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Feedback Linearizable Feedforward Systems: A Special Class

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***Abstract*—The problem of feedback linearizability of systems in feedforward form is addressed and an algorithm providing explicit coordinates change and feedback given. At each step, the algorithm replaces the involutive conditions of feedback linearization by some, easily checkable. We also reconsider type II subclass of linearizable strict feedforward systems introduced by Krstic and we show that it constitutes the only linearizable among the class of *quasilinear* strict feedforward systems. Our results allow an easy computation of the linearizing coordinates and thus provide a stabilizing feedback controller for the original system among others. We illustrate by few examples including the VTOL.**

***Index Terms*— Feedback linearization, linear ordinary differential equations (ODEs), linear systems, (strict) feedforward forms.**

I. Introduction

Linear systems constitute, without doubt, the most well-known class of control systems. Their importance resides in the fact that several physical systems can be modeled using linear dynamics making thus their analysis and design very simple. The controllability, observability, reachability, and realization of linear systems have been expressed in very simple algebraic terms. Another crucial property of linear controllable systems is that they can be stabilized by linear feedback controllers. Although not all systems can be modeled using linear dynamics, the approximation of nonlinear phenomena by linear models has proved to be a satisfactory tool for their local study. The drawback of linearization is that some important properties of a nonlinear system, like global controllability, might be lost by the operation. It is not however surprising that the question of transforming nonlinear control systems into linear ones has attracted much attention in the past thirty years. To give a brief account of that, consider a control system

$$\Sigma : \dot{x} = f(x) + g(x)u, (x, u) \in \Re^n \times \Re^m.$$

The two problems below were investigated in the early 80's.

Problem 1: When does there exist, locally or globally, a diffeomorphism $w = \varphi(x)$ giving rise to new coordinates system $w = (w_1, \ldots, w_n)^\top$ in which Σ takes the linear form

$$\Lambda : \dot{w} = Aw + bu, \quad w \in \Re^n, \, u \in \Re^m.$$

Problem 2: Can we find a change of coordinates $w = \varphi(x)$ coupled with an invertible feedback $u = \alpha(x) + \beta(x)v$ that transform Σ into a linear system $\Lambda : \dot{w} = Aw + bv$?

The first problem was pioneered by Krener [8] and completely solved by the author. The second problem was proposed and partially solved by Brockett [3] in the single-input case $(m = 1)$ with constant function β. The general case of *Problem 2* has been solved independently by Hunt and Su [5], and Jakubczyk and Respondek [7] who gave necessary and sufficient geometric conditions in terms of Lie brackets of vector fields defining the system (see Theorem II.1 below). Although those conditions did provide a way of testing the state and feedback linearizability of a system, they

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offer little on how to find linearizing change of coordinates (and feedback) except by solving a system of partial differential equations (PDEs). If *Problem 2* is solvable with a controllable pair (A, b), then the equilibrium of Σ can be stabilized by the feedback law $u = -\beta(x)^{-1}(\alpha(x) + \sum_{j=1}^{n} k_j\varphi_j(x))$, where the polynomial $p(\lambda) = \lambda^n + \sum_{j=1}^{n} k_j\lambda^{j-1}$ is Hurwitz. Hereafter, (A, b) is assumed to be always controllable, i.e., $\dim\operatorname{span}\{b, Ab, \ldots, A^{n-1}b\} = n$. Feedback linearization techniques have been used to improve the dynamical behavior of chaotic systems (see Lorenz system in [11], [14]) and have been applied to optimal control problems with a recent regain of interest. In [4], the authors used pseudospectral method to solve optimal control subject to feedback linearizable dynamics. Mayer's problem has been considered in [1] (see also [14]) and an optimal solution for globally feedback linearizable time-invariant systems obtained. Recall that Mayer's problem consists of finding $u(t)$ and $x(t)$ with $t \in [t_0, t_f]$ that minimize a functional cost $J = \Phi(x(t_f), t_f)$ subject to $\dot{x} = f(x) + g(x)u$ and inequality constraints $\tilde{s}(x, u) \leq 0$, $\tilde{c}(x) \leq 0$ with initial and terminal states satisfying $\Psi(x(t_0), x(t_f)) = 0$. Though these papers deal with feedback linearizable systems, dynamics are usually assumed linear, and linearizing coordinates left to be found. In [9], [10] Krstic studied *Type I* and *Type II* subclasses of *strict feedforward systems* and showed they are linearizable by giving explicit coordinates changes. A single-input control system Σ is in *feedforward form* (FF)-form if

$$\Sigma_{\mathrm{FF}} : \dot{x} = f(x) + g(x)u \triangleq \begin{cases} f_j(x) = f_j(x_j, \ldots, x_n), \ 1 \leq j < n \\ g_j(x) = g_j(x_j, \ldots, x_n), \ 1 \leq j < n \end{cases}$$

and Σ is in *strict feedforward form* (SFF)-form Σ_{SFF} if

$$\dot{x} = f(x) + g(x)u \triangleq \begin{cases} f_j(x) = f_j(x_{j+1}, \ldots, x_n), \ 1 \leq j \leq n \\ g_j(x) = g_j(x_{j+1}, \ldots, x_n), \ 1 \leq j \leq n \\ f_n(x) = 0, g_n \in \Re^* = \Re \setminus \{0\}. \end{cases}$$

For those classes, Krstic mentioned the difficulty, in general, of systematically finding linearizing coordinates. Inspired by his work, we extended the two classes to all state (resp. feedback) linearizable (SFF)-systems and provided a state [20] (resp. feedback [21]) linearizing coordinates. Let us first mention that (SFF)-systems have been introduced as early as in the papers of Teel [22], [23] and have been followed since by a growing literature [2], [9], [10], [13], [16], [20]. What made (SFF)-systems appealing is that a stabilizing feedback controller can always be constructed when their linear approximation around the equilibrium is controllable [22], [23]. *Problem 1* were tackled in [17] for the feedforward case. If there is a component f_j or g_j that is nonlinear with respect to the variable x_j, then finding an explicit linearizing coordinates becomes extremely difficult and might necessitate solving PDEs. For that reason we restricted our study to a special class of (FF)-systems, called *feedforward-nice*, for which the components f_j and g_j are affine with respect to the variable x_j for all $1 \leq j \leq n$. In Section II we give our first result as an algorithm allowing to construct explicitly feedback linearizing coordinates in a finite number of steps. We show, at each step, that the involutivity conditions of Theorem II.1 reduce to $\partial^2 f/\partial x_j^2$ being proportional to $\partial f/\partial x_j$, $j = n, \ldots, 3$. A change of coordinates, followed by feedback, is thus constructed (bottom up) to cancel all terms containing the variable x_j except for the corresponding component f_{j-1} that is normalized. In Section III we consider *Type II* systems and give necessary and sufficient conditions for their linearization. We illustrate by few examples including the vertical take-off and landing aircraft. Let mention that for general control systems different algorithms have been obtained in [12] and recently in [18], [19]. While the former uses the integration of a system of 1-forms the latter papers are based on an explicit solving of the flow box theorem and they all differ with the method for feedforward systems.

II. Main Results

For reasons mentioned previously, we deal with a subclass of (FF)-systems for which the components $f_j(x_j, \ldots, x_n)$ and $g_j(x_j, \ldots, x_n)$ are affine with respect to x_j for $1 \leq j \leq n$. This subclass, Σ_{FFnice} : $\dot{x} = f(x) + g(x)u$, call it *feedforward-nice* is hereby described in the x-coordinates by

$$\begin{cases} f_j(x) = x_j \tilde{f}_j(x_{j+1}, \ldots, {}_n) + \hat{f}_j(x_{j+1}, \ldots, x_n), \ 1 \leq j \leq n \\ g_j(x) = x_j \tilde{g}_j(x_{j+1}, \ldots, x_n) + \hat{g}_j(x_{j+1}, \ldots, x_n), \ 1 \leq j \leq n \end{cases}$$

with $\hat{f}_j(0) = 0$, $f_n = 0$, and $g_n \in \Re^*$. We suppose that Σ_{FFnice} is linearly controllable and, in particular, $(\partial \hat{f}_j/\partial x_{j+1})(0) \neq 0$ for $1 \leq j \leq n-1$. We recall briefly the main result of [17].

A. State Linearization

Consider Σ_{FFnice} and apply successively changes of coordinates $z = \phi(x)$

$$\begin{cases} z_j = x_j, \ j \neq k, \\ z_k = x_k \tilde{\phi}_k(x_{k+1}, \ldots, x_n) + \hat{\phi}_k(x_{k+1}, \ldots, x_n) \end{cases} \tag{II.I}$$

with

$$\tilde{\phi}_k = \exp\left(-\int_0^{x_n} \tilde{g}_k(x_{k+1}, \ldots, x_{n-1}, \varepsilon), \mathrm{d}\varepsilon \right)$$

$$\hat{\phi}_k = -\int_0^{x_n} \hat{g}_k(x_{k+1}, \ldots, x_{n-1}, \varepsilon)\tilde{\phi}_k(x_{k+1}, \ldots, x_{n-1}, \varepsilon)\mathrm{d}\varepsilon$$

starting from $k = n-1$ down to $k = 1$ to bring Σ_{FFnice} into a control-normalized form $\bar{\Sigma}_{\mathrm{FFnice}} : \dot{z} = \bar{f}(z) + \bar{g}(z)u$

$$\begin{cases} \bar{f}_j(z) = z_j \tilde{\bar{f}}_j(z_{j+1}, \ldots, z_n) + \hat{\bar{f}}_j(z_{j+1}, \ldots, z_n), \quad 1 \leq j < n \\ \bar{f}_n(z) = 0, \quad \bar{g}(z) = (0, \ldots, 0, 1)^\top \in \Re^n. \end{cases}$$

Ignoring $\dot{z}_n = u$ (the only component that depends on u), $\bar{\Sigma}_{\mathrm{FFnice}}$ appears as a system in $\Re^{n-1}$ whose control input is z_n. If the system is not affine in z_n, then $\bar{\Sigma}_{\mathrm{FFnice}}$ (hence Σ_{FFnice}) is not linearizable. Otherwise, the same operation is repeated for $\bar{\Sigma}_{\mathrm{FFnice}}$, then for its transforms, so on.

B. Feedback Linearization

Now we analyze the subclass Σ_{FFnice} under the feedback group

$$\Gamma : \begin{cases} z = \varphi(x) \\ u = \alpha(x) + \beta(x)v. \end{cases}$$

Notice that φ cannot take any particular form as (II.1) simply because the feedback destroys the (FFnice)-structure unless $\alpha(x) = \alpha x_n$ and $\beta(x) = \beta$, where $\alpha \in \Re$, $\beta \in \Re^*$. Before we proceed, recall the following from [5] and [7] (see also [6]).

Theorem II.1: System $\Sigma : \dot{x} = f(x) + g(x)u$ (not necessarily Σ_{FFnice}) is locally equivalent, via a feedback transformation Γ, to a linear controllable system $\Lambda : \dot{w} = Aw + bv$ if and only if (F1) $\dim\operatorname{span}\mathcal{D}^{n-1} = n$; (F2) $[\mathcal{D}^k, \mathcal{D}^k] \subset \mathcal{D}^k, 0 \leq k < n$.

Here $\mathcal{D}^k = \operatorname{span}\{g, ad_f g, \cdots, ad_f^k g\}$ and *(F2)* stands for involutivity conditions that are equivalent to the involutivity of $\mathcal{D}^{n-2}$ (see e.g. [7]). The main result of this section follows.

Theorem II.2: Assume Σ_{FFnice} is $\mathcal{F}$-linearizable. We can bring Σ_{FFnice} into a linear controllable system $\Lambda : \dot{w} = Aw + bu$ via an explicit feedback transformation Γ whose components are obtained by composing, differentiating, inverting, and integrating the components of $\Sigma_{\rm FFnice}$. Moreover, the algorithm giving Γ involves a maximum of $n(n+3)/2$ steps.

The proof is constructive and allows to compute, explicitly, a linearizing change of coordinates and feedback without solving the corresponding PDEs. It appears in the proof that the coordinates changes are globally defined though the inverse of the feedback is only locally guaranteed. Using this equivalent linear form Λ, it is proved in [14] that Mayer's optimal solution for globally feedback linearizable systems always lies on a constraint arc which allows to characterize and build such optimal solution for single-input systems. For multi-input systems, efficient numerical procedures (like pseudospectral method [4]) can be developed. The main difficulty, however, remains in finding linearizing coordinates which underlines the importance of our results. Before we proceed for the proof, we make the following assumption for simplicity.

Assumption 1: For any continuously differentiable function $w_k = \theta(z_1,\ldots,z_k)$ such that $\theta(0) = 0$ and $(\partial\theta/\partial z_k)(0) \neq 0$ we can *explicitly* find an inverse $z_k = \mu(z_1,\ldots,z_{k-1},w_k)$, i.e., a continuously differentiable function μ, locally around the origin, such that $\theta(z_1,\ldots,z_{k-1},\mu(z_1,\ldots,z_{k-1},w_k)) = w_k$. $\diamond$

Let us make clear that what is assumed is not the existence of the function μ, which is guaranteed by the implicit function theorem, but rather the possibility of explicitly computing μ. The assumption simplifies the problem but is not necessary; indeed, instead of taking $\Sigma_{\rm FFnice}$ into a linear form $\dot w = Aw + bu$, we can always bring it, via explicit coordinates change and feedback, to the feedback form (see [21] for (SFF)-forms)

$$\begin{cases} \dot w_j h = w_j \tilde F_j(w_{j+1}) + \hat F_j(w_{j+1}),\ 1 \le j \le n-1 \\ \dot w_n = v. \end{cases} \tag{II.2}$$

Proof of Theorem II.2: Let $\bar\Sigma_{\rm FFnice} : \dot z = \bar f(z) + \bar g(z)u$. Since

$$\bar f_{n-1} = z_{n-1}\tilde{\bar f}_{n-1}(z_n) + \hat{\bar f}_{n-1}(z_n) \text{ with } \frac{\partial \hat{\bar f}_{n-1}}{\partial z_n}(0) \neq 0$$

using **Assumption 1** we can apply the change of coordinates

$$w = \psi(z) \triangleq \begin{cases} w_j = \psi_j(z) = z_j,\ j \neq n-1 \\ w_n = \psi_n(z) = z_{n-1}\tilde{\bar f}_{n-1}(z_n) + \hat{\bar f}_{n-1}(z_n) \end{cases}$$

and the feedback $v = (\partial \bar f_{n-1}/\partial z_{n-1})(z)\bar f_{n-1}(z_{n-1},z_n) + (\partial \bar f_{n-1}/\partial z_n)u$ to transform the system such as $\bar f_{n-1}(z_{n-1},z_n) = z_n$.

Step 1: Involutivity of $\mathcal{D}^1$ implies $[\bar g, ad_{\bar f}\bar g] = \gamma_1 ad_{\bar f}\bar g + \gamma_0 \bar g$, where γ_1 and γ_0 are smooth functions. Since $\bar g = \partial/\partial z_n$, $\bar f_n(z) = 0$, and $\bar f_{n-1}(z) = z_n$, it follows that $\gamma_0 = \gamma_1 = 0$. Thus the involutivity of $\mathcal{D}^1$ reduces to the simple algebraic condition (necessary for $\mathcal{F}$-linearization)

$$(\mathcal{F}\mathcal{L}_n) \Longrightarrow \frac{\partial^2 \bar f_j}{\partial z_n^2} = 0 \quad \text{for all } 1 \le j \le n-1$$

that is $\bar f_j(z) = z_j \tilde{\bar f}_j(z_{j+1},\ldots,z_n) + \hat{\bar f}_j(z_{j+1},\ldots,z_n)$ are affine in the variable z_n for all $1 \le j \le n$

$$\begin{aligned} \bar f_j(z) =& z_j\left(\tilde c_j(z_{j+1},\ldots,z_{n-1}) + z_n \tilde b_j(z_{j+1},\ldots,z_{n-1})\right) \\ &+ \hat c_j(z_{j+1},\ldots,z_{n-1}) + z_n \hat b_j(z_{j+1},\ldots,z_{n-1}) \\ =& z_j \tilde c_j(z_{j+1},\ldots,z_{n-1}) + \hat c_j(z_{j+1},\ldots,z_{n-1}) \\ &+ z_n\left(z_j \tilde b_j(z_{j+1},\ldots,z_{n-1}) + \hat b_j(z_{j+1},\ldots,z_{n-1})\right). \end{aligned}$$

If $(\mathcal{F}\mathcal{L}_n)$ fails the algorithm ends: the system is not linearizable. If it holds, then we annihilate all terms containing z_n in all components (except $\bar f_{n-1}(z_n) = z_n$) using $n-2$ substeps.

General Substep: Assume for some $1 \le i \le n-2$, that $\bar\Sigma_{\rm FFnice}$ has been transformed, by coordinates changes and feedback to

$$\bar f_j(z) = \begin{cases} z_j \tilde c_j(z_{j+1},\ldots,z_{n-1}) \\ \quad + \hat c_j(z_{j+1},\ldots,z_{n-1}) \\ \quad + z_n\left(z_j \tilde b_j(z_{j+1},\ldots,z_{n-1})\right. \\ \quad\quad \left. + \hat b_j(z_{j+1},\ldots,z_{n-1})\right) & \text{if } 1 \le j \le i, \\ z_j \tilde c_j(z_{j+1},\ldots,z_{n-1}) \\ \quad + \hat c_j(z_{j+1},\ldots,z_{n-1}) & \text{if } i+1 \le j \le n. \end{cases}$$

It is straightforward that $w = \psi(z)$ whose components are

$$\begin{cases} w_j = z_j, \quad j \neq i \\ w_i = z_i \tilde\psi_i(z_{i+1},\ldots,z_{n-1}) + \hat\psi_i(z_{i+1},\ldots,z_{n-1}) \end{cases}$$

where $\tilde\psi_i(\cdot) = \exp(-\int_0^{z_{n-1}} \tilde b_i(z_{i+1},\ldots,z_{n-2},\varepsilon)\mathrm{d}\varepsilon)$ and

$$\hat\psi_i(\cdot) = -\int_0^{z_{n-1}} \hat b_i(z_{i+1},\ldots,z_{n-2},\varepsilon)\tilde\psi_i(z_{i+1},\ldots,z_{n-2},\varepsilon)\mathrm{d}\varepsilon$$

cancels $z_n(z_i\tilde b_i(z_{i+1},\ldots,z_{n-1}) + \hat b_i(z_{i+1},\ldots,z_{n-1}))$ in the component $\bar f_i(z)$ still preserving the (FFnice)-form. We transform $\Sigma_{\rm FFnice}$ via $\psi = \psi^{n-2} \circ \cdots \circ \psi^1$, with ψ^i as above, into

$$\bar\Sigma_{\rm FFnice} : \begin{cases} \bar f_j(z) = z_j \tilde c_j(z_{j+1},\ldots,z_{n-1}) + \hat c_j(z_{j+1},\ldots,z_{n-1}) \\ \bar f_{n-1}(z) = z_n, \bar f_n(z) = 0, \bar g(z) = (0,\ldots,0,1)^\top. \end{cases}$$

Remark that at each substep the inverse of the diffeomorphism is easily computable. Indeed, for the general substep the inverse $z = \psi^{-1}(w)$ is such that $z_j = w_j$ for $j \neq i$ and

$$z_i = \left[w_i - \hat\psi_i(w_{i+1},\ldots,w_{n-1})\right] \times \exp \int_0^{w_{n-1}} \tilde b_i(w_{i+1},\ldots,w_{n-2},\varepsilon)\mathrm{d}\varepsilon.$$

Since $\bar f_{n-2}(z) = z_{n-2}\tilde c_{n-2}(z_{n-1}) + \hat c_{n-2}(z_{n-1})$ with $(\partial \hat c_{n-2}/\partial z_{n-1})(0) \neq 0$, we can use the assumption to explicitly find the inverse of $w_{n-1} = z_{n-2}\tilde c_{n-2}(z_{n-1}) + \hat c_{n-2}(z_{n-1})$.

Then apply the change of coordinates and feedback

$$\begin{cases} w_j = z_j, j = 1,\ldots,n-2, & w_n = \dot w_{n-1} \\ w_{n-1} = z_{n-2}\tilde c_{n-2}(z_{n-1}) + \hat c_{n-2}(z_{n-1}), & v = \dot w_n \end{cases}$$

to normalize $\bar f_{n-2} = z_{n-1}$. This last substep can be omitted in case the system will be brought to the feedback form (II.2).

General Step: Define the projection $\pi : \Re^n \to \Re^{n-1}$ as $\pi(z_1,\ldots,z_n) = (z_1,\ldots,z_{n-1})^\top$. The projection $\pi(\bar\Sigma_{\rm FFnice})$ is a (FFnice)-system in $\Re^{n-1}$ with control input $v = z_n$. The necessary condition for linearization becomes

$$(\mathcal{F}\mathcal{L}_{n-1}) \Longrightarrow \frac{\partial^2 \bar f_j}{\partial z_{n-1}^2} \equiv 0, \quad \text{for all } 1 \le j \le n-2.$$

We repeat the same procedure as in *Step 1* to construct coordinates change that annihilates all terms containing the variable z_{n-1}. The diffeomorphism obtained is thus extended in $\Re^n$ by adding the nth component as identity. Then a feedback is applied as above to normalize the 2nd last component (in this case $\bar f_{n-3}(z) = z_{n-2}$). The procedure will be repeated $(n-2)$ times as long as the corresponding $(\mathcal{F}\mathcal{L}_n)$ conditions are satisfied for the corresponding system. The diffeomorphisms ϕ and ψ taking $\Sigma_{\rm FFnice}$ into a control-normalized $\bar\Sigma_{\rm FFnice}$ and $\bar\Sigma_{\rm FFnice}$ into the Brunovský canonical form are obtained by integrations, derivations, compositions of the components of $\Sigma_{\rm FFnice}$, and

$\bar{\Sigma}_{\mathrm{FFnice}}$, respectively. The feedback is obtained in general by inverting a nonlinear function (**Assumption 1**) but can be circumvented if we transform the system into (II.2). One can easily check that there is a maximum of $n+n+(n-1)+\cdots+1=n(n+3)/2$ substeps, a difference of n substeps with the $\mathcal{S}$-linearization due to the use of feedback. $\square$

Example II.3: Let us consider the system in (FFnice)-form

$$\Sigma_{\mathrm{FFnice}}:\begin{cases}\dot{x}_1=\sin\frac{x_2}{1+x_4^2}+x_4\sin x_3+\frac{x_2}{1+x_4^2}u, & \dot{x}_3=\sin x_4\\ \dot{x}_2=\left(1+x_4^2\right)\sin x_3+\frac{2x_2x_4}{1+x_4^2}u, & \dot{x}_4=u\end{cases}.$$

We rectify $g(x)=((x_2/(1+x_4^2)),(2x_2x_4/(1+x_4^2)),0,1)^\top$ and put Σ_{FFnice} in $\bar{\Sigma}_{\mathrm{FFnice}}$. Since $g_2(x)=(2x_2x_4/(1+x_4^2))=x_2\tilde{g}_2(x_3,x_4)+\hat{g}_2(x_3,x_4)$ with $\tilde{g}_2(x_3,x_4)=2x_4/(1+x_4^2)$ and $\hat{g}_2(x_3,x_4)=0$, a change of coordinates annihilating the component g_2 is

$$\begin{cases}\tilde{x}_1=x_1, & \tilde{x}_3=x_3\\ \tilde{x}_2=x_2\exp\left(\int_0^{x_4}\tilde{g}_2(x_3,\varepsilon)\mathrm{d}\varepsilon\right)=\frac{x_2}{1+x_4^2}, & \tilde{x}_4=x_4\end{cases}$$

with inverse

$$\begin{cases}x_1=\tilde{x}_1, & x_3=\tilde{x}_3\\ x_2=\tilde{x}_2\left(1+\tilde{x}_4^2\right), & x_4=\tilde{x}_4.\end{cases}$$

It transforms the system into

$$\begin{cases}\dot{\tilde{x}}_1=\sin\tilde{x}_2+\tilde{x}_4\sin\tilde{x}_3+\tilde{x}_2u, & \dot{\tilde{x}}_3=\sin\tilde{x}_4\\ \dot{\tilde{x}}_2=\sin\tilde{x}_3, & \dot{\tilde{x}}_4=u.\end{cases}$$

Now $g(\tilde{x})=(\tilde{x}_2,0,0,1)^\top$ and $g_1=\tilde{x}_2=\tilde{x}_1\tilde{g}_1(\tilde{x})+\hat{g}_1(\tilde{x})$ with $\tilde{g}_1=0$ and $\hat{g}_1=\tilde{x}_2$. Thus the change of coordinates

$$\begin{cases}z_1=\tilde{x}_1\tilde{\phi}_1(\tilde{x})+\hat{\phi}_1(\tilde{x})=\tilde{x}_1-\tilde{x}_2\tilde{x}_4\\ z_j=\tilde{x}_j, & j\neq 1\end{cases}$$

with $\tilde{\phi}_1(\tilde{x})=\exp(-\int_0^{\tilde{x}_4}\tilde{g}_1(\tilde{x}_2,\tilde{x}_3,\varepsilon)\mathrm{d}\varepsilon)=1$ and

$$\hat{\phi}_1(\tilde{x}_2,\tilde{x}_3,\tilde{x}_4)=\int_0^{\tilde{x}_4}\hat{g}_1(\tilde{x}_2,\tilde{x}_3,\varepsilon)\tilde{\phi}_1(\tilde{x}_2,\tilde{x}_3,\varepsilon)\mathrm{d}\varepsilon=-\tilde{x}_2\tilde{x}_4$$

annihilates g_1 and brings Σ_{FFnice} into $\bar{\Sigma}_{\mathrm{FFnice}}$:

$$\bar{\Sigma}_{\mathrm{FFnice}}:\begin{cases}\dot{z}_1=\sin z_2, & \dot{z}_3=\sin z_4\\ \dot{z}_2=\sin z_3, & \dot{z}_4=u.\end{cases}$$

The change of coordinates $w=\psi(z)$

$$\begin{cases}w_1=z_1, & w_3=\cos z_2\sin z_3\\ w_2=\sin z_2, & w_4=-\sin z_2\sin^2 z_3+\cos z_2\cos z_3\sin z_4\end{cases}$$

and feedback $v=(\partial w_4/\partial z_2)\sin z_3+(\partial w_4/\partial z_3)\sin z_4+(\partial w_4/\partial z_4)u$ linearizes the system: $\dot{w}_1=w_2$, $\dot{w}_2=w_3$, $\dot{w}_3=w_4$, $\dot{w}_4=v$. The feedback law ($p(\lambda)=\lambda^4+k_1\lambda^3+k_2\lambda^2+k_3\lambda+k_4$ Hurwitz)

$$u=-\frac{\sin z_3\frac{\partial\psi_4}{\partial z_2}+\sin z_4\frac{\partial\psi_4}{\partial z_3}+k_1\psi_1(z)+\cdots+k_4\psi_4(z)}{\cos z_2\cos_3\cos z_4}$$

re-expressed in original coordinates, locally asymptotically stabilizes the system on $(-\epsilon,\epsilon)^n$ for $0<\epsilon<\pi/2$. $\diamond$

Example II.4: Consider a simplified model of a VTOL with dynamics [15] (see Fig. 1.)[1]

[1]A special thank you to Linda G. for her invaluable assistance.

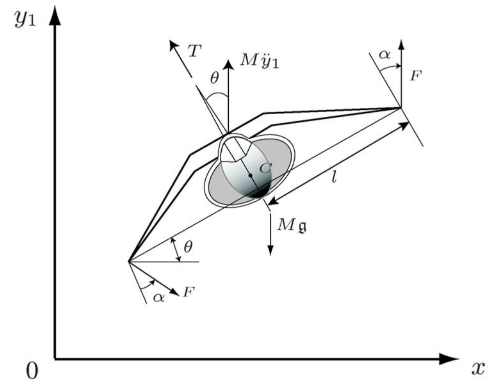

Fig. 1. Forces acting on the aircraft.

$$\begin{cases}\dot{x}_1=x_2, & \dot{x}_2=-\sin(\theta_1)\frac{T}{M}+\cos(\theta_1)\frac{2\sin\alpha}{M}F\\ \dot{y}_1=y_2, & \dot{y}_2=-\cos(\theta_1)\frac{T}{M}+\sin(\theta_1)\frac{2\sin\alpha}{M}F-\mathfrak{g}\\ \dot{\theta}_1=\theta_2, & \dot{\theta}_2=\frac{2l}{J}\cos\alpha F\end{cases}\tag{II.3}$$

where M, J, l and $\mathfrak{g}$ denote the mass, moment of inertia, distance between wingtips and gravitational acceleration. The control inputs are the thrust T, and the rolling moment due to the torque F, whose direction forms a fixed angle α with the horizontal body axis. The position of center mass and the roll angle with respect to the horizon are (x_1,y_1), and θ_1 while (x_2,y_2) and θ_2 stand for their respective velocities. Thus the system is a two-input control system in (SFF)-form. Let fix $T=-M(\mathfrak{g}-y_2)\cos\theta_1$ so as to compensate the force due to acceleration (sign minus denotes opposite direction to that force) and normalize M, J, and l to 1. Taking $u=F$ as control input, and $x_3=y_1$, $x_4=y_2$, $x_5=\theta_1$, $x_6=\theta_2/2\cos\alpha$, the system rewrites $\Sigma:\dot{x}=f(x)+g(x)u$, $x\in\Re^6$, $u\in\Re$ with

$$\begin{aligned}f(x)=&\,(x_2,(\mathfrak{g}-x_4)\sin x_5\cos x_5,x_4,\\&-\mathfrak{g}\sin^2 x_5-x_4\cos^2 x_5,2\cos\alpha x_6,0)^\top\\ g(x)=&\,(0,2\sin\alpha\cos x_5,0,2\sin\alpha\sin x_5,0,1)^\top.\end{aligned}$$

This system is in (FFnice)-form. Since g_1, g_3, g_5 are zero and $g_6=1$, only two substeps are required to rectify g.

To cancel $g_4(x)=2\sin\alpha\sin x_5$, observe that $g_4(x)=x_4\tilde{g}_4(x_5,x_6)+\hat{g}_4(x_5,x_6)$ with $\tilde{g}_4=0$ and $\hat{g}_4=2\sin\alpha\sin x_5$. Thus $\phi_4(x)=x_4\tilde{\phi}_4(x)+\hat{\phi}_4(x)$ with $\tilde{\phi}_4(x)=1$, $\hat{\phi}_4(x)=-\int_0^{x_6}2\sin\alpha\sin x_5\mathrm{d}\varepsilon=-2\sin\alpha(\sin x_5)x_6$. It follows that the change of coordinates (and its inverse):

$$z=\phi(x)\triangleq\begin{cases}z_j=x_j, & j\neq 4,\\ z_4=x_4-2\sin\alpha(\sin x_5)x_6\end{cases}$$
$$x=\psi(z)\triangleq\begin{cases}x_j=z_j, & j\neq 4,\\ x_4=z_4+2\sin\alpha(\sin z_5)z_6\end{cases}$$

transform the system Σ_{FFnice} into (keep same notation)

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=(\mathfrak{g}-x_4-2\sin\alpha(\sin x_5)x_6)\sin x_5\cos x_5\\ \quad+2\sin\alpha\cos x_5u\\ \dot{x}_3=x_4+2\sin\alpha(\sin x_5)x_6\\ \dot{x}_4=-\mathfrak{g}\sin^2 x_5-(x_4+2\sin\alpha(\sin x_5)x_6)\cos^2 x_5\\ \quad-2\sin\alpha(\cos x_5)x_6^2\\ \dot{x}_5=(2\cos\alpha)x_6\\ \dot{x}_6=u\end{cases}$$

with control vector field $g(x) = (0, 2\sin\alpha\cos x_5, 0, 0, 0, 1)^\top$. To annihilate $g_2(x)$, we calculate $\phi_2(x) = x_2\tilde{\phi}_2(x) + \hat{\phi}_2(x)$. Thus the change of coordinates (and its inverse)

$$z = \phi(x) \triangleq \begin{cases} z_j = x_j, \quad j \neq 2, \\ z_2 = x_2 - 2\sin\alpha(\cos x_5)x_6 \end{cases}$$
$$x = \psi(z) \triangleq \begin{cases} x_j = z_j, \quad j \neq 2 \\ x_2 = z_2 + 2\sin\alpha(\cos z_5)z_6 \end{cases}$$

transform the latter system into $\bar{\Sigma}_{\mathrm{FFnice}} : \dot{z} = \bar{f}(z) + \bar{g}(z)u$

$$\begin{cases} \dot{z}_1 = z_2 + 2\sin\alpha(\cos z_5)z_6 \\ \dot{z}_2 = (\mathfrak{g} - z_4 - 2\sin\alpha(\sin z_5)z_6)\sin z_5\cos z_5 \\ \quad +2\sin\alpha(\sin z_5)z_6^2 \\ \dot{z}_3 = z_4 + 2\sin\alpha(\sin z_5)z_6 \\ \dot{z}_4 = -\mathfrak{g}\sin^2 z_5 - (z_4 + 2\sin\alpha(\sin z_5)z_6)\cos^2 z_5 \\ -2\sin\alpha(\cos z_5)z_6^2 \\ \dot{z}_5 = (2\cos\alpha)z_6 \\ \dot{z}_6 = u. \end{cases}$$

Since $\partial^2\bar{f}_1/\partial z_6^2 = 0\cdot(\partial\bar{f}_1/\partial z_6)$ while $\partial^2\bar{f}_2/\partial z_6^2 \neq 0\cdot(\partial\bar{f}_2/\partial z_6)$, condition $(\mathcal{F}\mathcal{L}_6) \implies (\partial^2\bar{f}_j/\partial z_6^2) = \gamma_1(z)(\partial\bar{f}_j/\partial z_6), 1 \leq j \leq 5$ fails because $\gamma_1(z)$ is not unique. Thus, the system is not $\mathcal{F}$-linearizable. $\diamond$

III. Linearizable Systems of Type II

Consider *Type II* (SFF)-systems described in [10] by

$$\begin{cases} \dot{x}_j = x_{j+1} + g_j(x_{j+1}, \ldots, x_n)u, \ 1 \leq j < n \\ \dot{x}_n = u \end{cases} \tag{III.1}$$

where $g_1, \ldots, g_{n-1}$ are smooth functions, $g_i(0) = 0$. *Type II* have a linear drift $f(x) = Ax$ while *Type I* take the form

$$\begin{cases} \dot{x}_1 = x_2 + \sum_{j=2}^{n-1}\pi_i(x_i)x_{i+1} + \pi_n(x_n)u \\ \dot{x}_j = x_{j+1}, \quad j = 2, \ldots, n-1 \\ \dot{x}_n = u \end{cases}$$

and were introduced in [9], [10] (see also [17], [20] beyond that class), where it is showed that a linearizing coordinates is

$$w = \varphi(x) \triangleq \begin{cases} w_1 = x_1 - \sum_{i=2}^{n}\int_0^{x_i}\pi_i(\varepsilon)\mathrm{d}\varepsilon \\ w_j = x_j, \quad j = 2, \ldots, n. \end{cases}$$

For *Type II* systems, Krstic [10] defined recursively the functions $\mu_n(x_n), \ldots, \mu_2(x_n)$ for $i = n, \ldots, 2$ as

$$\mu_i = \frac{1}{x_n}\int_0^{x_n}\left(g_{i-1}(0, \ldots, 0, \varepsilon) - \sum_{j=i+1}^{n}\mu_j(\varepsilon)g_{i+n-j}(0, \ldots, 0, \varepsilon)\right)\mathrm{d}\varepsilon. \tag{III.2}$$

Next, he defined $\gamma_k(x)_n)$ for $k = 1, \ldots, n-2$ as

$$\gamma_k = \sum_{l=1}^{k-1}\gamma_l(x_n)\mu_{l+n+1-k}(x_n) + \mu'_{n+1-k}(x_n). \tag{III.3}$$

He then obtained the following result.

Theorem III.1 (Krstic [10]): If

$$g_i(x_{i+1}, \ldots, x_n) = \sum_{j=i+1}^{n-1}\gamma_{j-i}(x_n)x_j + g_i(0, \ldots, 0, x_n) \tag{III.4}$$

$\forall x, 1 \leq i \leq n-2$, then the diffeomorphic transformation

$$\begin{cases} w_i = x_i - \sum_{j=i+1}^{n-1}\mu_{i+n+1-j}(x_n)x_j, \ 1 \leq i \leq n-1 \\ w_n = x_n \end{cases} \tag{III.5}$$

converts the *(SFF)*-system (III.1) into the chain of integrators $\dot{w}_1 = w_2, \ldots, \dot{w}_{n-1} = w_n, \dot{w}_n = u$. Thus $u = -\sum_{i=1}^{n}C_n^{i-1}w_i$ globally asymptotically stabilizes the origin of (III.1).

Observe that (III.4) is in the form

$$g_i(x_{i+1}, \ldots, x_n) = \sum_{j=i+1}^{n}\beta_{i,j}(x_n)x_j, \quad 1 \leq i \leq n-1 \tag{III.6}$$

with $\beta_{i,j}(x_n) = \gamma_{j-i}(x_n)$ and $\beta_{i,n}(x_n) = g_i(0, \ldots, 0, x_n)$. We say that (III.1) is *quasilinear* if (III.6) holds for some functions $\beta_{i,j}(x_n)$. Clearly, (III.1) given by (III.4) is quasilinear and linearizable provided the coefficients γ_k satisfy (III.3) with μ_k given by (III.2). *Does any linearizable quasilinear system satisfy* (III.2)-(III.3)-(III.4)? We will show that the answer is yes. Notice that (III.1)–(III.6) can be put in the compact form

$$\dot{x} = Ax + g(x)u = Ax + bu + B(x_n)xu \tag{III.7}$$

with (A, b) the Brunovský pair, and $B = (\beta_{i,j}(x_n))_{1\leq i<j\leq n}$ a *upper triangular nilpotent matrix* (*u.t.n.m*) whose entries are smooth functions of the variable x_n. The main result of this section is the characterization of the linearizability of (III.7).

Theorem III.2: Let (III.1)–(III.6) be quasilinear or in compact form (III.7). The three conditions below are equivalent.

(i) System (III.7) is linearizable by change of coordinates.

(ii) There exists an invertible (*u.t.m*) $M = (\alpha_{i,j}(x_n))_{1\leq i\leq j\leq n}$ s. t.

$$\begin{cases} AM = MA, \\ Mb + (MB + M')x = b \quad \text{for all } x \in \Re^n. \end{cases} \tag{III.8}$$

(iii) System (III.7) is in the form (III.4) with γ_k given by (III.3) and μ_k given by (III.2).

Theorem III.2 establishes the fact that the linearizability conditions (III.6) with (III.3), (III.4) obtained by Krstic are necessary and sufficient for the quasilinear systems (recall not all *Type II* systems are linearizable). As a consequence we deduce the following algorithm whose importance is two fold: (a) it allows to test whether a quasilinear system is linearizable or not, and (b) when linearizable, it gives a linearizing change of coordinates. Let us recall that a matrix $M = (\alpha_{i,j}(x_n))_{1\leq i\leq j\leq n}$ is said to be in *Toeplitz form* if $\forall 1 \leq i \leq j \leq n$

$$\alpha_{i,j}(x_n) = \alpha_{i+1,j+1}(x_n) = \cdots = \alpha_{i+n-j,n}(x_n). \tag{III.9}$$

Linearization Procedure: Consider the (SFF)-system (III.7).

Step 1) Take $\hat{B}$ as the matrix B with the last row and last column deleted. If $\hat{B}$ is not a (*u.t.n.m*) in *Toeplitz form*, then STOP, the system is not linearizable.

Step 2) If $\hat{B}$ is a (*u.t.n.m*) in *Toeplitz form*, solve $\hat{M}\hat{B} + \hat{M}' = 0$ to get $\hat{M}$. Then compute the function $\alpha_{1,n}(x_n)$ via

$$\alpha_{1,n}(x_n) = -\frac{1}{x_n}\sum_{k=1}^{n-1}\int_0^{x_n}\alpha_{1,k}(\varepsilon)\beta_{k,n}(\varepsilon)\varepsilon\mathrm{d}\varepsilon. \tag{III.10}$$

Since M should be *Toeplitz*, the obtention of $\hat{M}$ and $\alpha_{1,n}$ gives M by adding to $\hat{M}$ the last column and row accordingly.

Step 3) If B_n (resp. M_n) is the last column of B (resp. M), check if $M_n + x_n(MB_n + M'_n) = b$. If no, the system is not linearizable and if yes, it is by $w = M(x_n)x$.

Proof of Theorem III.2: We will show (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

- (iii) $\Rightarrow$ (i) is direct and can be deduced from Theorem III.1.
- (i) $\Rightarrow$ (ii) We proved in [20] that (III.1) in $\Re^3$, is linearizable iff $g_1 = x_2\gamma_1(x_3) + \theta_1(x_3)$, where $\gamma_1(x_3) = (d/dx_3)((1/x_3)\int_0^{x_3} g_2(\varepsilon)\mathrm{d}\varepsilon)$. The (*u.t.m*) $M = (\alpha_{i,j}(x_3))_{1\le i\le j\le 3}$
$$\alpha_{i,i} = 1, \quad 1 \le i \le 3, \quad \alpha_{1,2}(x_3) = -\frac{1}{x_3}\int_0^{x_3} g_2(\varepsilon)\mathrm{d}\varepsilon$$
$$\alpha_{1,3}(x_3) = -\frac{1}{x_3}\left(\int_0^{x_3}\gamma_1(\varepsilon)\mathrm{d}\varepsilon - \int_0^{x_3}\alpha_{1,2}(\varepsilon)g_2(\varepsilon)\mathrm{d}\varepsilon\right)$$
satisfies (III.8) and defines $w = M(x_3)x$ that linearizes the system. Assume the implication true for systems in $\Re^{n-1}$. Let (III.1) be an n-dimensional system linearizable by $w = \varphi(x) = (\varphi_1(x_1,\ldots,x_n), \varphi_2(x_2,\ldots,x_n), \ldots, \varphi_n(x_n))^\top$. Set $\rho(x_1,\ldots,x_n) = (x_2,\ldots,x_n)^\top$. The projection $\rho(\Sigma)$ is linearizable by $\rho(\varphi(x)) = (\varphi_2(x_2,\ldots,x_n),\ldots,\varphi_n(x_n))^\top$. By induction $\varphi_i(x) = x_i + \sum_{j=i+1}^n \alpha_{i,j}(x_n)x_j$ for $i \ge 2$ with $\alpha_{i,j} = \alpha_{i-1,j-1}$. Since $w = \varphi(x)$ linearizes the system, $\dot{w} = (\partial\varphi/\partial x)Ax + (\partial\varphi/\partial x)bu + (\partial\varphi/\partial x)B(x_n)xu = A\varphi(x) + bu$. Thus $A\varphi(x) = L_{Ax}\varphi(x)$ which implies $\varphi_2(x) = L_{Ax}\varphi_1(x)$. Also $\varphi_2(x) = x_2 + \sum_{j=3}^n \alpha_{2,j}(x_n)x_j = L_{Ax}(x_1 + \sum_{j=2}^{n-1}\alpha_{1,j}(x_n)x_j)$. We then deduce that $\varphi_1(x) = x_1 + \sum_{j=2}^{n-1}\alpha_{1,j}(x_n)x_j + \theta(x_n)$. The change of coordinates $w = \varphi(x)$ is in the form $w = M(x_n)x$, where $M(x_n)$ satisfies (III.8) necessarily.
- (ii) $\Rightarrow$ (iii) Consider the system (III.7) and assume that $M = (\alpha_{i,j}(x_n))_{1\le i\le j\le n}$ exists and satisfies (III.8). Define μ_k and γ_k such that for $1 \le i < j \le n-1$ we have $\beta_{i,j}(x_n) = \gamma_{j-i}(x_n)$ and for $1 \le i < j \le n$ we have $\alpha_{i,j}(x_n) = -\mu_{i+1+n-j}(x_n)$. This is possible provided that (III.9) is satisfied. That's the only point we need to clarify because $Mb + (MB + M')x = b$ for all $x \in \Re^n$ is already equivalent to (III.2)-(III.3)-(III.4). Condition (III.9) is satisfied since $AM = MA$. Let $\hat{M}$ (resp. $\hat{B}$) denote the matrice M (resp. B) with the last row and last column deleted. $Mb + (MB + M')x = b$ for all $x \in \Re^n$ implies that $\hat{M}\hat{B} + \hat{M}' = 0$. $\hat{M}$ being in *Toeplitz form* (hence is $\hat{M}'$) and invertible, thus $\hat{B} = \hat{M}^{-1}\hat{M}'$ is also in *Toeplitz form.* □

Example III.3: Reconsider Example II.3 and let $a = 0$, that is
$$\begin{cases} \dot{x}_1 = x_2 + \left(\frac{1}{2}x_2 - \frac{1}{12}x_3x_4\right)u, & \dot{x}_3 = x_4 + x_4u \\ \dot{x}_2 = x_3 + \frac{1}{2}x_3u, & \dot{x}_4 = u. \end{cases}$$

This is a quasilinear (SFF)-system $\dot{x} = Ax + bu + B(x_4)xu$ in $\Re^4$, where $B = (\beta_{i,j})$ is a (*u.t.n.m*) with $\beta_{1,4} = \beta_{2,4} = 0$
$$\beta_{1,2}(x_4) = \beta_{2,3}(x_4) = \frac{1}{2}, \quad \beta_{1,3}(x_4) = -\frac{1}{2}, \quad \beta_{3,4}(x_4) = 1.$$

We look for an invertible (*u.t.m*) $M = (\alpha_{i,j})$ in *Toeplitz form*, i. e., $\alpha_{i,i} = 1, 1 \le i \le 4$; $\alpha_{1,2} = \alpha_{2,3} = \alpha_{3,4}$ and $\alpha_{1,3} = \alpha_{2,4}$ such that $\hat{M}\hat{B} + \hat{M}' = 0$. We deduce the system of ordinary differential equations (ODEs): $\alpha'_{1,3}(x_4) - (1/12)x_4 + (1/2)\alpha_{1,2}(x_4) = 0$, $\alpha'_{1,2}(x_4) + (1/2) = 0$ which yields $\alpha_{1,2} = -(1/2)x_4$ and $\alpha_{1,3} = (1/6)x_4^2$. Integrating (III.10) for $\beta_{1,4} = \beta_{2,4} = 0$, $\beta_{3,4} = 1$ gives $\alpha_{1,4}(x_4) = -(1/24)x_4^3$. Hence the change of coordinates $w = M(x_4)x$ defined by
$$\begin{cases} w_1 = x_1 - \frac{1}{2}x_2x_4 + \frac{1}{6}x_3x_4^2 - \frac{1}{24}x_4^4, & w_3 = x_3 - \frac{1}{2}x_4^2 \\ w_2 = x_2 - \frac{1}{2}x_3x_4 + \frac{1}{6}x_4^3, & w_4 = x_4 \end{cases}$$
linearizes the system (compare with Example II.3 when $a = 0$).

IV. Conclusion

In this technical note we considered a subclass of single input control systems in feedforward form, namely (FFnice), for which we gave a feedback linearizing algorithm. The importance of the algorithm is two fold: (a) it replaces the solving of the PDEs by simple algebraic operations involving differentiating, integrating, and composing functions only; (b) it does not require an *a priori* checking of the feedback linearization conditions of Theorem II.1 (which is usually very hard). Indeed, at each step those conditions are replaced with the fact that the second derivative of a certain vector field is proportional to its first derivative with respect to the same variable. Moreover, the algorithm allows to compute a stabilizing feedback controller for the original system. The drawback of the linearization technique in general is that some global properties can be lost during the operation. For feedforward systems however, the transformations we obtained are globally defined though their inverses might not be. A generalization of our results to multi-input control systems is under investigation for (SFF) and (FFnice) systems. The VTOL example, which is a 2-input control system, falls in that class of (FF)-systems. As mentioned earlier, when the system is not (FFnice), that is, at least one component is nonlinear with respect to the corresponding variable, then the PDEs cannot be separated into ODEs. Extending those results outside the class of (FF)-systems requires a different approach and is already challenging for single-input systems. We are currently working in that direction with promising results and we are expecting that the new algorithms will cover all linearizable systems.

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A Pontryagin's Maximum Principle for Non-Zero Sum Differential Games of BSDEs with Applications

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***Abstract*—This technical note is concerned with a maximum principle for a new class of non-zero sum stochastic differential games. The most distinguishing feature, compared with the existing literature, is that the game systems are described by backward stochastic differential equations (BSDEs). This kind of games are motivated by some interesting phenomena arising from financial markets and can be used to characterize the players with different levels of utilities. We establish a necessary condition and a sufficient condition in the form of maximum principle for open-loop equilibrium point of the foregoing games respectively. To explain the theoretical results, we use them to study a financial problem.**

***Index Terms*—Backward stochastic differential equation (BSDE), non-zero sum stochastic differential game, open-loop equilibrium point, Pontryagin's maximum principle, portfolio choice.**

I. Introduction

A. Basic Notations and Problem Formulation

THROUGHOUT this technical note, we denote by $\mathbb{R}^k$ the k-dimensional Euclidean space, $\mathbb{R}^{k\times l}$ the collection of $k \times l$ matrices. For a given Euclidean space, we denote by $\langle\cdot,\cdot\rangle$ (respectively, $|\cdot|$) the scalar product (respectively, norm). The superscript τ denotes the transpose of vectors or matrices. Let $(\Omega,\ \mathcal{F},\ (\mathcal{F}_t),\ \mathbb{P})$ be a complete filtered probability space equipped with a natural filtration

$$\mathcal{F}_t = \sigma\{W(s); 0 \le s \le t\}, \quad t \in [0,T]$$

where $(W(\cdot))$ is an $\mathbb{R}^m$-valued standard Brownian motion defined on the space, $T > 0$ is a fixed time horizon, and $\mathcal{F} = \mathcal{F}_T$. If $x : [0,T] \times \Omega \to \mathbb{R}^l$ is an $\mathcal{F}_t$-adapted square-integrable process (i.e., $\mathbb{E}\int_0^T |x(t)|^2 dt < +\infty$), then we write $x \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$. For simplicity, sometimes we throw away $\mathbb{R}^l$ when there is no danger of confusion.

In this technical note, we study a class of non-zero sum differential games of BSDEs, which is inspirited by some interesting financial phenomena. For simplicity, we only consider the case of two players, which is similar for n players. Let us now give a detailed formulation of the problem. Consider the following BSDE:

$$\begin{cases} -dy^{v_1,v_2}(t) = f(t, y^{v_1,v_2}(t), z^{v_1,v_2}(t), v_1(t), v_2(t))dt \\ \qquad\qquad -z^{v_1,v_2}(t)dW(t), \\ y^{v_1,v_2}(T) = \xi. \end{cases} \tag{1}$$

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