Z}$.

Jantzen (C1) Theorem. Let ρ_0, ρ be irreducible unitarizable supercuspidal representations of $GL(p_0, F)$, $GL(p, F)$, resp.; σ an irreducible supercuspidal representation of S_m. Further, suppose that ρ satisfies (C1). Let $\pi = \nu^{\alpha}\delta(\rho_0, k) \rtimes \delta(\rho, l; \sigma), \alpha \in \mathbb{R}$.

1. Suppose $\rho_0 \cong \rho$. Then, π is reducible if and only if
$$\alpha \in \left\{ \frac{-k+1}{2}, \frac{-k+1}{2} + 1, ..., \frac{k-1}{2} \right\}$$
$$\cup \left\{ \pm(l + \frac{k+1}{2}), \pm(l + \frac{k+1}{2} - 1), ..., \pm(l + \frac{-k+3}{2}) \right\}$$
(noting that the sets are not necessarily disjoint) with the exception that if $k = 2l+1$ and $\alpha = 0$, there is irreducibility (i.e., $\delta(\rho, 2l+1) \rtimes \delta(\rho, l; \sigma)$ is irreducible).

2. Suppose $\rho_0 \ncong \rho$. Then, π is reducible if and only if $\nu^{\alpha}\delta(\rho_0, k) \rtimes \sigma$ is reducible.

Remark. For part two of the previously stated theorem:
If $\rho_0 \ncong \widetilde{\rho_0}$ then $\nu^{\alpha}\delta(\rho_0, k) \rtimes \sigma$ is irreducible.
Jantzen also considers the cases where ρ_0 satisfies (C1/2), (C1), or (C0), resp., and then gives the reducibility points of $\nu^{\alpha}\delta(\rho_0, k) \rtimes \sigma$ in his paper.

Here is the result we will need:

Proposition 4.2 (C1)**.** *Suppose σ is an irreducible supercuspidal representation of S_m and ρ is a representation of $GL(p,F)$ satisfying (C1). Let*

$$\pi = \delta(\rho, n) \rtimes \sigma, \quad n \geq 2.$$

Then π is irreducible if and only if $n \in 2\mathbb{Z}$.

4.6 Mœglin and Tadić's Results

In a paper by Colette Mœglin certain invariants which play an important role in the theory of automorphic forms are described. In a joint paper with Marko Tadić a further description of these invariants is given. They have shown that under a basic assumption (which is very natural), these invariants classify irreducible square integrable representations of classical p-adic groups (modulo cuspidal data).

Here is a brief description of these invariants:

Let π be an irreducible square integrable representation of a classical p-adic group. To π Mœglin attaches a triple

$$(Jord(\pi), \epsilon_\pi, \pi_{cusp}).$$

Such triples satisfying certain requirements are called **admissible triples**. For the sake of simplicity, we will only give the description in the case of symplectic groups.

Let ρ be an irreducible supercuspidal representation of $GL(p, F)$. Denote by $\delta(\rho, n)$ the unique irreducible subrepresentation of the balanced segment representation

$$[\nu^{-\frac{n-1}{2}}\rho, \nu^{\frac{n-1}{2}}\rho].$$

Let θ be an irreducible squre integrable representation of S_q. We shall now consider the parabolically induced representation

$$\pi = \delta(\rho, n) \rtimes \theta,$$

induced from a suitable parabolic subgroup.

Definition. Let θ be an irreducible squre integrable representation of S_q. $Jord(\theta)$ can be defined as a set of all pairs (ρ, n) such that

- $\rho \cong \tilde{\rho}$,
- π is irreducible,

- $\pi_k = \delta(\rho, n+2k) \rtimes \theta$ is reducible for some $k \in \mathbb{N}$.

Definition (Basic Assumption (for S_q))**.** Let ρ be an irreducible self-dual supercuspidal representation of $GL(p,F)$ and let σ be an irreducible supercuspidal representation of S_q. Then

$$\nu^\alpha \rho \rtimes \sigma$$

reduces for some $\alpha \geq 0$. This α is unique and is denoted by $\alpha(\rho,\sigma)$. In particular, $\alpha(\rho, 1_{S_0}) \in \{0, \frac{1}{2}, 1\}$. The basic assumption in the case of S_q is equivalent to

$$\alpha(\rho,\sigma) - \alpha(\rho, 1_{S_0}) \in \mathbb{Z}.$$

Also, this yields the following dimension relation:

$$\sum_{(\rho,a)\in Jord(\pi)} a \dim \rho = 2q+1; \quad Sp(2q)^\wedge = SO(2q+1),$$

where $\dim \rho$ is the dimension of ρ in the sense of the segment length.

4.7 Sally and Tadić's Results

Irreducible Square Integrable Representations for GS_2 with Minimal Parabolic Support. In keeping with the notation for Sp_4, we denote GSp_4 by GS_2.

1. The representation $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$, where $\sigma \in (F^\times)^\wedge$, has a unique subrepresentation which will be denoted σSt_{GS_2}. This subrepresentation is square-integrable. For different σ's we get subrepresentations which are not equivalent.

2. For each character $\xi_0 \in (F^\times)^\wedge$ of order two and each $\sigma \in (F^\times)^\wedge$, the representation $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$ has a unique irreducible subrepresentation. Denote it by $\delta([\xi_0,\, \nu\xi_0],\, \nu^{-1/2}\sigma)$. This representation is square integrable. The only non-trivial equivalences among such representations are

$$\delta([\xi_0,\, \nu\xi_0],\, \nu^{-1/2}\sigma) \cong \delta([\xi_0,\, \nu\xi_0],\, \nu^{-1/2}\xi_0\sigma).$$

The square integrable representations defined in (i) and in (ii) are disjoint groups of representations. They exhaust all square integrable representations of GS_2 which are supported in the minimal parabolic subgroups.

Irreducible Square Integrable Representations for S_2 with Minimal Parabolic Support. Let $\sigma \in S_2^{\sim}$. Then σ is isomorphic to a subrepresentation of $\pi|_{S_2}$ for some $\pi \in GS_2^{\sim}$. Moreover, if σ is square-integrable (resp. tempered, unitary, supercuspidal), then one may choose π to be square-integrable (resp. tempered, unitary, supercuspidal).

1. For each $\xi_0 \in (F^\times)^\wedge$ of order two, the representation $\nu\xi_0 \times \xi_0 \rtimes 1$ has exactly two irreducible subrepresentations. They are square integrable and they are not equivalent. Denote them by $\delta'(\xi_0)$ and $\delta''(\xi_0)$. Then we have

$$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)|_{S_2} = \delta'(\xi_0) \oplus \delta''(\xi_0).$$

2. If δ is an irreducible square integrable representation of S_2 which is supported in the minimal parabolic subgroups, then δ is either the Steinberg representation or it is a representation considered in (1). We have

$$\sigma St_{GS_2}|_{S_2} \cong St_{S_2}.$$

Irreducible Square Integrable Representations for S_2 Supported in $P^S_{(1)}$. Let $\sigma \in C^u(SL(2,F))$ and let χ be a character of $F^\times$ of order two which belongs to $(F^\times/F^\times_\sigma)^\wedge$. Then $\nu\chi \rtimes \sigma$ contains a unique irreducible subrepresentation. This subrepresentation is irreducible. For different pairs (σ, χ) as above, one gets square integrable subrepresentations which are not isomorphic. Each irreducible square integrable representation of S_2 supported in $P^S_{(1)}$ is isomorphic to some square integrable representation as above.

Irreducible Square Integrable Representations for S_2 Supported in $P^S_{(2)}$. Let $\rho \in C^u(GL(2,F))$. Suppose that $\rho \cong \tilde{\rho}$ and $\omega_\rho = 1_{F^\times}$. Then $\nu^{1/2}\rho \rtimes 1_{F^\times}$ contains a unique irreducible subrepresentation. This subrepresentation is square integrable. For different ρ as above, we get square integrable representations which are not isomorphic. Each square integrable representation which is supported in $P^S_{(2)}$, is isomorphic to a square integrable representation as above.

5 RESULTS

We first consider discrete series representations supported in the minimal parabolic subgroup (description given by Sally and Tadić). First, consider the Steinberg representation

$$St_{S_2} = \delta(1_{F^\times}, 2; 1_{S_0}) = \delta(\nu^2 \times \nu \rtimes 1_{S_0}) \hookrightarrow \nu^2 \times \nu \rtimes 1_{S_0}$$

Theorem 5.1. *$Jord(St_{S_2}) = \{(1_{F^\times}, 5)\}$.*

Proof. The representation $\nu^\beta 1_{F^\times} \rtimes 1_{S_0}$ of S_1 is reducible if and only if $\beta = \pm 1$. Thus, $1_{F^\times}$ satisfies (C1). Then by the exceptional case of part 1 of the (C1) theorem we have

$$\delta(1_{F^\times}, 5) \rtimes St_{S_2}$$

is irreducible, and thus $(1_{F^\times}, 5) \in Jord(St_{S_2})$ if indeed $\delta(1_{F^\times}, 5 + 2k) \rtimes St_{S_2}$ is reducible for some k.

Notice that according to the natural hypothesis assumed by Mœglin and Tadić, $(1_{F^\times}, 5)$ should be the only element (if it is an element, which it is). To show this we need to find $k \in 1 + 2\mathbb{Z}$ such that $\alpha = 0$ and

$$\pi = \nu^\alpha \delta(\rho, k) \rtimes St_{S_2}$$

is reducible ($\rho = 1_{F^\times}$), and also we need to consider part 2 of Jantzen's (C1) theorem ($\rho \neq 1_{F^\times}$). It is clear, that $k = 5$ is the only solution to the first part. For the second part we have

$$\pi \text{ is reducible} \iff \delta(\rho, k) \rtimes 1_{S_0} \text{ is reducible}$$

which is always the case since $\delta(\rho, k)$ is a balanced segment. More specifically, $\chi_{i+1} = \nu^{\pm 1}\chi_i, \forall i$.

Therefore, $Jord(St_{S_2}) = \{(1_{F^\times}, 5)\}$. $\square$

Now we consider irreducible square-integrable representations of $Sp(4)$ supported in $P^S_{(1)}$. Sally and Tadić showed that each irreducible square-integrable representation of S_2 supported in $P^S_{(1)}$ is isomorphic to some square-integrable representation of the form

$$\theta_{(1)} = \delta(\chi, 1; \sigma)$$

where $\sigma \in C^u(SL(2, F))$ and χ is a nontrivial character of $F^\times$ of order two which belongs to $(F^\times / F^\times_\sigma)^\wedge$. Moreover, for different irreducible square-integrable representations supported in $P^S_{(1)}$ the pairs (χ, σ) are determined uniquely.

Theorem 5.2. *We have*

$$Jord(\theta_{(1)}) = \{(\chi, 3)\} \cup \bigcup_{\rho \not\cong \chi} (\{\rho\} \times Jord_\rho(\sigma)).$$

Proof. By the reducibility results of Walspurger/Shahidi in 4.2, χ satisfies (C1) relative to σ. Following the notation of the (C1) theorem we let $\pi = \nu^\alpha \delta(\rho, k) \rtimes \theta_{(1)}$. First consider case 1 of the (C1) theorem: We have $\rho \cong \chi, \alpha = 0$ and by the exceptional subcase we have that π is irreducible when $k = 3$. Hence $(\chi, 3) \in Jord(\theta_{(1)})$.

In the other subcase we get π -irreducible when $k \in 2\mathbb{Z}^+$. But since we have irreducibility for $k = 2m$ and for $k = 2m + 2n, \forall n$, all of those k are excluded from $Jord(\theta_{(1)})$ by definition. Having exhausted (C1) case 1 we now consider case 2 of the same theorem.

In case 2 we have $\pi = \nu^\alpha \delta(\rho, k) \rtimes \theta_{(1)}$ where $\rho \not\cong \chi$ and we again are only concerned with $\alpha = 0$. By the (C1) theorem π is reducible when $\delta(\rho, k) \rtimes \sigma$ is reducible. Thus, $\{\rho\} \times Jord_\rho(\sigma) \in Jord(\theta_{(1)})$ by the definition of $Jord_\rho(\sigma)$. $\square$

Remark. Notice that if we assume that ρ from the above theorem also satisfies (C1) or (C0) then $\delta(\rho, k) \rtimes \sigma$ is irreducible for all even values of $k \geq 2$ by Jantzen's propositions on reducibility points. Thus, we can say that k must be odd. In fact, for both cases, $k = 1$. This follows from the fact that $\delta(\rho, k) \rtimes \sigma$ is reducible for all odd values of $k \geq 3$ by Jantzens's reducibility propositions [see pages 16,25,27].

Thus, after exhausting all choices of σ we have

$$Jord(\theta_{(1)}) = \{(\chi, 3)\} \cup \bigcup_{\rho_\sigma \ncong \chi} \{(\rho_\sigma, 1)\}.$$

Now we consider irreducible square-integrable representations of $Sp(4)$ supported in $P^S_{(2)}$. Sally and Tadić showed that each irreducible square-integrable representation of S_2 supported in $P^S_{(2)}$ is isomorphic to some square-integrable representation of the form

$$\theta_{(2)} = \delta(\rho, 1; 1_{S_0})$$

where $\rho \in C^u(GL(2, F)), \rho \cong \tilde{\rho}, \omega_\rho = 1_{F^\times}$, and $\delta(\rho, 1; 1_{S_0}) \hookrightarrow \nu^{1/2}\rho \rtimes 1_{S_0}$. For different ρ we have different $\theta_{(2)}$ and this list is exhaustive of the square integrable representations with support $P^S_{(2)}$.

Theorem 5.3. *We have*

$$Jord(\theta_{(2)}) = \{(\rho, 2), (1_{F^\times}, 1)\}.$$

Proof. The proof is of the same style as the previous proofs.

We have that ρ satisfies (C$\frac{1}{2}$) relative to 1_{S_0}. Let $\pi = \nu^\alpha \delta(\rho_0, k) \rtimes \delta(\rho, 1; 1_{S_0})$. We will check reducibilities given by case 1 of the (C1/2) theorem first.

In the exceptional case we get straight away that $(\rho, 2) \in Jord(\theta_{(2)})$, since for $\alpha = 0$ and $k = 4$ we get that π is reducible. For $\alpha = 0$ and k odd we get that π is irreducible in case 1 of the (C1/2) theorem. But since it is irreducible for all odd k we have $(\rho_0, k) \notin Jord(\theta_{(2)}), \forall k \in 2\mathbb{Z}^+ + 1$.

In case 2 we have $\rho_0 \ncong \rho$. We also have the relation

$$\pi \text{ is reducible } \iff \nu^\alpha \delta(\rho_0, k) \rtimes 1_{S_0} \text{ is reducible}$$

in case 2.

First, we will treat the case where ρ_0 is a representation of $GL(p_0, F)$, where $p_0 > 1$. We have that

$$\delta([\nu^{-\frac{k-1}{2}}\rho_0, \nu^{\frac{k-1}{2}}\rho_0]) \rtimes 1_{S_0}$$

is reducible since $\nu^j\rho_0 \rtimes 1_{S_0}$ is reducible, for some j.

Now consider ρ_0 a representation of $GL(1, F)$. Again we have

$$\delta([\nu^{-\frac{k-1}{2}}\rho_0, \nu^{\frac{k-1}{2}}\rho_0]) \rtimes 1_{S_0}$$

is reducible since $\nu^j\rho_0 \rtimes 1_{S_0}$ is reducible, for some j, except in the case $\rho_0 = 1_{F^\times}$ and $k = 1$.

Thus, $Jord(\theta_{(2)}) = \{(\rho, 2), (1_{F^\times}, 1)\}$.

$\square$

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