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Finite Element Analysis: Mathematical Theory and Applications

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FINITE ELEMENT ANALYSIS

MATHEMATICAL THEORY AND APPLICATIONS.

by

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Submitted in Partial Fulfillment of the Requirements for the
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This paper discusses the mathematical theory of finite elements. Using the concepts of inner product spaces, the mathematics of finite element analysis is explained in the context of function spaces. The finite elements as well as the finite element space will be rigorously defined. Furthermore, examples from research in the field of engineering will be explained and viewed through a mathematical lens. This research paper seeks to firmly bridge the applications with mathematical content for the research scientist who generally focuses on the applications. The paper will begin with a review of some basic building blocks for the finite element space followed by validation of the process and construction of finite elements. The use of these elements will be compared. A trial will be done on the computer using a simple differential equation where the use of different finite elements will be investigated for accuracy to validate the roof functions used in finite element analysis. Finally a short review of literature illustrating the technique will be presented.

DEDICATION

This research paper is dedicated to my eldest brother and sister, Sheila Lewis and Shelby Wilson. The two of you have helped me in many ways over the years. When I decided to leave my full time job and go back to school, the two of you supported that decision whole heartedly. I love you both.

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CHAPTER 1

MATHEMATICAL THEORY

1.1 FUNCTION SPACES

Finite Element Analysis has become an indispensible tool for many engineers and other scientists. The sophistication of the method, its accuracy, simplicity, and computability all make it a widely used tool in the engineering modeling and design process. This paper will discuss finite element analysis from mathematical theory to applications. For purposes of analysis of the method, it is easier to study theory along side applications. This hopefully gives the reader an opportunity to draw direct connections between application and theory, putting the mathematics into context. For the basis of understanding the mathematical theory, we will utilize a one dimensional problem. However, all the concepts and proofs can be easily transformed to multidimensional situations with a few adjustments.

In many cases, the solution to even second order differential equations can be quite complicated and an alternative method to computing an exact answer would be needed. Finite element analysis provides the tools necessary to approximate the solution. We will consider a simple example to help illustrate the theory. Consider, the second-order linear differential equation $-y''+y=x$ on the domain $[0,1]$. In this case an exact solution using traditional methods for solving differential equations can be found. Applying methods traditionally learned in standard differential courses we find that the exact solution that satisfies $y(0)=0$ and $y(1)=0$ to be $y(x) = x - \frac{sinh(x)}{sinh(1)}$. However, we will utilize this example for explanation purposes and as a comparison for the accuracy of the method.We will return to this equation in section chapter 4.

Before we begin, let us build the mathematical framework and key ideas needed for the theoretical foundation of finite element anlaysis. Consider the following definitions.

Definition. (Vector Space) Let V be a set. Let F be a field of scalars. We call the set V a vector space over F if V is an abelian group under an operation, which we will denote '+' and for every $\lambda, \mu \in F$ and for every $v_1, v_2 \in V$

(i) $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$

(ii) $(\lambda + \mu)v_1 = \lambda v_1 + \mu v_1$

(iii) $\lambda(\mu v_1) = (\lambda\mu)v_1$

(iv) $1v_1 = v_1$

For purposes of our work, all vector spaces will be over the real numbers.

Definition. (Inner Product) Let V be a vector space. Let $\langle *, * \rangle$ be a real valued function $\langle *, * \rangle : V \times V \to \mathbb{R}$,defined on $V \times V$. We call $\langle *, * \rangle$ an inner product if the following criteria are met: For all $v_1, v_2 \in V$ and for all $\lambda \in \mathbb{R}$,

(i) $\langle v_1, v_1 \rangle \geq 0$ with equality $\Leftrightarrow v_1 = 0$

(ii) $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$

(iii) $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$

(iv) $\langle \lambda v_1, v_2 \rangle = \lambda \langle v_1, v_2 \rangle$

We call the vector space V together with the inner product $(V, \langle, \rangle)$ an inner product space.

Definition. (Norm) Let V be a vector space. Let $\|*\|$, $\|*\| : V \to \mathbb{R}$ be a real valued function defined on the set V. We call the function $\|*\|$ a norm if the following criteria are met:

For all $v_1, v_2 \in V$ and for all $\lambda \in \mathbb{R}$

(i) $\|v_1\| \geq 0$ with equality $\Leftrightarrow v_1 = 0$

(ii) $\|\lambda v_1\| = |\lambda| \, \|v_1\|$

(iii) $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

We call the vector space V together with a defined norm $(V, \|*\|)$ a normed vector space.

Let's take a second to solidify some of these definitions with concrete examples.

Example 1.1.1. An example of a vector space is the set of $m \times n$ matrices. Using addition and scalar multiplication defined component wise, one can easily check that the set of all $m \times n$ matrices meet the criteria for a vector space.

Example 1.1.2. An example of a norm is the absolute value function on the reals.

Example 1.1.3. An example of an inner product is $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Here we use the Riemann Integral to define an inner product on a function space.

Proposition 1.1.1. *Let $(H, \langle , \rangle)$ be an inner product space. Then H becomes a normed inner product space with $\|h\| = \sqrt{\langle h, h \rangle}$. We will call this the norm induced by the inner product.*

Proof. To prove the first property of norm, $\|h\| \geq 0$ for all $h \in H$ with equality if and only if $h = 0$. Suppose that $\|h\| = \sqrt{\langle h, h \rangle}$. Then, $\|h\|^2 = \langle h, h \rangle$. By application of property (i) of inner products, $\|h\|^2 = \langle h, h \rangle \geq 0$. By properties of square root function $\|h\| \geq 0$. Now suppose that $\|h\| = 0$. Then, $\|h\|^2 = \langle h, h \rangle = 0$. By property (i) of inner products $\langle h, h \rangle = 0 \, if \; and \; only \; if \; h = 0$.

To prove the second property of norm, $\|\lambda h\| = |\lambda| \, \|h\| \;\; \forall \lambda \in \mathbb{R}, \;\; \forall h \in H$, notice that, $\|\lambda h\| = \sqrt{\langle \lambda h, \lambda h \rangle}$. Then, by property (iv) of inner products, $\sqrt{\lambda^2 \langle h, h \rangle} = \sqrt{\lambda^2}\sqrt{\langle h, h \rangle} = |\lambda| \, \|h\|$.

To prove third property of norms (triangle inequality), $\|h_1 + h_2\| \leq \|h_1\| + \|h_2\|$ $\forall h_1, h_2 \in H$, notice, $\|h_1 + h_2\|^2 = \langle h_1 + h_2, h_1 + h_2 \rangle$. By property (ii) and (iii) of inner products,

$$\begin{aligned} \|h_1 + h_2\|^2 &= \langle h_1, h_1 \rangle + 2 \langle h_1, h_2 \rangle + \langle h_2, h_2 \rangle \\ &\leq \langle h_1, h_1 \rangle + 2 |\langle h_1, h_2 \rangle| + \langle h_2, h_2 \rangle \end{aligned}$$

Thus, by the Cauchy-Schwarz inequality,

$$\langle h_1, h_1 \rangle + 2 |\langle h_1, h_2 \rangle| + \langle h_2, h_2 \rangle \leq \|h_1\|^2 + 2 \|h_1\| \|h_2\| + \|h_2\|^2$$

$$\|h_1\|^2 + 2 \|h_1\| \|h_2\| + \|h_2\|^2 = (\|h_1\| + \|h_2\|)^2$$

Now we have the following inequality, $(\|h_1 + h_2\|)^2 \leq (\|h_1\| + \|h_2\|)^2$. Taking the square root of both sides of the equation gives

$\|h_1 + h_2\| \leq \|h_1\| + \|h_2\|$.

1.2 VARIATIONAL FORM OF DIFFERENTIAL EQUATIONS

Before we begin, let's first describe some notation. In what follows C will denote the set of all continuous functions and L will denote the set of all polynomials of degree 1. Superscripts will denote the order of the smoothness of the function. For example $C^2([0,1])$ is the set of all second order differentiable functions defined on the closed interval $[0,1]$.

When we discuss the weak or variational form of an object in mathematics it generalizes the standard form and exists in context where the standard form may not. For example the weak derivative of a function may exist where the standard form does not. In addition, in many cases the adjective 'weak' implies the use of the inner product. For example, we say a sequence of vectors (v_n) in a vector space converges weakly to the vector $w \in V$ if and only if $\lim_{n \to \infty} \langle v_n, u \rangle = \langle w, u \rangle$ for all vectors $u \in V$.

To restate our problem in its weak or variational form we will utilize the properties of the inner product. For purposes of what is to follow, we will use two inner products. The inner product which we will define as $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$, standard integration and the 'energy' inner product defined as $\langle f, g\rangle_e = \int_0^1 f'(x)g'(x)dx$ where $'$ represents the derivative operator.

Suppose there is a linear differential relationship between the two functions y and f which exist in the function space H. Suppose we consider non-zero functions $h \in H$ where $H = \{h \in L_2(0,1) \ s.t.\ h(0) = 0 \text{ and} \langle \text{h}, \text{h}\rangle < \infty\}$. Suppose the following inner product relationship is true, $\langle Ly, h\rangle = \langle f, h\rangle$ where L is some defined linear operator (differential operator in the case of differential equations), y is an undetermined differentiable function, and f is some determined function involved in the differential relationship. Using what we know about inner products we can restate the problem of finding y as the following:

Find some $y \in H$ such that $\langle Ly, h\rangle - \langle f, h\rangle = \langle Ly - f, h\rangle = 0 \ \forall\, h \in H$.

This allows the opportunity to utilize a systematic check/search of all possible functions, meeting the given criteria, that may solve the relationship or equation.

Proposition 1.2.1. *Let $\langle *, *\rangle_e$ be defined on $C^2([0,1])$ where $\langle f, g\rangle_e := \int_0^1 f'(x)g'(x)dx$. Then $\langle *, *\rangle_e$ is an inner product on the set H defined above. We will refer to this inner product as the energy inner product.*

Proof. To prove property (i) of inner product, suppose $h_1(x) \in C^2([0,1])$ with $h(0) = 0$. Then, $(h_1'(x))^2$ is non-negative and continuous. Thus, $\int_0^1 (h_1'(x))^2 dx \geq 0$. Now suppose $\int_0^1 (h_1'(x))^2 = 0$. This implies that $h_1'(x) = 0$, for all $x \in [0,1]$, which implies that $h_1(x)$ is constant. Since $h_1(0) = 0$, then $h_1(x) = 0$ for all $x \in [0,1]$.

Note: The inner product described here is only valid for a specific set of functions. The Neumann and Dirichlet boundary conditions become important assumptions for our analysis. Here we have assumed that our function was such that $h(0) = 0$ and $h(1) = 0$

To prove property (ii) of inner product, notice $\langle h_1, h_2\rangle_e = \int_0^1 h_1'(x)\, h_2'(x)dx$.

By commutativity of multiplication for functions

$$\int_0^1 h_1'(x)\, h_2'(x)dx = \int_0^1 h_2'(x)h_1'(x)dx = \langle h_2, h_1 \rangle_e.$$

To prove property (iii) of inner product, notice that

$$\langle h_1 + h_3, h_2 \rangle_e = \int_0^1 (h_1' + h_3')(x)h_2'(x)dx = \int_0^1 h_1'h_2'(x) + h_3'h_2'(x)dx$$

. Using the linearity of integration we have

$$\int_0^1 h_1'h_2'(x) + h_3'h_2'(x)dx = \int_0^1 h_1'h_2'(x)dx + \int_0^1 h_3'h_2'(x)dx = \langle h_1, h_2 \rangle_e + \langle h_3, h_2 \rangle_e$$

To prove property (iv) of inner product,

$$\langle \lambda h_1, h_2 \rangle_e = \int_0^1 \lambda h_1'h_2'(x)dx = \lambda \int_0^1 h_1'h_2'(x)dx = \lambda \langle h_1, h_2 \rangle_e$$

To consider an approximate solution to any problem in FEA we must have some criteria for measuring the closeness of our approximations. To do this we need a measure of distance. In mathematics one way to generate a measure of distance is the norm. In a normed vector space $d(u, v) = \|u - v\|$. It was shown previously that the described relationship $\|h\| = \sqrt{\langle h, h \rangle}$ is indeed a norm. We will utilize this norm within our calculations. Based on the previous proof we can easily show that $\|h\|_E = \sqrt{\langle h, h \rangle_e}$ is also a valid norm since $\langle , \rangle_e$ also meets the criteria for being an inner product.

Let us illustrate this theory in the context of an example. Consider, $-y'' = x$ We can rewrite this problem in terms of a differential operator D, as $(-D^2)y = x$, or equivalently, $Ly = f$ where L represents the linear operator $-D^2$ and $f = x$. To restate the problem in its varitional or weak form, suppose there is a non zero function $h \in H$ where $H = \{h \in L^2(0, 1) \text{ s.t. } h(0) = 0 \text{ and } \langle h, h \rangle_e < \infty\}$. Suppose the following inner product relationship is true, $\langle Ly, h \rangle = \langle f, h \rangle$. Then by properties of inner products $\langle Ly, h \rangle - \langle f, h \rangle = \langle Ly - f, h \rangle = 0$. Using our defined inner product we find that $\int_0^1 (-y'' - x) * hdx = 0$. So using inner products the problem can be restated in the following way:

Find a $y \in H$ such that $\int_0^1 (-y'' - x)hdx = 0$ for all $h \in H$.

Utilizing integration by parts and applying the boundary conditions, we find that $\int_0^1 -y''hdx$ becomes $\int_0^1 y'\, h'dx$ - $\int_0^1 x\, h\, dx = 0$. This can be written in terms the two defined inner products as $\langle y, h \rangle_e - \langle x, h \rangle = 0$

Theorem 1.2.2. *Suppose $y \in C^2([0,1])$ and $f \in C^0([0,1])$, with $y(0) = 0$. Let y be such that $\langle y, h \rangle_e = \langle f, h \rangle$ for all $h \in H$, the function space defined previously. Then y solves the equation, $-y'' = f$.*

Proof: Let h be contained in the space $H \cap U_1 \subset U$ where U_1 is the set of functions with continuous 1st order derivatives. Suppose h meets all boundary requirements. Suppose $\int_0^1 fhdx = \int_0^1 y'h'dx$. Using integration by parts with $\int_0^1 y'h'dx$ where $w = y'$, $dz = h'dx$, $dw = y''$ and $z = h$, we find that $\int_0^1 y'h'dx = y'h|_0^1 - \int_0^1 hy''dx = y'(1)h(1) - \int_0^1 hy''dx$. Since $y(0) = 0$ and we assumed that $h(x)$ has the same boundary conditions then $h(0) = 0$. This statement must be true for all h in the subspace with $h(0) = h(1) = 0$. Finally we have that $\int_0^1 y'h'dx = \int_0^1 y''hdx = \langle -y'', h \rangle$. Using what we know about inner products we find that $\langle f, h \rangle - \langle -y'', h \rangle = 0$ and $\langle f - (-y''), h \rangle = 0$. Let $w = f + y''$, then w is in $C^0([0,1])$. Suppose $w \neq 0$. Since w is continuous we find that $\exists\, (x_0, x_1)\ s.t.\, \forall\, x\ \in (x_0, x_1)$ either $w(x) > 0$ or $w(x) < 0$.

Then notice that $h(x) = (x - x_0)^2 (x - x_1)^2$ on $[x_0, x_1]$ with $h(x) = 0$ outside of the given interval, meets the requirements for being in H. Now we have the following:

(i) $w(x)$ is continuous by continuity of $-y''$ and f

(ii) $h(x)$ is such that $\langle h, h \rangle_e < \infty$. Notice that after reduction of h and application of chain and multiplication rules for derivatives, you obtain a fourth degree polymomial, which has finite 'area' on any finite sub interval.

(iii) $h(x)$ is continuous

Now consider $\langle w, h\rangle$. We find that,

$$\int_0^1 (f + y'')(x - x_0)^2(x - x_1)^2 dx = \int_{x_0}^{x_1} (f + y'')(x - x_0)^2(x - x_1)^2 dx \ > \ 0$$

Since this is not equal to zero, we must have a contradiction and w must be equal to zero. This implies that $f + y'' = 0$ and that $f = -y''$.

Remark. In the previous proof we utilized the assumption that $y(0) = 0$ which is the essential or Dirichlet boundary condition. A similar proof, can be done to prove that the approximation solves the original equation with natural or Neumann boundary condition of $y'(1) = 0$.

Now that we have established that the weak form solves the original equation, we can utilize the weak form of the differential equation to estimate local solutions using peice-wise polynomial approximations in the finite element method. However, before we do this we must verify the uniqueness of our proposed solution. Previously, we have considered infinite dimensional spaces. For purposes of finite element analysis we will truncate our dimensions. Consider the set $S_n \subset H$ a finite dimensional subspace of H. We can consider a finite degree polymonial approximation to each of our involved quantities. We can write a new statement as follows:

Find $y_s \in S_n$ such that y_s satisfies, $\int_0^1 y_s' h_s' + y_s h_s \, dx = \int_0^1 x h_s \, dx \; \forall h_s \in S_n$.

As we build on our analysis we can begin to specify characteristics of our subspaces, keeping in mind that all subspaces must contain the same properties of the original space.

Proposition 1.2.3. *Suppose $\langle y_s, h_s\rangle = \langle f, h_s\rangle$ for all h_s in the subspace S_n. Then for any given function $f \in L^2(0,1)$, there is only one function y_s which satisfies the supposed statement.*

Proof: Since S_n is a finite dimensional vector space we can write S_n in terms of a basis, $\{\Phi_i : 1 \leq i \leq n\}$. Let $y_s = \sum_{j=1}^{n} y_j \Phi_j$ where y_j is some constant, $K_{ij} = \langle \Phi_i, \Phi_j \rangle_e$, and

$F_i = \langle f, \Phi_i \rangle$. Using this we can again rewrite the expression as $KY = F$ where K is a square matrix and $Y = (y_j)$. Proving uniqueness in the context of a square matrix is to prove that the matrix K has an inverse. For the inverse of a matrix to exist, the determinant of the matrix must be non-zero, which implies row or column independence. Row independence implies that if $KV = 0$, then it must be that $V = 0$. So let's assume that $KV = 0$ but $V \neq 0$. Then $v = \sum_{j=1}^{n} v_j \Phi_j$ with $V = (v_j)$ and the equivalence of $KY = F$ and $\langle y_s, h_s \rangle_e = \langle f, h_s \rangle$ allows the following representation:

$$KV = \begin{bmatrix} \langle \Phi_1, \Phi_1 \rangle_e \cdots \langle \Phi_1, \Phi_n \rangle_e \\ \vdots \\ \langle \Phi_n, \Phi_1 \rangle_e \cdots \langle \Phi_n, \Phi_n \rangle_e \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Performing the KV will result in the following:

Take for illustration the 1st row of matrix K multiplied by V.

$$v_1 \langle \Phi_1, \Phi_1 \rangle_e + v_2 \langle \Phi_1, \Phi_2 \rangle_e + \cdots + v_n \langle \Phi_1, \Phi_n \rangle_e = 0$$

After applying properties of the inner product, we have $\langle \Phi_1, V \rangle_e = 0$. Thus we have that $\langle \Phi_j, V \rangle_e = 0$ for all j. Multiplying by v_j and summing over all j would result in the following $\langle V, V \rangle_e = 0$. This is true if $v'(x) = 0$. If $v'(x) = 0$ then we have that $v(x)$ is a constant function. Recall that $v(0) = 0$ according to our original assumptions for function to be in H, and thus $v(x) = 0$ everywhere. This is a contradiction to our original statement that $v \neq 0$. Therefore K is invertible and the solution to our equation exists and is unique.

1.3 KEY POINTS

In this section we have attempted to develop some key concepts that would be helpful in analyzing the Finite Element Method for approximating differential equations. We have shown the following three main points.

(1) Given a second order differential equation, we can convert the equation into a valid weak or inner product form for analysis

(2) Any solution to the weak form of the equation is also a solution to the original differential equaliton. An important note is that the weak form solution minimizes the weak equation in the given function space.

(3) For a differential equation, if a solution to the weak form exists, the solution is unique.

The remainder of this paper will discuss the finite element space.

CHAPTER 2

FINITE ELEMENT SPACE

2.1 DEFINITION AND CONSTRUCTION OF FINITE ELEMENTS

Now that we have an understanding of inner product spaces, we will take a more intimate look at finite element spaces. In the previous section we saw that the original equation existed in a function space with continuous 2nd order derivatives, a norm to measure distance and an inner product. Also, we saw some sub-spaces including V, and S_n. The finite element space is a finite dimensional inner product space. This will be the sub-space for our analysis. In this case we will define a finite dimensional space of continuous and piecewise differentiable functions. We wish to approximate the solution within the context of this space. The finite element space is constructed from a set of finite elements, defined on sub-intervals of the domain. The sub-intervals constructed form a partition $x_0 < x_1 < x_2 < x_3 < \cdots < x_n$ where $x_0 = 0$ and $x_n = 1$. Once we have partitioned the domain, we need to define a set of simple functions/elements to choose from to approximate the solution over the interval. Recall from the introduction that we can construct all elements of the finite dimensional space using a set of basis functions defined on the subintervals. We will call these basis functions 'finite elements'. The space for which these elements form a basis is called the finite element space.

We show the properties of finite elements as defined by Ciarlet, 1978.

Definition. Finite Element [1: p 69]

> Let $K \subset \mathbb{R}^n$ be a bounded closed set with nonempty interior and piece-wise smooth boundary (the element domain)
>
> Let P be a finite dimensional space of functions defined on K (the space of shape functions)

Let $N = \{N_1, N_2, \ldots, N_k\}$ be the basis for the set of nodal variables P'.

We call (K, P, N) a finite element.

A finite element is defined here in $\mathbb{R}^n$. Since we are working in $\mathbb{R}^1$ the following example is important to illustrate the definition of a finite element and finite element space.

Example 2.1.1 (1: p 70)**.** Let $K = [0, 1]$, P be the set of linear polynomials, and $N = \{N_1, N_2\}$ where $N_1(v) = v(0)$ and $N_2(v) = v(1)$ for all $v \in P$. Then (K, P, N) is a finite element and the nodal basis consists of $\Phi_1(x) = 1 - x$ and $\Phi_2(x) = x$. This is the 1-dimensional Lagrange element.

More generally,let the domain $[0, 1]$ be partitioned such that $0 = x_0 < x_1 \cdots < x_n = 1$ and let S_n be the vector space of functions v such that, $v \in C^0([0, 1])$, $v|_{[x_i, x_{i+1}]}$ is a linear polynomial, for $i = 1, \ldots, n$, and $v(0) = 0$. Consider the following roof functions,

$$\Phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_i} & \text{for} \quad x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{h_{i+1}} & \text{for} \quad x_i \le x \le x_{i+1} \\ 0 & \quad \text{otherwise} \end{cases}$$

Here $h_i = x_i - x_{i-1}$ is the fixed interval size, $x_i \in [0, 1]$ with $0 = x_0 < x_1 < \cdots < x_n = 1$. In general h should be small. Notice the domain is closed and bounded by definition of the function. Notice $\Phi_i(x_{i-1}) = 0$, the same is true for x_{i+1}. This is the element domain. The functions $\frac{x - x_{i-1}}{h_i}$ and $\frac{x_{i+1} - x}{h_{i+1}}$ are the shape functions. These functions are piece-wise linear and smooth. Finally we will show the set of all such functions form a basis for the finite element space. That is, the set $\{\Phi_i : i = 1, 2, \ldots, n\}$ is linearly independent and spans the finite element space S_n.

Note that as we change the partition, the dimension of the subspace S_n will change as well.

Proposition 2.1.1. *The previously described roof function form a basis for the finite element space.*

Proof Suppose that $\sum_{i=1}^{n} c_i \Phi_i = \widehat{0}$ where $\widehat{0}$ is the zero function. By definition of the finite element $\Phi_j(x_i) = 0$ for all $i \neq j$ and $\Phi_i(x_i) = 1$ for all i. Consider the point $x_k \in [0,1]$. We have that

$$\sum_{i=1}^{n} c_i \Phi_i(x_k) = c_1\Phi_1(x_k) + c_2\Phi_2(x_k) + \cdots + c_k\Phi_k(x_k) + \cdots + c_n\Phi_n(x_k) = \widehat{0}(x_k).$$

This will simplify to $c_k\Phi_k(x_k) = \widehat{0}(x_k)$ which proves that $c_k = 0$.

Given $f \in C^0([0,1])$ with $f(0) = 0$, we define the interpolant of f to be $f_p = \sum_{i=1}^{n} f(x_i)\Phi_i$. Notice the interpolant is a projection of f onto the subspace S_n. To show $\{\Phi_i : i = 1, \ldots, n\}$ spans S_n we must show that if $f \in S$, then f is a linear combination of the Φ_i's. Since f_p is defined to be a linear combination of Φ_i's, if we can prove that whenever $f \in S$, $f = f_p$, then we are done. Notice that $f - f_p \in S$. Therefore, by definition of S_n, $(f - f_p)(0) = 0$, $f - f_p \in C^0([0,1])$, and $f - f_p$ is linear on $[x_i, x_{i+1}]$ for all $i \in Z$.

$$(f - f_p)(x_i) = f(x_i) - f_p(x_i) = f(x_i) - \sum_{j=1}^{n} f(x_j)\Phi_j(x_i)$$

Notice that $\Phi_i(x_i) = 1$ for all i and $\Phi_i(x_j) = 0$ for all $i \neq j$. Thus the previous statement reduces to $f(x_i) - f_p(x_i) = 0$ which proves that $f = f_p$ and the set spans the function space.

Now that we have defined the finite element and the finite element space. Let's take a look back at our original problem. Using the roof functions described above we can partition the domain, approximate all of our function from the original equation with elements from the finite element space, and create a new expression.

By approximating in our subspace, we find that $y \approx y_s = \sum_{i=1}^{n} \alpha_i \Phi_i$, $v \approx v_s = \sum_{i=1}^{n} \beta_i \Phi_i$. Our objective is as follows:

Find $v_s \in S$ **such that** $\langle y_s, v_s \rangle_e + \langle y_s, v_s \rangle = \langle x, v_s \rangle$**.**

2.2 KEY POINTS

In this section we have

(1) Defined/Constructed a subspace of our defined H space which is finite dimensional and contains a defined norm and inner product.

(2) Defined/constructed a set of elements which can be used as a basis in the finite element space.

(3) Proved the roof functions described are a basis for the set S_n defined above.

Keep in mind that the roof/hat function is not the only finite element function. There are multiple other types including triangular and rectangular finte element basis functions.

Appropriate choice of basis or element functions can change the accuracy of results. This will be discussed in the following section.

CHAPTER 3

ERROR OF THE FINITE ELEMENT METHOD

Previously we defined an energy inner product on our space and we demonstrated that $\|h\|_e = \sqrt{\langle h, h\rangle_e}$ for all $h \in H$ was a valid norm. In this section we will use this norm to determine some error bounds on the finite element method. Now, y_s is the projection of y onto the subspace S_n. Thus, $y - y_s$ is orthogonal to everything in S_n. We will use this fact in the following proof.

Proposition 3.0.1. $\|y - y_s\|_e = \min\{\|y - v\|_e \; : \; v \; \in \; S\}$

Proof. $\|y - y_s\|_e^2 = \langle y - y_s, y - y_s\rangle_e$ rewriting the quantity for the second part as $y - v + v - y_s$ we obtain $\|y - y_s\|_e^2 = \langle y - y_s, y - v\rangle_e + \langle y - y_s, v - y_s\rangle_e$. Notice that the second part will be zero because $v - y_s \in S$. Thus, $\|y - y_s\|_e^2 = \langle y - y_s, y - v\rangle_e$. By the Cauchy Schwartz inequality $\|y - y_s\|_e^2 \leq \|y - y_s\|_e \|y - v\|_e$. Notice this will be true for all v. Dividing both sides by $\|y - y_s\|_e$ we find, $\|y - y_s\|_e \leq \|y - v\|_e$ for any $v \in S$

Taking the infimum of the right hand side gives us the following:

$$\|y - y_s\|_e \leq inf\{\|y - v\|_e \; : v \in S\} \leq \|y - y_s\|_e$$

Thus, $\|y - y_s\|_e = inf\{\|y - v\|_e \; : v \in S_n\}$. Since this infimum is obtained at $y_s \in S_n$, we can replace inf with min.

This is the basic error for our approximation using the energy norm. Notice that the error is bounded. Assuming that the infimum is small enough we can find a good approximation to our solution. However, how can we guarantee that the infimum will be small? This will depend mainly on two things: the size of the intervals h used to discretize the domain, and the shape of the finite elements used. An example of this will be shown in chapter 4. What is left to show is that the error in the L_2 norm will be smaller.

Theorem 3.0.2. The following inequality holds: $\|y - y_s\| \leq \|y - y_s\|_e$

Proof. Let w be the solution to the following differential equation $-w'' = y - y_s$ on the interval $[0, 1]$ with boundary conditions same as y. Taking the norm of both sides results in the following, $\|y - y_s\| = \langle y - y_s, y - y_s \rangle$. Replacing $y - y_s$ with $-w''$ and using integration by parts with $u = y - y_s$, $du = (y - y_s)'$, $dv = -w''$ and $v = w'$, we can rewrite the previous equation as $\|y - y_s\|^2 = \langle y - y_s, w \rangle_e$. Suppose $v \in S$. Then, $\langle y - y_s, v \rangle = 0$ and $\langle y - y_s, w \rangle_e = \langle y - y_s, w - v \rangle_e$ for all $v \in S$. Then we have the following, $\|y - y_s\|^2 = \langle y - y_s, w \rangle_e = \langle y - y_s, w - v \rangle_e$. Applying the Cauchy Schwarz inequality we find that $\|y - y_s\|^2 \leq \|y - y_s\|_e \|w - v\|_e$. Dividing both sides by $\|y - y_s\|$, replacing $y - y_s$ with $-w''$ and taking the infimum over all $v \in S$, we have

$$\|y - y_s\| \leq \|y - y_s\|_e \inf_{v \in S} \frac{\|w - v\|_e}{\|-w''\|}$$

Now,provided that we can approximate w close enough by some function $v \in S$ and the following approximation assumption holds, $\inf_{v \in S_n} \|w - v\|_e \leq \epsilon \|-w''\|$, we get that $\|y - y_s\| \leq \epsilon \|y - y_s\|_e$.

The next result is useful to bound the error as a function of the mesh size of the partition. It shows that as the mesh size decreases, or as h decreases, the error in the approximation of $y_s \in S_n$, decreases.

Theorem 3.0.3. *Let $h = max_{1 \leq i \leq n}(x_i - x_{i-1})$. Then $\|y - y_s\|_e \leq Ch \|y''\|$ for all $y \in H$ where C is independent of h and y*

Proof. Recall the interpolant described previously. We can identify the function $y_s(x)$ with it's interpolant. Then for any subinterval $[x_j, x_{j-1}]$ we can write the following equivalent statement,

$$\int_{x_{j-1}}^{x_j} (y - y_p)'(x)^2 dx \leq c(x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} y''(x)^2$$

Notice here that we have defined the statement of our proof piecewise. By definition of the two norms, it is sufficient to prove the inequality piecewise. Let,$e = y - y_s$ represent the error function. Then $e' = (y - y_s)'$ and $e'' = y'' - y_s''$, but the linearity of y_s on the subinterval means $y_s'' = 0$. Therefore we have,

$$\int_{x_{j-1}}^{x_j} e'(x)^2 dx \leq c(x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} e''(x)^2 dx$$

Using $\hat{x} = \frac{x - x_{j-1}}{x_j - x_{j-1}}$ we can write the following, $\int_0^1 e'(\hat{x})^2 d\hat{x} \leq c \int_0^1 e''(\hat{x})^2 d\hat{x}$.

Now that we have rewritten the proposition in a form that does not depend on h, we will prove the inequality to be true for some c.

Utilizing Rolle's Theorem, $e'(\xi) = 0$ for some ξ on the interval $(0, 1)$. Thus, $e'(z) = \int_\xi^z e''(\hat{x}) d\hat{x}$ and $|e'(z)| = \left| \int_\xi^z e''(\hat{x}) d\hat{x} \right|$. By the Cauchy Schwartz inequality,

$$|e'(z)| \leq \left| \int_\xi^z 1 d\hat{x} \right|^{1/2} \left| \int_\xi^z e''(\hat{x})^2 d\hat{x} \right|^{1/2}$$

$$|e'(z)| \leq |z - \xi|^{1/2} \left| \int_\xi^z e''(\hat{x})^2 d\hat{x} \right|^{1/2}$$

$$|e'(z)| \leq |z - \xi|^{1/2} \left(\int_0^1 e''(\hat{x})^2 d\hat{x} \right)^{1/2}$$

Squaring both sides and integrating with respect to z will give the following results for c.

$$C = c = \sup_{0<\xi<1} \int_0^1 |z - \xi| \, dy = 1/2$$

Let h represent the size of $[x_{i-1}, x_i]$ for the domain $[0, 1]$. Let $\epsilon \geq 0$ be some small value. Assuming we are in a space where ϵ can be arbitrarily small, which is the case for the reals, we have the following: $\|y - y_s\|_e \leq \epsilon \|y - v\|_e$. Notice that this decreases the error in our approximation. Letting $\hat{h} = \epsilon h$, we can make our error significantly small.

3.1 KEY POINTS

In this section we have

(1) Utilized the energy norm as an estimate of error for the finite element method.

(2) Proved that the L_2 error is bounded.

(3) Proved two methods of controlling error using mesh size h or various finite elements.

The two methods for decreasing error described in this section do not exhaust the techniques for error reduction in the finite element method. Other techniques include, weighted residual formulations that assume an approximate solution for the governing differential equation, subdoman methods that force the intergral of the error function to be zero over some selected subintervals, and least square methods that require the error to be minimized with respect to the unknow coefficients in the assumed solution. [3: pp 43-8]

CHAPTER 4

WORKED EXAMPLES AND APPLICATIONS

4.1 WORKED EXAMPLES WITH MATLAB APPROXIMATION

Now we return to our stated example, $-y'' + y = x$. We can rewrite this statement using the definitions of y_s and v_s as follows

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\int \alpha_i\beta_j \frac{d\Phi_i}{dx}\frac{d\Phi_j}{dx} + \sum_{i=1}^{n}\sum_{j=1}^{n}\int \alpha_i\beta_j\Phi_i\Phi_j = \sum_{i=1}^{n}\int x\alpha_i\Phi_i.$$

Using the previous information, we can state the previous problem as,

find $v_s \in S$ such that $\langle y_s, v_s\rangle_e + \langle y_s, v_s\rangle = \langle x, v_s\rangle$.

Notice that this can be simplified in the following way:

Consider only the prime inner product portion of the equation. When we expand this form we get,

$$\int \left(\alpha_1\beta_1\Phi_1'^2 + \alpha_1\beta_2\Phi_1'\Phi_2' + ... + \alpha_1\beta_n\Phi_1'\Phi_n' + ... + \alpha_n\beta_n\Phi_n'^2\right)\,dx$$

Similary for the inner product of y_s and v_s we will find the following:

$$\int \left(\alpha_1\beta_1\Phi_1^2 + \alpha_1\beta_2\Phi_1\Phi_2 + ... + \alpha_1\beta_n\Phi_1\Phi_n + ... + \alpha_n\beta_n\Phi_n^2\right)\,dx$$

Finally for the inner product of x and v_s we find the following $\int x(\beta_1\Phi_1 + ... + \beta_n\Phi_n)\,dx$

We can group all terms according to their β index. This allows us to write the left hand side of the statement in the following way.

$$\sum_{i=1}^{n}\beta_i\left(\sum_{j=1}^{n}\alpha_j\int\left(\Phi_j'\Phi_i' + \Phi_j\Phi_i\right)dx\right) = \sum_{i=1}^{n}\beta_i\int x\Phi_i$$

This form lends itself to computer programming analysis because it can easily be written in terms of matrices and vectors. The entire analysis is reduced to the following statement.

$\sum_{j=1}^{n} \beta_j (\sum_{i=1}^{n} K_{i,j}\alpha_i - F_j) = 0$ where K is called the stiffness matrix and F is called the load vector, with $K_{i,j} = \int \Phi'_j \Phi'_i + \Phi_i \Phi_j$ and $F_i = \int x\Phi_i$

Notice that y_s is completely determined by the coefficients α_i. Therefore if we know these coefficients, we know y_s. These can be found by manipulating the previous equation. We cannot manipulate the α's. However we can control our trail functions. So let's assume that for any given k, $\beta_k = 1$ and $\beta_i = 0$ for all $i \neq k$. Now we find that we have a system of equations given by the following: for any $i \in [1, n]$, $F_i = \sum_{j=1}^{n} K_{i,j}\alpha_j$ Given this we can determine the values of α_j by inverting the matrix $K_{i,j}$ and multiplying the vector F_i. Finally we get that $\alpha_j = \sum_{i=1}^{n} K_{j,i}^{-1} F_i$

In general when constructing a finite element analysis approximation we are given values at various nodes. This informaiton is observed through experimentation. For example, given that a specific differential equation governs a process, during an experiment a researh scientist may, based on observations, determine values of the solution at various time points. These values can be used in the approximation of the differential equation.

Consider the problem at hand with domain $[0, 1]$ and $h = 1/3$, using the given basis functions (hat functions). Our first task is to determine the stiffness matrix for the approximation. This approximation will be done using MATLAB. See Appendix for MATLAB code. This would result in the following approximation graph.

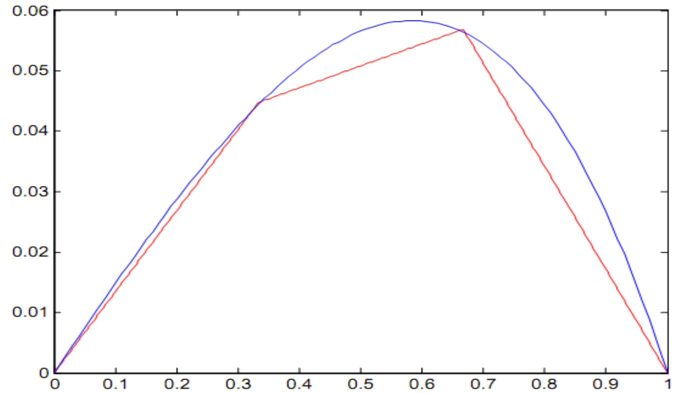

Figure 4.1. Approximation of exact solution with $h = 1/3$.

Notice that this approximation is poor. However, increasing the value of h we can see that we will get progressively better approximations.

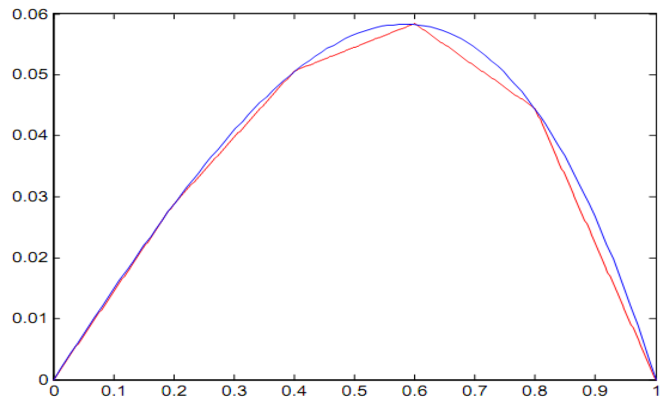

Figure 4.2. Approximation of exact solution with $h = 1/5$.

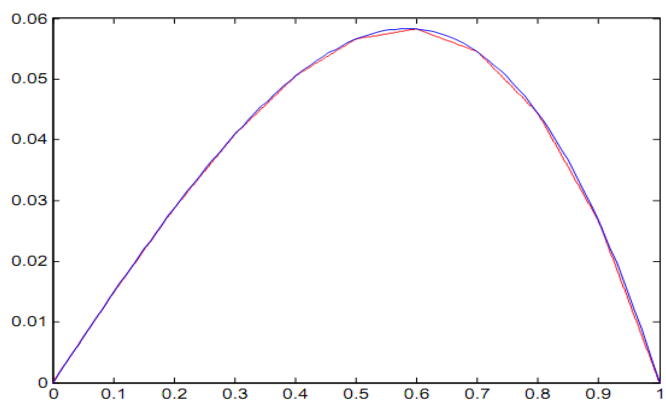

Figure 4.3. Approximation of exact solution with $h = 1/10$.

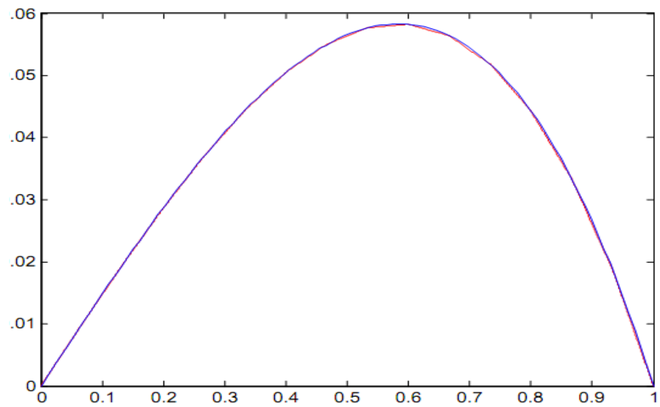

Figure 4.4. Approximation of exact solution with $h = 1/15$.

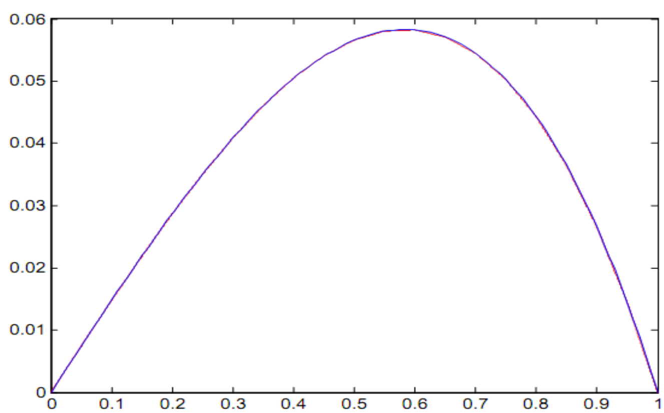

Figure 4.5. Approximation of exact solution with $h = 1/20$.

As can be seen from the figures, there is not much visual difference between the exact solution and the approximation for $h = 20$. The same program was run using $h = 100$. This approximation took more computer time. This is one important issue for research scientist who tend to have time constraints and the question of how accurate an approximation can get in the shortest amount of time is an increasingly pressing one. Much of the current research in software for finite element analysis is related to optimizing the accuracy without increasing the run-time for the process.

4.2 APPLICATIONS

Very few applications of the finite element analysis method involve such a simple equation or analysis. However the same theory applies. Finite element analysis is now widely used in almost every arena, there are researchers who utilize finite element analyis to improve finte element analysis. One body of research of particular interest to me as a biomechanist, is the use of finite element analysis in building and analyzing mathematical models of biological processes in bone. Researchers like Yoo and Jasiuk, are working to build computer models that accurately predict the growth and modulation of bone tissue, as well as pinpointing areas of high stress due to certain physical activities. In the past this research would have to be done using an animal model or using human bones. Today these types of experiments are increasingly being done by computers with fairly high levels of accuracy. Yoo and Jasiuk, 2006, used finite element analysis techniques to model the strain in a section of bone under uniaxial extension, hydrostatic deformation, shear deformation, torsion, and bending. After analysis using FEA software, the group found that the results from the model were in agreement with result for physical materials previously tested.[4] Kumar et al., 2009 used finite element stress analysis coupled with an evolution model to simulate the response of bone to mechanical loading. They found, after comparing to experimental observations, that the model results were consistent with expermental observations.[2] These results support the validity of mathematical modeling in biological sciences.

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APPENDICES

APPENDIX I

One Dimensional Finite Element Analysis MATLAB code

```
%%This file is a simulation of finite element analysis technique.
%%The first portion of this program is setting up for the problem.
%%-y" + y = x; y(0)=0 and y(1)=0;
%%The solution to the following differential equation is
%% y(x)= x -sinh(x)/sinh(1)
%%The following code shows a graphical solution to the equation
format rational
%%xd will be the domain intervals, yd will be the solution
xd=linspace(0,1,100);
yd=xd-(sinh(xd)/sinh(1));
yalta=((exp(1)-exp(-1))/2);
yalternative= xd - ((exp(xd)-exp(-xd))/2);
%%Notice the boundary conditions have been met.
%%The following code is to set up the hat functions.
%% We will utilize hi to be the interval size.
%%Here we assume constant interval size
hi=input('please specify a step size between 0 and 1 in the form 1/n')
%%We define xa to be the descretized domain in vector form
xa=[0:1:(1/hi)];
%%getting the proper interval values for our subregions of the domain
nodevalues=xa.*hi;
%%getting the element matrix so that the left and right hand positions are
%%organized properly. The first column will be the left hand end points
%%the second column will be the right hand end points.
elementr=zeros(1/hi,2);
```

```
elementr(1,1)=1;
elementr(1,2)=2;
for i=2:1/hi
elementr(i,1)=elementr(i-1,1)+1;
elementr(i,2)=elementr(i-1,2)+1;
end
%%This matrix is to store my hat functions
syms o
HatM=[o];
%%Now to program the hat functions to do this we will use two functions.
for j=1:1/hi
leftpoint=nodevalues(elementr(j,1))
rightpoint=nodevalues(elementr(j,2))
x=linspace(leftpoint,rightpoint,100);
%These few lines of code were to check that the hat functions were working.
f1=hat1(x,leftpoint,rightpoint);
f2=hat2(x,leftpoint,rightpoint);
%plot(x,f1, 'b')
%hold on
%plot(x,f2, 'r')
%This code will replace elements of the H
syms x
f1f=hat1(x,leftpoint,rightpoint);
f2f=hat2(x,leftpoint,rightpoint);
%f2p=diff(hat2(x,leftpoint,rightpoint));
%f1p=diff(hat1(x,leftpoint,rightpoint));
HatM(1,j)=f1f;
```

```
HatM(2,j)=f2f;
end
%%This code is to determine the K element matrix
HatM;
HatM(1,1)=0;
HatM(2,(1/hi))=0;
HatMp=diff(HatM);
%%This will set up a matrix to store K-values
K=zeros((1/hi)-1,(1/hi)-1);
%%Calculation of K(i,i) for stiffness matrix
for i=1:(1/hi)-1
for j=i:(1/hi)-1
if i==j
pKa=((HatMp(2,j))ˆ 2)+((HatM(2,j))ˆ 2)
p1Ka=expand(pKa)
prKa=int(p1Ka)
preKa=subs(prKa,nodevalues(j+1))-subs(prKa,nodevalues(j))
pKb=((HatMp(1,j+1))ˆ 2)+((HatM(1,j+1))ˆ 2)
p1Kb=expand(pKb)
prKb=int(p1Kb)
preKb=subs(prKb,nodevalues(j+2))-subs(prKb,nodevalues(j+1))
K(i,j)=preKa+preKb
elseif abs(i-j)¿1
K(i,j)=0
else
pKc=((HatMp(1,j))*(HatMp(2,j)))+((HatM(1,j))*(HatM(2,j)))
p1Kc=expand(pKc)
```

```
prKc=int(p1Kc)
preKc=subs(prKc,nodevalues(j+1))-subs(prKc,nodevalues(j))
K(i,j)=preKc
K(j,i)=K(i,j)
end
end
end
%%The following code will calculate the values for F
F=zeros((1/hi)-1,1);
for i=1:(1/hi)-1
pfa=x*HatM(2,i)
prfa=int(pfa)
prefa=subs(prfa,nodevalues(i+1))-subs(prfa,nodevalues(i))
pfb=x*HatM(1,i+1)
prfb=int(pfb)
prefb=subs(prfb,nodevalues(i+2))-subs(prfb,nodevalues(i+1))
F(i)=prefa+prefb
end
%%This is the code for the approximation with the hand calculations.
alpha=(K^ -1)*F
%%Plot of the approximation
for i=1:(1/hi)
x=linspace(nodevalues(i),nodevalues(i+1),100);
if i==1
approximation=alpha(i)*HatM(2,1);
y=subs(approximation,x);
plot(x,y,'r')
```

```
hold on
elseif i==(1/hi)
approximation=alpha((1/hi)-1)*HatM(1,i);
y=subs(approximation,x);
plot(x,y, 'r')
hold on
else
approximation=alpha(i-1)*HatM(1,i) + alpha(i)*HatM(2,i);
y=subs(approximation,x);
plot(x,y, 'r')
hold on
end
end
plot(xd, yd)
```

Remark. This previously described progam will solve any differential equation of the form $-y''+y = f(x)$ for any function $f(x)$. For example by modifying a few lines of code, we can approximate a solution to $-y'' + y = x^3$ using the same program.

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