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Descent Construction for *GSpin* Groups – Odd Cuspidal Case

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DESCENT CONSTRUCTION FOR GSPIN GROUPS– ODD CUSPIDAL CASE

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Abstract. In this paper we provide an extension of the theory of *descent* of Ginzburg-Rallis-Soudry to the context of "almost symplectic" representations, that is representations τ with the property that the exterior square L-function twisted by some Hecke character ω has a pole. Our theory supplements the recent work of Asgari-Shahidi on the functorial lift from $GSpin_{2n+1}$ groups to GL_{2n}.

1. Introduction

The theory of descent for symplectic cuspidal representations of the general linear group $GL_{2n}(\mathbb{A})$ was developed in a sequence of remarkable works [GRS1]-[GRS5]. In these works the authors constructed in an explicit way a space $\sigma(\pi)$ of cuspidal automorphic functions on $SO_{2n+1}(\mathbb{A})$ which weakly lifts to a cuspidal self-dual representation π of $GL_{2n}(\mathbb{A})$ with the property that $L(\pi, \wedge^2, s)$ has a pole at $s = 1$. In [C-K-PS-S2] the method of converse theorem is used to show the existence of a weak functorial lift from generic cuspidal automorphic representations of classical groups to automorphic representations of the general linear group. The combination of these methods allows the authors of [GRS4] to describe the image of the functorial lift of [C-K-PS-S1].

Thus, the conjunction of the descent method with the method of the converse theorem provides a very detailed description of the image of functoriality corresponding to the standard embedding of ${}^LG \to GL_N(\mathbb{C})$ with G a classical group. For an excellent survey we refer the reader to [So1].

Key words and phrases. Langlands functoriality, descent, unipotent integration.

Recently, Asgari and Shahidi proved in [Asg-Sha1] the existence of weak functorial lift from GSpin groups to the general linear group. Later, in the special case of $GSp(4)$ they were able to show in [Asg-Sha2] that this weak functorial lift is in fact strong in an appropriate sense.

In this paper we extend the descent method of Ginzburg, Rallis, and Soudry to GSpin groups. As a bonus we can complete the results of Asgari and Shahidi and describe the cuspidal image of their functorial lift from $GSpin_{2n+1}$, for $n \geq 2$.

Let us briefly review the method:

We begin with an irreducible unitary cuspidal automorphic representation τ of $GL_{2n}(\mathbb{A})$. We first relate the property of essential self-duality to the existence of a pole of an L-function of τ, and then construct an Eisenstein series with the L-function appearing in the constant term. In fact there are two possibilities for what the L-function is, and hence two possibilities for the structure of the Eisenstein series, and we only consider one in these notes. Our Eisenstein series will be defined on the group $GSpin_{4n}$ induced from a Levi M isomorphic to $GL_{2n} \times GL_1$. Now, a pole of the relevant L-function allows us to construct a residue representation $\mathcal{E}_{-1}(\tau, \omega)$ of $GSpin_{4n}$. Next, we give an embedding of $GSpin_{2n+1}$ into $GSpin_{4n}$, and construct, using formation of Fourier coefficient, a space of functions $DC_\omega(\tau)$ on this subgroup of $GSpin_{4n}$. We prove that $DC_\omega(\tau)$ is nonzero, and that all of the functions in it are cuspidal. It follows that it decomposes as a direct sum of irreducible automorphic cuspidal representations of $GSpin_{2n+1}$. We then show that each of these irreducible constituents lifts weakly to τ by the functorial lifting associated to the inclusion

$$^L(GSpin_{2n+1}) = GSp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}) = {}^LGL_{2n}.$$

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Without David Ginzburg and David Soudry's many careful and patient explanations of the "classical" case– $\omega = 1$– this work would not have been completed. It is important to point out that not all of the arguments shown to us have appeared in print. Nevertheless, in each case the specialization of our arguments to the case $\omega = 1$ may be correctly attributed to Ginzburg, Rallis, Soudry (with any errors or stylistic blemishes introduced being our own responsibility).

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2. The main result

Let $G = GSpin_{2n+1}$ and let $H = GL_{2n}$. Consider the inclusion

$$^LG =^L (GSpin_{2n+1}) = GSp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}) = {}^LGL_{2n} =^L H.$$

We denote this map r. Also, if $\pi \cong \otimes_v' \pi_v$ is an automorphic representation of a group $G'(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of a number field F, then the semisimple conjugacy class in the L-group ${}^LG'$ associated to the local representation π_v at an unramified place v will be denoted t_{π_v}. We say that an automorphic representation σ of $G(\mathbb{A})$ is a weak lift of the automorphic representation τ of $H(\mathbb{A})$ if for almost all places, $r(t_{\sigma_v}) = t_{\tau_v}$.

To formulate our main result we introduce the notion of η symplectic representations. Let τ be an irreducible automorphic cuspidal representation of GL_{2n}. Suppose that τ is essentially self-dual, i.e. that the contragredient $\tilde{\tau}$ of τ is isomorphic to $\tau \otimes \eta$ for some Hecke character η. It follows from [Ja-Sh2] (see remark (4.11) pp. 553-54) that $L(s, \tau \times \tau \otimes \eta)$ has a simple pole at $s = 1$. Now, $L(s, \tau \times \tau \otimes \eta)$ is the Langlands L function of the representation $\tau \boxtimes \eta$ (exterior tensor product) of the group $GL_{2n}(\mathbb{A}) \times GL_1(\mathbb{A})$ associated to the representation of the L group $GL_{2n}(\mathbb{C}) \times GL_1(\mathbb{C})$ (finite Galois form) on $M_{2n \times 2n}(\mathbb{C})$ in which $GL_{2n}(\mathbb{C})$ acts by $g \cdot X = gX\ {}^t g$ and $GL_1(\mathbb{C})$ acts by scalar multiplication. But this representation is reducible, decomposing into the subspaces of skew-symmetric and symmetric matrices. We denote the associated L functions $L(s, \tau, \wedge^2 \times \eta)$ and $L(s, \tau, sym^2 \times \eta)$ respectively. The local factors at finite ramified places may be defined using the local Langlands classification ([L2],[H-T],[Henn1]) and the definition of an Artin L function attached to a finite dimensional representation of the Weil group [Tate1], or they may be defined as in [Sha2]. By [Henn2] these two definitions agree. Then we have

$$L(s, \tau \times \tau \otimes \eta) = L(s, \tau, \wedge^2 \times \eta)L(s, \tau, sym^2 \times \eta).$$

As both of the L functions on the right-hand side are obtainable via the Langlands-Shahidi method, neither may vanish at $s = 1$ (see [Gel-Sha] §2.6 p. 84). Thus, exactly one of these two L functions has a simple pole at $s = 1$ while the other is holomorphic and nonvanishing. Similarly, if $\tilde{\tau}$ is not isomorphic to $\tau \otimes \eta$ then they are both holomorphic at $s = 1$. (This requires the extension of [Ja-Sh2] remark (4.11) to *completed* L functions– i.e., the statement that none of the local L functions has a pole at $s = 1$. The requisite facts about local L functions are well-known and a proof is reviewed at the end of Theorem 4.0.3.) One may prove the second assertion using results of Langlands via the method explained on p. 840 of [Kim].

We will say that τ is η-symplectic in case $L(s, \tau, \wedge^2 \times \eta)$ has a pole at $s = 1$ and η-orthogonal otherwise. We also define "almost symplectic" to mean "η-symplectic for some η," and "almost orthogonal" similarly.

Remarks 2.0.1. (1) *There is another natural notion of "orthogonal/symplectic representation." Specifically, one could say that an automorphic representation is orthogonal/symplectic if the space it acts on supports an invariant symmetric/skew-symmetric form. The two notions appear to be related, but do not coincide. See* [PraRam].

(2) *There is a third approach to defining a local factor for $L(s, \tau, \wedge^2 \times \eta)$, which is to apply the "gcd" construction described in* [Gel-Sha] *section I.1.6, p. 17, to the integrals in* [Ja-Sh1]. *As far as we know this is not written down anywhere.*

(3) *An integral representation for $L(s, \tau, sym^2)$ was given in* [BG]. *The problem of extending this to $L(s, \tau, sym^2 \times \eta)$ has been considered by Banks* [Banks1, Banks2]. *Nontrivial technical difficulties arise, particularly in the case we consider, when τ is defined on GL_{2n}* [Banks3].

(4) *Let AS denote the functorial lift constructed in* [Asg-Sha1]. *It is shown in* [Asg-Sha1] *that $AS(\pi)$ is nearly equivalent to $\widetilde{AS(\pi)} \otimes \omega_\pi$, where ω_π denotes the central character of the representation π. (Of course, this means that they are the same space of functions when $AS(\pi)$ is cuspidal.) Thus, in practice it turns out to make sense to use $\eta = \omega^{-1} (= \bar{\omega})$.*

Theorem 2.1. *Let τ be an irreducible cuspidal automorphic $\bar{\omega}$- symplectic representation of $GL_{2n}(\mathbb{A})$. Then there exists an irreducible generic cuspidal automorphic representation σ of $GSpin_{2n+1}(\mathbb{A})$ such that*

- *σ weakly lifts to τ, and*
- *the central character ω_σ of σ is ω.*

Remark 2.0.2. *The case $n = 1$ is trivial because $GSpin_3 = GSp_2 = GL_2$, so the inclusion r is simply the identity map, and one may take $\sigma = \tau$. Henceforth, we assume $n \geq 2$. The careful reader will find places where this assumption is crucial to the validity of the argument.*

Corollary 2.2. *The cuspidal image of the functorial lift AS described in Theorem 1.1 (p. 140) of* [Asg-Sha1] *is exactly the set of almost symplectic cuspidal automorphic representations of $GL_{2n}(\mathbb{A})$.*

3. Notation

3.1. **General.** Throughout most of the paper, F will denote a number field. In Appendix I, it will be a non-Archimedean local field of characteristic zero.

We denote by J the matrix, of any size, with ones on the diagonal running from upper right to lower left, and by J' the matrix $\left(\begin{smallmatrix} & J \\ -J & \end{smallmatrix}\right)$ of any even size. We also employ the notation ${}^t g$ for the transpose of g and ${}_t g$ for the "other transpose" $J\ {}^t g J$. We employ the shorthand $G(F\backslash\mathbb{A}) = G(F)\backslash G(\mathbb{A})$, where G is any F-group.

We denote the Weyl group of the reductive group G by W_G or by W, when the meaning is clear from context.

If π is an automorphic or local representation, then $\tilde{\pi}$ is the contragredient, and ω_π the central character.

3.2. **Similitude groups and GSpin groups.** We first define the *similitude orthogonal and symplectic groups* to be

$$GO_m = \{g \in GL_m : gJ\ {}^t g = \lambda(g)J \text{ for some } \lambda(g) \in \mathbb{G}_m\},$$

$$GSp_{2n} = \{g \in GL_{2n} : gJ'\ {}^t g = \lambda(g)J' \text{ for some } \lambda(g) \in \mathbb{G}_m\}.$$

For each of these groups the map $g \mapsto \lambda(g)$ is a rational character called the *similitude factor*. If m is odd then GO_m is in fact isomorphic to $SO_m \times GL_1$. This case will play no further role. The group GO_{2n} is disconnected; indeed the subgroup generated by SO_{2n} and $\left\{\left(\begin{smallmatrix} \lambda I_n & \\ & I_n \end{smallmatrix}\right) : \lambda \in \mathbb{G}_m\right\}$ is a connected index two subgroup, which we denote GSO_{2n}.

We shall now define GSpin groups as the groups whose duals are the similitude classical groups $GSp_{2n}(\mathbb{C}), GSO_{2n}(\mathbb{C})$. Thus we write down the based root data, but employ notation appropriate to the application in which what we write down will arise as the dual of something.

The groups GSp_{2n} and GSO_{2n} share a maximal torus, consisting of matrices of the form

$$\mathrm{diag}(t_1, \ldots, t_n, \lambda t_n^{-1}, \ldots, \lambda t_1^{-1}).$$

The coordinates used just above correspond to a choice of $\mathbb{Z}$-bases for the lattices of characters and cocharacters. For $i = 1$ to n, let e_i^* denote the character that sends this torus element to t_i for $i = 1$ to n and e_0^* being the map that sends it to the similitude factor, λ. Let $\{e_i : i = 0 \text{ to } n\}$ denote the dual basis for the cocharacter lattice. Let $X^\vee$ denote the character lattice and X the cocharacter lattice. Each similitude classical group has a Borel subgroup equal to the set of upper triangular matrices which are in it. In each case we employ this choice of Borel, and let $\Delta^\vee$ denote the set of simple roots and Δ the set of simple coroots. Then we easily compute that for GSp_{2n}

$$\Delta^\vee = \{e_i^* - e_{i+1}^*,\ i = 1 \text{ to } n-1\} \cup \{2e_n^* - e_0^*\}.$$

$$\Delta = \{e_i - e_{i+1},\ i = 1 \text{ to } n-1\} \cup \{e_n\}.$$

and for GSO_{2n}

$$\Delta^\vee = \{e_i^* - e_{i+1}^*,\ i = 1 \text{ to } n-1\} \cup \{e_{n-1}^* + e_n^* - e_0^*\}.$$

$$\Delta = \{e_i - e_{i+1},\ i = 1 \text{ to } n-1\} \cup \{e_{n-1} + e_n\}.$$

We now define $GSpin_{2n+1}$ to be the $F-$split connected reductive algebraic group having based root datum dual to that of GSp_{2n}, and $GSpin_{2n}$ to be the one having based root datum dual to that of GSO_{2n}. We have here used the fact that F-split connected reductive algebraic groups are classified by based root data, for which see p.274 of [Spr].

To save space, the group $GSpin_m$ will usually be denoted G_m.

Observe that in either the odd or even case e_0^* is a generator for the lattice of cocharacters of the center of G_m.

Because we define G_m in the manner we do, it comes equipped with a choice of Borel subgroup and maximal torus, as do various reductive subgroups we shall consider below. In each case, we denote the Borel subgroup of the reductive group G by $B(G)$, and the maximal torus by $T(G)$.

A straightforward adaptation of the proof of Theorem 16.3.2 of [Spr] shows that there exist surjections $G_m \to SO_m$ defined over F. We fix one such and denote it pr. We require that pr is such that $\mathrm{pr}(B(G_m))$ consists of upper triangular matrices.

An alternative description of the same group as a quotient of $Spin_m \times GL_1$ is given in [Asg]. Proposition 2.4 on p. 678 of [Asg] shows that the two definitions are equivalent.

For those familiar with the construction of $Spin_m$ as a subgroup of the multiplicative group of a Clifford algebra, we remark that there is a third construction of $GSpin_m$ as the slightly larger group obtained by including the nonzero scalars in the Clifford algebra as well. In this guise, it is sometimes referred to as the "Clifford group." (See, e.g., [I] p.999.) This description will not play a role for us.

We will construct an Eisenstein series on G_{2m} induced from a standard parabolic $P = MU$ such that M is isomorphic to $GL_m \times GL_1$. There are two such parabolics. We choose the one in which we delete the root $e_{m-1} + e_m$ and the coroot $e^*_{m-1} + e^*_m - e^*_0$ from the based root datum. *We shall refer to this parabolic as the "Siegel."*

Remark 3.2.1.
- *We can identify the based root datum of the Levi M with that of $GL_m \times GL_1$ in such a fashion that e_0 corresponds to GL_1 and does not appear at all in GL_m. We can then identify M itself with $GL_m \times GL_1$ via a particular choice of isomorphism which is compatible this and with the usual usage of e_i, e_i^* for characters, cocharacters of the standard torus of GL_m.*
- *The lattice of rational characters of M is spanned by the maps $(g, \alpha) \mapsto \alpha$ and $(g, \alpha) \mapsto \det g$. Restriction defines an embedding $X(M) \to X(T)$, which sends these maps to e_0 and $(e_1 + \cdots + e_m)$, respectively. By abuse of notation, we shall refer to the rational character of M corresponding to e_0 as e_0 as well.*
- *The modulus of P is $(g, \alpha) \to \det g^{(m-1)}$.*

The group G_{2n} has an involution † which reverses the last two simple roots. The effect is such that

$$\mathrm{pr}({}^{\dagger}g) = \begin{pmatrix} I_{n-1} & & & \\ & & 1 & \\ & 1 & & \\ & & & I_{n-1} \end{pmatrix} \mathrm{pr}(g) \begin{pmatrix} I_{n-1} & & & \\ & & 1 & \\ & 1 & & \\ & & & I_{n-1} \end{pmatrix}.$$

As is well known, there is a group $Pin_{4n} \supset Spin_{4n}$ such that pr extends to a two-fold covering $Pin_{4n} \to O_{4n}$. The involution † can be realized as conjugation by a preimage of the above permutation matrix.

We also fix a maximal compact subgroup K_m of $G_m(\mathbb{A})$. Any which satisfies the conditions required in [MW1] (see pages 1 and 4) will do.

3.3. **Unramified Correspondence.**

Lemma 3.3.1. *Suppose that $\tau \cong \otimes'_v \tau_v$ is an $\bar{\omega}$-symplectic irreducible cuspidal automorphic representation of $GL_{2n}(\mathbb{A})$. Let v be a place such that τ_v is unramified. Let $t_{\tau,v}$ denote the semisimple conjugacy class in $GL_{2n}(\mathbb{C})$ associated to τ_v. Let $r : GSp_{2n}(\mathbb{C}) \to GL_{2n}(\mathbb{C})$ be the natural inclusion. Then $t_{\tau,v}$ contains elements of the image of r.*

Proof. For convenience in the application, we take GL_{2n} to be identified with a subgroup of the Levi of the Siegel parabolic as in section 3.2. Since τ_v is both unramified and generic, it is isomorphic to $\mathrm{Ind}_{B(GL_{2n})(F_v)}^{GL_{2n}(F_v)} \mu$ for some unramified character μ of the maximal torus $T(GL_{2n})(F_v)$ such that this induced representation is irreducible. (See [Car], section 4, [Z] Theorem 8.1, p. 195.) Let $\mu_i = \mu \circ e_i^*$.

Since $\tau \cong \tilde{\tau} \otimes \omega$, it follows that $\tau_v \cong \tilde{\tau}_v \otimes \omega_v$ and from this we deduce that $\{\mu_i : 1 \le i \le 2n\}$ and $\{\mu_i^{-1}\omega : 1 \le i \le 2n\}$ are the same set.

By Theorem 1, p. 213 of [Ja-Sh1], we have $\prod_{i=1}^{2n} \mu_i = \omega^n$.

Now, what we need to prove is the following: if S is a set of $2n$ unramified characters of F_v, such that

(1) $\prod_{i=1}^{2n} \mu_i = \omega^n$

(2) For each i there exists j such that $\mu_i = \mu_j^{-1}\omega$

then there is a permutation $\sigma : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ such that $\mu_{\sigma(i)} = \omega\mu_{2n-\sigma(i)}^{-1}$ for $i = 1$ to n. This we prove by induction on n. When $n = 1$, we know that $\mu_1 = \mu_i^{-1}\omega$ for $i = 1$ or 2. If $i = 2$ we are done, while if $i = 1$ we use $\omega = \mu_1\mu_2$ to obtain $\mu_1 = \mu_2$, and the desired assertion. Now, if $n > 1$ it is sufficient to show that there exist $i \ne j$ such that $\mu_i = \mu_j^{-1}\omega$. If there exists i such that $\mu_i \ne \mu_i^{-1}\omega$ then this is clear. On the other hand, there are exactly two unramified characters μ such that $\mu = \mu^{-1}\omega$. The result follows □

3.4. **Unipotent subgroups and their characters.** The kernel of pr consists of semisimple elements. In particular, the number of unipotent elements of a fiber is zero or one, and it's one if and only if the element of SO_m is unipotent. In other words, pr yields a bijection of unipotent elements (indeed, an isomorphism of unipotent subvarieties), and we may specify unipotent elements or subgroups by their images under pr . This also defines coordinates for any unipotent element or subgroup, which we use when defining characters. Thus, we write u_{ij} for the i, j entry of $\mathrm{pr}(u)$.

Above we fixed a specific isomorphism of a subgroup of G_{2m} with GL_m. If u is a unipotent element of of this subgroup this identification with an $m \times m$ matrix gives a second definition of u_{ij} This is not a problem, however, as the two definitions agree.

Most of the unipotent groups we consider are subgroups of the maximal unipotent of G_m consisting of elements u with $\mathrm{pr}(u)$ upper triangular. We denote this group $U_{\max}$. A complete set of coordinates is $\{u_{ij} : 1 \le i < j \le m - i\}$. We denote the opposite maximal unipotent by $\overline{U_{\max}}$. It consists of all unipotent elements of G_m such that $\mathrm{pr}(u)$ is *lower* triangular.

We fix once and for all a character ψ_0 of $\mathbb{A}/F$. We use this character together with the coordinates just above to specify characters of our unipotent subgroups.

When specifying subgroups of $U_{\max}$ and their characters, the restriction to $\{(i, j) : 1 \le i < j \le m - i\}$ is implicit.

It will also sometimes be necessary to describe unipotent subgroups such that only a few of the entries in the corresponding elements of SO_m are nonzero. For this purpose we introduce the notation $e'_{ij} = e_{ij} - e_{m+1-j,m+1-i}$. One may check that for all $i \ne j$ and $a \in F$, the matrix $I + ae'_{ij}$ is an element of $SO_m(F)$.

3.5. **"Unipotent periods".** We now introduce the framework within which, we believe, certain of the computations involved in the descent construction can be most easily understood.

Let G be a reductive algebraic group defined over a number field F. If U is a unipotent subgroup of G and ψ_U is a character of $U(F\backslash\mathbb{A})$, we define the *unipotent period* (U, ψ_U) associated to this pair to be given by the formula

$$\varphi^{(U,\psi_U)}(g) := \int_{U(F\backslash\mathbb{A})} \varphi(ug)\psi_U(u)du.$$

Clearly, φ must be restricted to a space of left $U(F)$-invariant functions such that the integral is defined (for example, because φ is smooth).

Let $\mathcal{U}$ denote the set of unipotent periods. For V a space of functions defined on $G(\mathbb{A})$, put

$$\mathcal{U}^{\perp}(V) = \{(U,\psi) \in \mathcal{U} : \varphi^{(U,\psi)} \equiv 0 \ \forall \varphi \in V\}.$$

When V is the space of a representation π we will employ also the notation $\mathcal{U}^{\perp}(\pi)$. We also employ the notation $(U,\psi) \perp V$ for $(U,\psi) \in \mathcal{U}^{\perp}(V)$ and similarly $(U,\psi) \perp \pi$.

We also require a vocabulary to express relationships among unipotent periods. We shall say that

$$(U,\psi_U) \in \langle (U_1,\psi_{U_1}),\dots,(U_n,\psi_{U_n})\rangle$$

if $V \perp (U_i,\psi_{U_i}) \forall i \Rightarrow V \perp (U,\psi_U)$. Clearly, if $(U_1,\psi_{U_1}) \in \langle (U_2,\psi_2),(U_3,\psi_3)\rangle$, and $(U_2,\psi_2) \in \langle (U_4,\psi_4),(U_5,\psi_5)\rangle$ then $(U_1,\psi_1) \in \langle (U_3,\psi_3),(U_4,\psi_4),(U_5,\psi_5)\rangle$.

We also use notation $(U_1,\psi_1)|(U_2,\psi_2)$, or the language "$(U_1,\psi_1)$ divides (U_2,ψ_2)," " (U_2,ψ_2) is divisible by (U_1,ψ_1) " for $(U_2,\psi_2) \in \langle (U_1,\psi_1)\rangle$. Finally, $(U_1,\psi_1) \sim (U_2,\psi_2)$ means $(U_1,\psi_1)|(U_2,\psi_2)$ and $(U_2,\psi_2)|(U_1,\psi_1)$. This is an equivalence relation and we shall refer to unipotent periods which are related in this way as "equivalent."

It is *sometimes* possible to compose unipotent periods. Specifically, if $f^{(U_1,\psi_1)}$ is left-invariant by $U_2(F)$, then one may consider $(f^{(U_1,\psi_1)})^{(U_2,\psi_2)}$. We denote the composite by $(U_2,\psi_2)\circ(U_1,\psi_1)$.

Now, suppose that U is the unipotent radical of a parabolic P of G with Levi M. The choice of ψ_0 gives rise to an identification of the space of characters of $U(F)\backslash U(\mathbb{A})$ with the F points of $\overline{U}/(\overline{U},\overline{U})$ which is compatible with the action of $M(F)$. Here $\overline{U}$ denotes the unipotent radical of the parabolic $\overline{P}$ of G opposite to P. For ϑ a character, let M^{ϑ} denote the stabilizer of ϑ (regarded as a point in $\overline{U}/(\overline{U},\overline{U})(F)$) in M. So M^{ϑ} is an algebraic subgroup of M defined over F.

Definition 3.5.1. *Then we define $FC^{\vartheta} : C^{\infty}(G(F\backslash\mathbb{A})) \to C^{\infty}(M^{\vartheta}(F\backslash\mathbb{A}))$ by*

$$FC^{\vartheta}(\varphi)(m) = \varphi^{(U,\vartheta)}(m) = \int_{U(F\backslash\mathbb{A})} \varphi(um)\vartheta(u)du.$$

This is certainly an $M^{\vartheta}(\mathbb{A})$-equivariant map.

4. Eisenstein series

Let τ be an irreducible cuspidal automorphic representation of GL_m.

We will construct an Eisenstein series on G_{2m} induced from the Siegel parabolic $P = MU$. Let s be a complex variable. Using the identification of M with $GL_m \times GL_1$ fixed in section 3.2 above, we define an action of $M(\mathbb{A})$ on the space of τ by

$$(g,\alpha)\cdot\varphi(g_1) = \varphi(g_1 g)\omega(\alpha)|\det g|^s.$$

We employ the "outer tensor product" notation for this representation of $M(\mathbb{A})$, denoting it $\tau \otimes |\det|^s \boxtimes \omega$.

For each s we have the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tau \otimes |\det|^s \boxtimes \omega$, (normalized induction) of $G_{2m}(\mathbb{A})$. The standard realization of this representation is action by right translation on the space $V^{(1)}(s,\tau,\omega)$ given by

$$\left\{ \tilde{F} : G_{2m}(\mathbb{A}) \to V_{\tau}, \text{ smooth } \middle| \tilde{F}((g,\alpha)h)(g_1) = \tilde{F}(h)(g_1 g)\omega(\alpha)|\det g|^{s+\frac{(m-1)}{2}} \right\}.$$

(The factor $|\det g|^{\frac{(m-1)}{2}}$ is equal to $|\delta_P|$, and makes the induction normalized.) A second useful realization is action by right translation on

$$V^{(2)}(s,\tau,\omega) = \left\{ f : G_{2m}(\mathbb{A}) \to \mathbb{C}, \middle| f(h) = \tilde{F}(h)(id), \tilde{F} \in V^{(1)}(s,\tau,\omega) \right\}.$$

(Here id denotes the identity element of $GL_m(\mathbb{A})$.)

These representations fit together into a fiber bundle over $\mathbb{C}$. So a section of this bundle is a function f defined on $\mathbb{C}$ such that $f(s) \in \mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tau \otimes |\det|^s \boxtimes \omega$ for each s. We shall only require the use of flat, K-finite sections, which are defined as follows. Take $f_0 \in \mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tau \boxtimes \omega$, K-finite, and define $f(s)(h)$ by

$$f(s)(u(g,\alpha)k) = f_0(u(g,\alpha)k)|\det g|^s$$

for $u \in U(\mathbb{A}), g \in GL_m(\mathbb{A}), \alpha \in \mathbb{A}^\times, k \in K$. This is well defined. (I.e., although g is not uniquely determined in the decomposition, $|\det g|$ is. Cf. the definition of m_P on p.7 of [MW1].)

Remark 4.0.2. *Clearly, the function f is determined by $f(s_0)$ for any s_0. In particular, any function of f may be regarded as a function of $f_{s_0} \in \mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tau \otimes |\det|^{s_0} \boxtimes \omega$, for any particular value of s_0. We have exploited this fact with $s_0 = 0$ to streamline the definitions.* A posteriori *it will become clear that the point $s = \frac{1}{2}$ is of particular importance, and we shall then switch to $s_0 = \frac{1}{2}$.*

For such f the sum

$$E(f)(g)(s) := \sum_{\gamma \in P(F)\backslash G(F)} f(s)(\gamma g)$$

converges for $\mathrm{Re}(s)$ sufficiently large ([MW1], §II.1.5, pp.85-86), and has meromorphic continuation to $\mathbb{C}$ ([MW1] §IV.1.8(a), IV.1.9(c),p.140). These are our Eisenstein series. We collect some of their well-known properties in the following theorem.

Theorem 4.0.3. (1) *The following are equivalent:*

(a) *There exist $f_0 \in \mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tau \boxtimes \omega$, and $g \in G_{2m}(\mathbb{A})$ such that $E(f)(g)$ has a pole at $s = \frac{1}{2}$.*

(b) *The representation τ is $\bar{\omega}$-symplectic.*

(2) *In the case when the equivalent conditions above hold, the pole at $s = \frac{1}{2}$ is simple.*

(3) *Let us now assume the equivalent conditions of (1) hold, and regard f as a function of $f_{\frac{1}{2}} \in \mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$. If $E_{-1}(f_{\frac{1}{2}})(g) := \lim_{s \to \frac{1}{2}} (s - \frac{1}{2}) E(f)(g)(s)$, then $E_{-1}(f)$ is an L^2 function for all $f_{\frac{1}{2}} \in \mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$.*

(4) *The function E_{-1} is an intertwining operator from $\mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$ into the space of L^2 automorphic forms.*

(5) *If $\mathcal{E}_{-1}(\tau, \omega)$ is the image of E_{-1}, and ψ_{LW} is the character of $U_{\max}$ given by $\psi_{LW}(u) = \psi_0(\sum_{i=1}^{m-1} u_{i,i+1})$, then $(U_{\max}, \psi_{LW}) \notin \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$.*

Proof. Let w denote the shortest element of $W_M w_\ell W_M$ where $W_M = M \cap W_{G_m}$ is the Weyl group of GL_m embedded as the intersection of the Weyl group of G_{2m} with the Levi M of P and w_ℓ is the longest element of G_{2m}. Let $\dot{w}$ denote any representative for w in $G_{2m}(F)$, and let $U_w = U_{\max} \cap \dot{w}^{-1} \overline{U_{\max}} \dot{w}$ The standard intertwining operator $M(s)$ is an operator from $\mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} (\tau \otimes |\det|^s \boxtimes \omega)$ to $\mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} (\tau \otimes |\det|^s \boxtimes \omega) \circ Ad(\dot{w})$.

Lemma 4.0.4. $(\tau \otimes |\det|^s \boxtimes \omega) \circ Ad(\dot{w}) \cong \tilde{\tau} \otimes \omega \otimes |\det|^{-s} \boxtimes \omega$.

Proof. We compute the automorphism of the character and cocharacter lattices given by $Ad(\dot{w})$. It is determined by the fact that it permutes the roots and coroots, and the condition that it sends every simple root except $e_{m-1} + e_m$ to a positive root, and sends $e_{m-1} + e_m$ to a negative root. The only such map is the one given by

$$e_i \mapsto -e_{m+1-i} \quad i > 0; \qquad e_0 \mapsto e_0 + e_1 + \cdots + e_m;$$
$$e_i^* \mapsto -e_{m+1-i}^* + e_0^* \quad i > 0; \qquad e_0^* \mapsto e_0^*.$$

It then follows from Theorem 16.3.2 of [Spr] that up to inner automorphism $Ad(\dot{w})(g,\alpha)$ is given by $({}_tg^{-1}, \alpha \cdot \det g)$, whence

$$(\tau \otimes |\det|^s \boxtimes \omega) \circ Ad(\dot{w}) \cong (\tilde{\tau} \otimes \omega \otimes |\det|^s \boxtimes \omega).$$

Here we have used the fact that the contragredient $\tilde{\tau}$ of τ may be realized as an action on the same space of functions as τ via $g \cdot \varphi(g_1) = \varphi(g_1\, {}_tg^{-1})$. This follows from strong multiplicity one and the analogous statement for local representations, for which see [GK75] page 96, or [BZ1] page 57. □

Corollary 4.0.5. *It $\tau \cong \tilde{\tau} \otimes \omega$ (for example, if τ is $\bar{\omega}$-symplectic) then $(\tau \otimes |\det|^s \boxtimes \omega) \circ Ad(\dot{w}) \cong \tau \ \otimes |\det|^{-s} \boxtimes \omega$.*

The operator $M(s)$ is defined for $\mathrm{Re}(s)$ large by the integral (cf. [MW1] II.1.6)

$$(M(s)f(s))(g) = \int_{U_w(\mathbb{A})} f(s)(\dot{w}ug)du,$$

and elsewhere by meromorphic continuation ([MW1] IV.1.8 (b)).

It is an application of [MW1] IV.1.9 and II.1.7 that if Q is a standard parabolic of G_{2m} and $f_0 \in \mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tau \boxtimes \omega$, then the constant term of $E(f_0, s)$ along Q (cf. [MW1] I.2.6) is trivial unless $Q = P$ in which case it is

$$f(s) + M(s)f(s).$$

It follows from [MW1] I.4.10, that $E(f)$ has a pole at s_0 if and only if $M(s)f(s)$ does. We show below that for $s_0 = \frac{m}{2}$ this is the case if and only if τ is $\bar{\omega}$-symplectic. This will complete the proof of item (1).

Item (2) is an application of [MW1] IV.1.11 (c).

Since $f(s)$ is clearly entire, it now follows from [MW1] I.4.11 that when $E(f)$ has a residue at s_0 with $\mathrm{Re}(s_0) > 0$, this residue is L^2. Item (3) follows.

It follows from [MW1] IV.1.9 (b)(i) applied to $(s - \frac{1}{2})E(f)$ (which is valid by IV.1.9 (d)) that the residue is an automorphic form. To complete the proof of (4), let $\rho(g)$ denote right translation. It is clear that for values of s in the domain of convergence, $(s - \frac{1}{2})E(\rho(g)f)(s) = (s - \frac{1}{2})\rho(g)(E(f)(s))$. By uniqueness of analytic continuation, the equality also holds at values of s where both sides are defined by analytic continuation, including $\frac{1}{2}$. The action of the universal enveloping algebra at the infinite places is dispatched in the same manner.

Similarly, in the case when $M(s)f(s)$ has a pole at $\frac{1}{2}$, we may continue $(s - \frac{1}{2})M(s)f(s)$ to $\frac{1}{2}$, where it gives a nontrivial element of $\mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tilde{\tau} \otimes \omega \otimes |\det|^{-\frac{1}{2}} \boxtimes \omega$. (Cf. [MW1] IV.1.4.) Of course, we may also write "$\mathrm{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tau \otimes |\det|^{-\frac{1}{2}} \boxtimes \omega$," since it will be shown below that this only occurs when τ is $\bar{\omega}$-symplectic. Item (5) now follows from the genericity of τ.

What remains is to complete the proof that $M(s)$ has a pole at $\frac{1}{2}$ iff τ is $\bar{\omega}$-symplectic.

Now, let $\tilde{M}(s)$ denote the analogue of $M(s)$ defined using $V^{(1)}(s, \tau, \omega)$. It maps into the space $V^{(3)}(-s, \tilde{\tau} \otimes \omega, \omega)$ given by

$$\left\{ \tilde{F} : G_{2m}(\mathbb{A}) \to V_\tau, \text{ smooth } \middle| \tilde{F}((g,\alpha)h)(g_1) = \omega(\alpha \det g)|\det g|^{-s+\frac{(m-1)}{2}} \tilde{F}(h)(g_1\, {}_tg^{-1}) \right\}.$$

Fix realizations of the local representations τ_v and an isomorphism $\iota : \otimes_v' \tau_v \to \tau$. Define, for each v, $V^{(1)}(s, \tau_v, \omega_v)$ to be

$$\left\{ \tilde{F}_v : G_{2m}(F_v) \to V_{\tau_v}, \text{ smooth } \middle| \tilde{F}_v((g,\alpha)h) = \omega_v(\alpha)|\det g|_v^{s+\frac{(m-1)}{2}} \tau_v(g)\tilde{F}_v(h) \right\},$$

$V^{(3)}(s, \tilde{\tau}_v \otimes \omega_v, \omega_v)$ to be

$$\left\{ \tilde{F}_v : G_{2m}(F_v) \to V_{\tau_v}, \text{ smooth } \middle| \tilde{F}_v((g,\alpha)h) = \omega_v(\alpha \det g)|\det g|_v^{s+\frac{(m-1)}{2}} \tau_v({}_tg^{-1})\tilde{F}_v(h) \right\}.$$

Then the formula

$$\tilde{\iota}(\otimes_v \tilde{F}_v)(g) = \iota(\otimes'_v \tilde{F}_v(g_v))$$

defines maps

$$\otimes'_v V^{(1)}(s,\tau_v,\omega_v) \to V^{(1)}(s,\tau,\omega),$$
$$\otimes'_v V^{(3)}(s,\tilde{\tau}_v \otimes \omega_v,\omega_v) \to V^{(3)}(s,\tilde{\tau}\otimes\omega,\omega),$$

both of which we denote by $\tilde{\iota}$.

It is known that each map is, in fact, an isomorphism. For the benefit of the reader we sketch an argument. On pp. 307 of [Sha1] certain explicit elements of (a generalization of) $V^{(1)}(s,\tau,\omega)$ are constructed as integrals involving matrix coefficients. Using Schur orthogonality, one may check that $\tilde{F}$ is expressible in this form iff both the K-module it generates and the $K \cap M(\mathbb{A})$-module it generates are irreducible. It is clear that such vectors span the space of all K-finite vectors. On the other hand the (finite dimensional) space of matrix coefficients of this irreducible representation of K is spanned by those that factor as a product of matrix coefficients of local representations, and these are clearly in the image of $\tilde{\iota}$.

For $\tilde{F}_v \in V^{(1)}(s,\tau_v,\omega_v)$, let

$$A_v(s)\tilde{F}_v(g) = \int_{U_w(F_v)} \tilde{F}_v(\dot{w}ug)du.$$

Then the following diagram commutes

$$\begin{array}{ccc}
\otimes'_v V^{(1)}(s,\tau_v,\omega_v) & \xrightarrow{A(s)} & \otimes'_v V^{(3)}(-s,\tilde{\tau}_v\otimes\omega_v,\omega_v) \\
\tilde{\iota}\Big\downarrow & & \tilde{\iota}\Big\downarrow \\
V^{(1)}(s,\tau,\omega) & \xrightarrow{\tilde{M}(s)} & V^{(3)}(-s,\tilde{\tau}\otimes\omega,\omega),
\end{array}$$

with $A(s) := \otimes_v A_v(s)$.

Now, $M(s)f(s)$ has a pole (i.e., there exists $g \in G_{4n}(\mathbb{A})$ such that $M(s)f(s)(g)$ has a pole) if and only if $\tilde{M}(s)\tilde{F}(s)$ has a pole (i.e., there exist $g \in G_{4n}(\mathbb{A})$ and $m \in M(\mathbb{A})$ such that $\tilde{M}(s)\tilde{F}(s)(g)(m)$ has a pole), where $\tilde{F}$ is the element of $V^{(1)}(s,\tau,\omega)$ such that $f(g) = \tilde{F}(g)(id)$.

We wish to show that there exists $\tilde{F}$ such that this is the case iff τ is $\bar{\omega}$-symplectic. Clearly, we may restrict attention to $\tilde{F}$ of the form $\tilde{\iota}(\otimes_v \tilde{F}_v)$.

Recall that for all but finitely many non-archimedean v, the space V_{τ_v} comes equipped with a choice of $GL_{2n}(\mathfrak{o}_v)$-fixed vector ξ°_v used to define the restricted tensor product.

If $\tilde{F} = \tilde{\iota}(\otimes_v \tilde{F}_v) \in V^{(1)}(s,\tau,\omega)$, then there is a finite set S of places, such that if $v \notin S$ then v is non-archimedean, τ_v is unramified, and $\tilde{F}_v(s) = \tilde{F}^\circ_{(s,\tau_v,\omega_v)}$ is the unique element of $V^{(1)}(s,\tau_v,\omega_v)$ satisfying $\tilde{F}_{(s,\tau_v,\omega_v)}(k) = \xi^\circ_v$ for all $k \in G_{4n}(\mathfrak{o}_v)$.

Now

$$A_v(s)\tilde{F}^\circ_{(s,\tau_v,\omega_v)} = \frac{L_v(2s,\tau_v,\wedge^2\times\bar{\omega}_v)}{L_v(2s+1,\tau_v,\wedge^2\times\bar{\omega}_v)}\tilde{F}^\circ_{(-s,\tilde{\tau}_v\otimes\omega_v,\omega_v)}.$$

(A proof of this appears in [L1], albeit not in this precise language. See especially pp. 25-27.) Thus,

$$A(s)\tilde{\iota}(\otimes_v\tilde{F}_v) = \frac{L^S(2s,\tau,\wedge^2\times\bar{\omega})}{L^S(2s+1,\tau,\wedge^2\times\bar{\omega})}\tilde{\iota}\left(\left(\bigotimes_{v\in S} A_v(s)\tilde{F}_v(s)\right)\otimes\left(\bigotimes_{v\notin S}\tilde{F}^\circ_{-s,\tilde{\tau}_v\otimes\omega_v,\omega_v}\right)\right).$$

To complete the proof of 1 (and of the theorem) we must show:

- **(i):** $A_v(s)$ is holomorphic and nonvanishing (i.e., not the zero operator) on $Ind_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})}\tau\otimes|\det|^s\boxtimes\omega$ at $s=\frac{1}{2}$, for all τ.

(ii): $L_v(s, \tau_v, \wedge^2 \times \bar{\omega}_v)$ is holomorphic and nonvanishing at $s = 1$, for all τ_v.
(iii): $L^S(s, \tau, \wedge^2 \times \bar{\omega})$ is holomorphic and nonvanishing at $s = 2$.

Item (iii) is covered by Proposition 7.3 of [Kim-Sh].

Items (i) and (ii) are essentially contained in Proposition 3.6, p. 153 of [Asg-Sha1]. Since what we need is *part* of the same information, presented differently, we repeat the part of the arguments we are using.

The nonvanishing part of (i) is a completely general fact (i.e., holds at least for any Levi of any split reductive group). For example, the only element of the arguments made on p. 813 of [GRS3] which is particular to the situation they consider there (the Siegel of Sp_{4n}) is the precise ratio of L functions appearing in the constant term.

Similarly, local L functions never vanish. At a finite prime the local L function is $P(q_v^{-s})^{-1}$ for some polynomial P, while at an infinite prime it is given in terms of the Γ function and functions of exponential type.

We turn to holomorphicity.

Lemma 4.0.6. *Let π_v be any representation of $GL_m(F_v)$, which is irreducible, generic, and unitary. Then there exist*

- *integers $k_1, \dots, k_r$ of such that $k_1 + \dots + k_r = m$,*
- *real numbers $\alpha_1, \dots, \alpha_r \in (-\frac{1}{2}, \frac{1}{2})$,*
- *discrete series representations δ_i of $GL_{k_i}(F_v)$ for $i = 1$ to r*

such that

$$\pi_v \cong Ind_{P_{(k)}(F_v)}^{GL_m(F_v)} \bigotimes_{i=1}^{r} (\delta_i \otimes |\det{}_i|^{\alpha_i}).$$

Here $P_{(k)}$ denotes the standard parabolic of GL_m with Levi consisting of block diagonal matrices with the block sizes $k_1, \dots, k_r$ (in that order), and $\det_i$ *denotes the determinant of the ith block.*

Remark 4.0.7. *In fact, one may prove a much more precise statement, but the above is what is needed for our purposes.*

Proof. This follows from the main theorem of [Tad2] (see p. 3) together with the fact that the representation denoted $u(\delta, m)$ in that paper is only generic if $m = 1$. For this latter statement see the "Proof of (a)⇒(f)" on p. 93 of [Vog] in the Archimedean case and Theorem 8.1 on p. 195 of [Z] in the non-Archimedean case. (For the notion of "highest derivative" see p. 452 of [BZ2]: a representation is generic iff its "highest derivative" is the trivial representation of the trivial group, which corresponds to the empty multiset under the Zelevinsky classification.) ☐

Continuing with the proof of 1, let $(k) = (k_1, \dots, k_r)$, $\delta = (\delta_1, \dots, \delta_r)$ and $\alpha = (\alpha_1, \dots, \alpha_r)$ be obtained from τ_v as just above, and let $\tilde{P}_{(k)}$ denote the standard parabolic of G_{2m} which is contained in the Siegel parabolic P such that $\tilde{P}_{(k)} \cap M = P_{(k)}$.

Then

$$Ind_{P(F_v)}^{G_{2m}(F_v)} \tau_v \otimes |\det|_v^s \boxtimes \omega_v^s \cong Ind_{\tilde{P}_{(k)}(F_v)}^{G_{2m}(F_v)} \boxtimes_{i=1}^r (\delta_i \otimes |\det{}_i|_v^{s+\alpha_i}) \boxtimes \omega_v.$$

This family (as s varies) of representations lies inside the larger family,

$$Ind_{\tilde{P}_{(k)}(F_v)}^{G_{2m}(F_v)} \boxtimes_{i=1}^r (\delta_i \otimes |\det{}_i|^{s_i}) \boxtimes \omega_v \qquad s = (s_1, \dots, s_r) \in \mathbb{C}^r,$$

and our intertwining operator $A_v(s)$ is the restriction, to the line $s_i = s + \alpha_i$ of the standard intertwining operator for this induced representation, which we denote $A_v(\underline{s})$. This operator is defined, for all $\text{Re}(s_i)$ sufficiently large, by the same integral as $A_v(s)$.

A result of Harish-Chandra says that "$\text{Re}(s_i)$ sufficiently large" can be sharpened to "$\text{Re}(s_i) > 0$." (This is because all δ_i are discrete series, although tempered would be enough.) This result is given in the p-adic case as [Sil] Theorem 5.3.5.4, and in the Archimedean case, [Kn] Theorem 7.22, p. 196.

Hence, the integral defining $A_v(s)$ converges for $s > \max_i(-\alpha_i)$, and in particular converges at $\frac{1}{2}$.

From the relationship between the local L functions and the so-called local coefficients, it follows that the local L functions are also holomorphic in the same region. For this relationship see [Sha3] for the Archimedean case and [Sha2], p. 289 and p. 308 for the non-Archimedean case.

This completes the proof of (i) and (ii), of (1), and of the theorem. □

5. Main Results

5.1. **Descent Construction.** Next we describe certain unipotent periods of G_{2m} which play a key role in the argument. For $1 \le \ell < m$, let N_ℓ be the subgroup of $U_{\max}$ defined by $u_{ij} = 0$ for $i > \ell$. (Recall that according to the convention above, this refers only to those i, j with $i < j \le m - i$.) This is the unipotent radical of a standard parabolic Q_ℓ having Levi L_ℓ isomorphic to $GL_1^\ell \times G_{2m-2\ell}$.

Let ϑ be a character of N_ℓ then we may define

$$DC^\ell(\tau, \omega, \vartheta) = FC^\vartheta \mathcal{E}_{-1}(\tau, \omega).$$

Theorem 5.1.1. *Let τ be an $\bar{\omega}$-symplectic irreducible cuspidal automorphic representation of $GL_{2n}(\mathbb{A})$. If $\ell \ge n$, and ϑ is in general position, then*

$$DC^\ell(\tau, \omega, \vartheta) = \{0\}.$$

Proof. By Theorem 4.0.3, (3) the representation $\mathcal{E}_{-1}(\tau, \omega)$ decomposes discretely. Let $\pi \cong \otimes'_v \pi_v$ be one of the irreducible components, and $p_\pi : \mathcal{E}_{-1}(\tau, \omega) \to \pi$ the natural projection.

Fix a place v_0 such which τ_{v_0} and π_{v_0} are unramified. For any $\xi^{v_0} \in \otimes'_{v \ne v_0} Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|_v^{\frac{1}{2}} \boxtimes \omega_v$ we define a map

$$i_{\xi^{v_0}} : Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0} \to Ind_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$$

by $i_{\xi^{v_0}}(\xi_v) = \iota(\xi_{v_0} \otimes \xi^{v_0})$, where ι is an isomorphism of the restricted product $\otimes'_v Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|_v^{\frac{1}{2}} \boxtimes \omega_v$ with the global induced representation $Ind_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$. Clearly

$$\mathcal{E}_{-1}(\tau, \omega) = E_{-1} \circ \iota(\otimes'_v Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|_v^{\frac{1}{2}} \boxtimes \omega_v).$$

For any decomposable vector $\xi = \xi_{v_0} \otimes \xi^{v_0}$,

$$p_\pi \circ E_{-1} \circ \iota(\xi) = p_\pi \circ E_{-1} \circ i_{\xi^{v_0}}(\xi_{v_0}).$$

Thus, π_{v_0} is a quotient of $Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0}$, and hence (since we took v_0 such that π_{v_0} is unramified) it is isomorphic to the unramified constituent ${}^{un}Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0}$.

Denote the isomorphism of π with $\otimes'_v \pi_v$ by the same symbol ι. This time, fix $\zeta^{v_0} \in \otimes'_{v \ne v_0} \pi_v$, and define $i_{\zeta^{v_0}} :^{un} Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0} \to \pi$. It follows easily from the definitions that

$$FC^\vartheta \circ i_{\zeta^{v_0}}$$

factors through the Jacquet module $\mathcal{J}_{N_\ell, \vartheta}({}^{un}Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0})$. In appendix 6 we show that this Jacquet module is zero. The result follows. □

Remark 5.1.2. *A general character of N_ℓ is of the form*

$$\psi(c_1u_{1,2}+\cdots+c_{\ell-1}u_{\ell-1,\ell}+d_1u_{\ell,\ell+1}+\cdots+d_{4n-2\ell}u_{\ell,4n-\ell}).$$

The Levi L_ℓ acts on the space of characters (cf. section 3.5). Over an algebraically closed field there is an open orbit, which consists of all those elements such that $c_i \neq 0$ for all i and ${}^t\underline{d}J\underline{d} \neq 0$. Here, $\underline{d}$ is the column vector ${}^t(d_1,\ldots,d_{4n-2\ell})$, and J is defined as in 3.1. Over a general field two such elements are in the same F-orbit iff the two values of ${}^t\underline{d}J\underline{d}$ are in the same square class.

Let ψ_ℓ be the character of N_ℓ defined by

$$\psi_\ell(u)=\psi_0(u_{12}+\cdots+u_{\ell-1,\ell}+u_{\ell,2n}-u_{\ell,2n+1}).$$

It is not hard to see that

- the stabilizer $L_\ell^{\psi_\ell}$ (cf. M^ϑ in definition 3.5.1) has two connected components,
- the one containing the identity is isomorphic to $G_{4n-2\ell-1}$,
- there is an "obvious" choice of isomorphism $inc: G_{4n-2\ell-1}\to(L_\ell^{\psi_\ell})^0$ having the following property: if $\{e_i^*: i=0 \text{ to } 2n\}$ is the basis for the cocharacter lattice of G_{4n} as in section 3.2, and $\{\bar{e}_i^*, i=0 \text{ to } n\}$ is the basis for that of G_{2n+1}, then

$$inc\circ e_i^* = \begin{cases} e_0^*, & i=0 \\ e_{n-1+i}^*, & i=1 \text{ to } n. \end{cases} \tag{5.1.3}$$

In the case when $\ell=2n-1$, $N_\ell=U_{\max}$, and ψ_ℓ is a generic character. The above remarks remain valid with the convention that $G_1=GL_1$.

Let

$$DC_\omega(\tau)=FC^{\psi_{n-1}}\mathcal{E}_{-1}(\tau,\omega).$$

It is a space of smooth functions $G_{2n+1}(F\backslash\mathbb{A})\to\mathbb{C}$, and affords a representation of the group $G_{2n+1}(\mathbb{A})$ acting by right translation, where we have identified G_{2n+1} with the identity component of $L_{n-1}^{\psi_{n-1}}$.

Theorem 5.1.4. *The space $DC_\omega(\tau)$ is a nonzero cuspidal representation of $G_{2n+1}(\mathbb{A})$, which supports a nonzero Whittaker integral. If σ is any irreducible automorphic representation contained in $DC_\omega(\tau)$, then σ lifts weakly to τ under the map r. Also, the central character of σ is ω.*

Remark 5.1.5. *Since $DC_\omega(\tau)$ is nonzero and cuspidal, there exists at least one irreducible component σ. In the case of orthogonal groups, one may show (*[So1]*, pp. 8-9, item 4) that all of the components are generic using the Rankin-Selberg integrals of* [Gi-PS-R],[So2]. *On the other hand, in the odd case, one may also show (*[GRS4]*, Theorem 8, p. 757, or* [So1] *page 9, item 6) using the results of* [Ji-So] *that $DC_\omega(\tau)$ is irreducible.*

Proof. The statements are proved by combining relationships between unipotent periods and knowledge about $\mathcal{E}_{-1}(\tau,\omega)$.

For genericity, let (U_1,ψ_1) denote the unipotent period obtained by composing the one which defines the descent with the one which defines the Whittaker function on G_{2n+1} embedded into G_{4n} as the stabilizer of the descent character. Thus U_1 is the subgroup of the standard maximal unipotent defined by the relations $u_{i,2n}=u_{i,2n+1}$ for $i=n$ to $2n-1$, and

$$\psi_1(u)=\psi(u_{1,2}+\cdots+u_{n-2,n-1}+u_{n-1,2n}-u_{n-1,2n+1}+u_{n,n+1}+\cdots+u_{2n-1,2n}).$$

Next, let U_2 denote the subgroup of the standard maximal unipotent defined by $u_{i,i+1}=0$ for i even and less than $2n$. (One may also put $\le 2n$: the equation $u_{2n,2n+1}=0$ is automatic for any element of $U_{\max}$.) The character ψ_2 depends on whether n is odd or even. If n is even, it is

$$\psi(u_{1,3}+u_{2,4}+\cdots+u_{2n-1,2n+1}),$$

while, if n is odd, it is

$$\psi(u_{1,3} + u_{2,4} + \cdots + u_{2n-3,2n-1} + u_{2n-2,2n+1} + u_{2n-1,2n}),$$

Finally, let U_3 denote the maximal unipotent, and ψ_3 denote

$$\psi_3(u) = \psi(u_{1,2} + \cdots + u_{2n-1,2n}).$$

Thus (U_3, ψ_3) is the composite of the unipotent period defining the constant term along the Siegel parabolic, and the one which defines the Whittaker functional on the Levi of this parabolic. By Theorem 4.0.3 (5) this period is *not* in $\mathcal{U}^\perp(\mathcal{E}_{-1}(\tau,\omega))$.

In the appendices, we show

(1) $(U_1,\psi_1)|(U_2,\psi_2)$, in Lemma 7.3.1, and
(2) $(U_3,\psi_3) \in \langle (U_2,\psi_2), \{(N_\ell,\vartheta) : n \le \ell < 2n \text{ and } \vartheta \text{ in general position.}\}\rangle$ in Lemma 7.3.2.

By Theorem 5.1.1 $(N_\ell,\vartheta) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau,\omega))$ for all $n \le \ell < 2n$ and ϑ in general position. It follows that $(U_1,\psi_1) \notin \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau,\omega))$. This establishes genericity (and hence nontriviality) of the descent.

Turning to cuspidality, we prove in the appendices an identity relating:

- Constant terms on G_{2n+1} embedded as $(L_{n-1}^{\psi_{n-1}})^0$,
- Descent periods in G_{4n},
- Constant terms on G_{4n},
- Descent periods on G_{4n-2k}, embedded in G_{4n} as a subgroup of a Levi.

To formulate the exact relationship we introduce some notation for the maximal parabolics of GSpin groups.

The group G_{2n+1} has one standard maximal parabolic having Levi $GL_i \times G_{2n-2i+1}$ for each value of i from 1 to n. Let us denote the unipotent radical of this parabolic by V_i^{2n+1}. We denote the trivial character of any unipotent group by $\mathbf{1}$.

The group G_{4n} has one standard maximal parabolic having Levi $GL_k \times G_{4n-2k}$ for each value of k from 1 to $2n-2$. We denote the unipotent radical of this parabolic by V_k.

(The group G_{4n} also has two parabolics with Levi isomorphic to $GL_{2n} \times GL_1$, but since they will not come up in this discussion, we do not need to bother over a notation to distinguish them.)

We prove in Lemma 7.3.4 that $(V_k^{2n+1}, \mathbf{1}) \circ (N_{n-1}, \psi_{n-1})$ is contained in

$$\langle (N_{n+k-1}, \psi_{n+k-1}), \{(N_{n+j-1}, \psi_{n+j-1})^{(4n-2k+2j)} \circ (V_{k-j}, \mathbf{1}) : \ \ 1 \le j < k\}\rangle,$$

where $(N_{n+j-1}, \psi_{n+j-1})^{(4n-2k+2j)}$ denotes the descent period, defined as above, but on the group $G_{4n-2k+2j}$, embedded into G_{4n} as a component of the Levi with unipotent radical V_{k-j}.

By Theorem 5.1.1 $(N_{n+k-1}, \psi_{n+k-1}) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau,\omega))$ for $k = 1$ to n. For all k, j such that $1 \le j < k \le n$, the period $(V_{k-j}, \mathbf{1})$ is the constant term along a parabolic which is not associated to P. Hence $(V_{k-j}, \mathbf{1}) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau,\omega))$ by [MW1] Proposition II.1.7. This shows that any nonzero function appearing in any of the spaces $DC_\omega^a(\tau)$ must be cuspidal. Such a function is also easily seen to be of uniformly moderate growth, being the integral of an automorphic form over a compact domain. In addition, such a function is easily seen to have central character ω, and any function with these properties is necessarily square integrable modulo the center ([MW1] I.2.12). It follows that each of the spaces $DC_\omega^a(\tau)$ decomposes discretely.

Now, suppose $\sigma \cong \otimes_v' \sigma_v$ is an irreducible representation which is contained in $DC_\omega(\tau)$. Let p_σ denote the natural projection $DC_\omega(\tau) \to \sigma$. Once again, by Theorem 4.0.3, (3) the representation $\mathcal{E}_{-1}(\tau,\omega)$ decomposes discretely. Let π be an irreducible component of $\mathcal{E}_{-1}(\tau,\omega)$ such that the restriction of $p_\sigma \circ FC$ to π is nontrivial. As discussed previously in the proof of Theorem 5.1.1, at all but finitely many v, τ is unramified at v and furthermore, π_v is the unramified constituent ${}^{un}Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \boxtimes \omega_v \otimes |\det|_v^{\frac12}$ of $Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \boxtimes \omega_v \otimes |\det|_v^{\frac12}$. If v_0 is such a place, the map $p_\sigma \circ FC \circ i_{\zeta^{v_0}}$, with $i_{\zeta^{v_0}}$ defined as in Theorem 5.1.1, factors through $\mathcal{J}_{N_{n-1},\psi_{n-1}}\left({}^{un}Ind_{P(F_v)}^{G_{4n}(F_v)}\tau_v\otimes|\det|_v^{\frac{1}{2}}\boxtimes\omega_v\right)$, and gives rise to a $G_{2n+1}(F_{v_0})$-equivariant map from this Jacquet-module onto σ_{v_0}.

To pin things down precisely, assume that τ_v is the unramified component of $Ind_{B(GL_{2n})(F_v)}^{GL_{2n}(F_v)}\mu$, and let $\mu_1,\dots,\mu_{2n}$ be defined as in the proof of Lemma 3.3.1. By Lemma 3.3.1, we may assume without loss of generality that $\mu_{2n+1-i}=\omega\mu_i^{-1}$ for $i=1$ to n.

We also need to refer to the elements of the basis of the cocharacter lattice of G_{2n+1} fixed in section 3.2. As in the remarks preceding the definition of $DC_\omega(\tau)$, we denote these $\bar{e}_0^*,\dots,\bar{e}_n^*$.

In the appendices, we show that

$$\mathcal{J}_{N_{n-1},\psi_{n-1}}\left({}^{un}Ind_{P(F_v)}^{G_{4n}(F_v)}\tau_v\boxtimes\omega_v\otimes|\det|_v^{\frac{1}{2}}\right)$$

is isomorphic as a $G_{2n+1}(F_v)$-module to $Ind_{B(G_{2n+1})(F_v)}^{G_{2n+1}(F_v)}\chi$ for χ the unramified character of $B(G_{2n+1})(F_v)$ such that

$$\chi\circ\bar{e}_i^*=\mu_i, i=1\text{ to }n, \chi\circ\bar{e}_0^*=\omega_v.$$

It follows that τ is a weak lift of σ associated to the map r. □

6. Appendix I: Local results on Jacquet Functors

In this appendix, F is a non-archimedean local field, on which we place the additional technical hypothesis

$$B(G_{2n-1})(F)G_{2n-1}(\mathfrak{o})=G_{2n-1}(F), \tag{6.0.6}$$

which is known (see [Tits], 3.9, and 3.3.2) to hold at all but finitely many non-Archimedean completions of a number field. Here, G_{2n-1} is identified with $(L_{n-1}^{\psi})^0$ is defined as in (5.1.3), and $\mathfrak{o}$ denotes the ring of integers of F.

Proposition 6.0.7. *Let $\tau=Ind_{B(GL_{2n})(F)}^{GL_{2n}(F)}\mu$, where μ satisfies $\mu\circ e_i^*=\omega\mu\circ e_{2n+1-i}^*$. Then for $\ell\geq n$ and ϑ in general postion, the Jacquet module $\mathcal{J}_{N_\ell,\vartheta}({}^{un}Ind_{P(F)}^{G_{4n}(F)}\tau\otimes|\det|^{\frac{1}{2}}\boxtimes\omega)$ is trivial.*

Proof. First, let $\mu_i:F\to\mathbb{C}$ be the unramified character given by $\mu_i=\mu\circ e_i^*$. By induction in stages,

$${}^{un}Ind_{P(F)}^{G_{4n}(F)}\tau\otimes|\det|^{\frac{1}{2}}\boxtimes\omega={}^{un}Ind_{B(G_{4n})(F)}^{G_{4n}(F)}\tilde{\mu},$$

where $\tilde{\mu}\circ e_i^*(x)=|x|^{\frac{1}{2}}\mu_i(x)$, for $i=1$ to $2n$ and $\tilde{\mu}\circ e_0^*=\omega$. By the definition of the unramified constituent

$${}^{un}Ind_{B(G_{4n})(F)}^{G_{4n}(F)}\tilde{\mu}={}^{un}Ind_{B(G_{4n})(F)}^{G_{4n}(F)}\tilde{\mu}',$$

where $\tilde{\mu}'\circ e_{2i-1}^*(x)=\mu_i(x)|x|^{\frac{1}{2}}$, and $\tilde{\mu}'\circ e_{2i}^*(x)=\mu_i(x)|x|^{-\frac{1}{2}}$, for $i=1$ to n, and $\tilde{\mu}'\circ e_0^*=\omega$. Now, it is well known that

$${}^{un}Ind_{B(GL_2)(F)}^{GL_2(F)}\mu|\ |^{\frac{1}{2}}\otimes\mu|\ |^{-\frac{1}{2}}=\mu\circ\det.$$

It follows that

$${}^{un}Ind_{B(G_{4n})(F)}^{G_{4n}(F)}\tilde{\mu}'={}^{un}Ind_{P_{2^{2n}}(F)}^{G_{4n}(F)}\hat{\mu},$$

where $P_{2^{2n}}$ is the parabolic of G_{4n} having Levi isomorphic to $GL_2^n\times GL_1$, such that the roots of this Levi are $e_1-e_2,e_3-e_4,\dots,e_{2n-1}-e_{2n}$, and $\hat{\mu}$ is the character given by $\hat{\mu}\circ e_{2i-1}^*=\hat{\mu}\circ e_{2i}^*=\mu_i,\hat{\mu}\circ e_0^*=\omega$.

The space $Ind_{P_{2^{2n}}(F)}^{G_{4n}(F)}\hat{\mu}$ has a filtration as a $Q_\ell(F)$-module, in terms of $Q_\ell(F)$-modules indexed by the elements of $(W\cap P_{2^{2n}})\backslash W/(W\cap Q_\ell)$. For any element x of $P_{2^{2n}}(F)wQ_\ell F)$ the module corresponding to w is isomorphic to $c-ind_{x^{-1}P_{2^{2n}}(F)x\cap Q_\ell(F)}^{Q_\ell(F)}\hat{\mu}\delta_{P_{2^{2n}}}^{\frac{1}{2}}\circ Ad(x)$. Here $Ad(x)$ denotes the map given by conjugation by x. It sends $x^{-1}P_{2^{2n}}(F)x \cap Q_\ell(F)$ into $P_{2^{2n}}(F)$. Also, here and throughout $c-ind$ denotes non-normalized compact induction. (See [Cass], section 6.3.)

Lemma 6.0.8. *The Weyl group of G_m is canonically identified with that of SO_m.*

Proof. For this lemma only, let T denote the torus of SO_m and $\tilde{T}$ that of G_m. Then the following diagram commutes:

$$\begin{array}{ccc} Z_{G_m}(\tilde{T}) & \longrightarrow & N_{G_m}(\tilde{T}) \\ \downarrow & & \downarrow \\ Z_{SO_m}(T) & \longrightarrow & N_{SO_m}(T). \end{array}$$

Both horizontal arrows are inclusions and both vertical arrows are pr. ☐

One easily checks that every element of the Weyl group of SO_{4n} is represented by a permutation matrix. We denote the permutation associated to w also by w. The set of permutations w obtained is precisely the set of permutations $w \in \mathfrak{S}_{4n}$ satisfying,

(1) $w(4n+1-i) = 4n+1-w(i)$ and

(2) $\det w = 1$ when w is written as a $4n \times 4n$ permutation matrix.

As representatives for the double cosets $(W \cap P_{2^{2n}})\backslash W/(W \cap Q_\ell)$ we choose the element of minimal length in each. As permutations, these elements have the properties

(3) $w^{-1}(2i) > w^{-1}(2i-1)$ for $i = 1$ to $2n$, and

(4) If $\ell \le i < j \le 4n+1-\ell$ and $w(i) > w(j)$, then $i = 2n$ and $j = 2n+1$.

Let I_w be the $Q_\ell(F)$-module obtained as

$$c - ind_{\dot{w}^{-1}P_{2^{2n}}(F)\dot{w} \cap Q_\ell(F)}^{Q_\ell(F)} \hat{\mu}\delta_{P_{2^{2n}}}^{\frac{1}{2}} \circ Ad(\dot{w})$$

using any element $\dot{w}$ of $\mathrm{pr}^{-1}(w)$.

A function f in I_w will map to zero under the natural projection to $\mathcal{J}_{N_\ell,\vartheta}(I_w)$ iff there exists a compact subgroup N_ℓ^0 of $N_\ell(F)$ such that

$$\int_{N_\ell^0} f(hn)\overline{\vartheta(n)}dn = 0 \qquad \forall h \in Q_\ell(F).$$

(See [Cass], section 3.2.) Let $\vartheta^h(n) = \vartheta(hnh^{-1})$. It is easy to see that the integral above vanishes for suitable N_ℓ^0 whenever

$$\vartheta^h|_{N_\ell(F)\cap w^{-1}P_{2^{2n}}(F)w} \text{ is nontrivial.} \tag{6.0.9}$$

Furthermore, the function $h \mapsto \vartheta^h$ is continuous in h, (the topology on the space of characters of $N_\ell(F)$ being defined by identifying it with a finite dimensional F-vector space, cf. section 3.5) so if this condition holds for all h in a compact set, then N_0^ℓ can be made uniform in h.

Now, ϑ is in general position. Hence, so is ϑ^h for every h. So, if we write

$$\vartheta^h(u) = \psi_0(c_1u_{1,2} + \cdots + c_{\ell-1}u_{\ell-1,\ell} + d_1u_{\ell,\ell+1} + \cdots + d_{2m-2\ell}u_{\ell,2m-\ell}),$$

we have that $c_i \neq 0$ for all i and ${}^t\underline{d}w\underline{d} \neq 0$.

Clearly, the condition (6.0.9) holds for all h unless

(5) $w(1) > w(2) > \cdots > w(\ell)$.

Furthermore, because ${}^t\underline{d}w\underline{d} \neq 0$, there exists some i_0 with $\ell+1 \le i_0 \le 2n$ such that $d_{i_0-\ell} \neq 0$ and $d_{4n+1+\ell-i_0} \neq 0$. From this we deduce that the condition (6.0.9) holds for all h unless w has the additional property

(6) There exists i_0 such that $w(\ell) > w(i_0)$ and $w(\ell) > w(4n+1-i_0)$.

However, if $\ell \geq n$ it is easy to check that no permutations with properties (1),(3) (5) and (6) exist.

Thus $\mathcal{J}_{N_\ell,\vartheta}(I_w) = \{0\}$ for all w and hence $\mathcal{J}_{N_\ell,\vartheta}({}^{un}Ind_{P(F)}^{G_{4n}(F)}\tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega) = \{0\}$ by exactness of the Jacquet functor. $\square$

Proposition 6.0.10. *Let $\tau = Ind_{B(GL_{2n})(F)}^{GL_{2n}(F)}\mu$, where μ satisfies $\mu \circ e_i^* = \omega\mu \circ e_{2n+1-i}$. Then the Jacquet module*

$$\mathcal{J}_{N_{n-1},\psi_{n-1}}\left({}^{un}Ind_{P(F)}^{G_{4n}(F)}\tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega\right)$$

is isomorphic as a $G_{2n+1}(F)$-module to a subquotient of $Ind_{B(G_{2n+1})(F)}^{G_{2n+1}(F)}\chi$ for χ the unramified character of $B(G_{2n+1})(F)$ such that

$$\chi \circ \bar{e}_i^* = \mu_i, i = 1 \text{ to } n, \chi \circ \bar{e}_0^* = \omega.$$

Proof. As before, we have

$${}^{un}Ind_{P(F)}^{G_{4n}(F)}\tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega = {}^{un} Ind_{P_{2^{2n}}}^{G_{4n}(F)}\hat{\mu},$$

and we filter $Ind_{P_{2^{2n}}(F)}^{G_{4n}(F)}\hat{\mu}$ in terms of $Q_{n-1}(F)$-modules I_w. This time, $\mathcal{J}_{N_{n-1},\psi_{n-1}}(I_w) = \{0\}$ for all w except one. This one Weyl element, which we denote w_0, corresponds to the unique permutation satisfying (1),(2),(3),(4) of the previous result, together with $w(i) = 4n - 2i + 1$ for $i = 1$ to $n-1$. Exactness yields

$$\mathcal{J}_{N_{n-1},\psi_{n-1}}\left({}^{un}Ind_{P(F)}^{G_{4n}(F)}\tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega\right) \cong \mathcal{J}_{N_\ell,\vartheta}(I_{w_0}).$$

(This is an isomorphism of $Q_{n-1}^{\psi_{n-1}}(F)$-modules, where $Q_{n-1}^{\psi_{n-1}} = N_{n-1} \cdot L_{n-1}^{\psi_{n-1}} \subset Q_{n-1}$, is the stabilizer of ψ_{n-1} in Q_{n-1} (cf. L^ϑ above).)

Now, recall that for each $h \in Q_{n-1}(F)$ the character $\psi_{n-1}^h(u) = \psi_{n-1}(huh^{-1})$ is a character of N_{n-1} in general position, and as such determines coefficients $c_1, \dots, c_{n-2}$ and $d_1, \dots, d_{2n+2}$ as in remark 5.1.2. Clearly,

$$Q_{n-1}^o := \{ h \in Q_{n-1}(F) |\, d_i \neq 0 \text{ for some } i \neq n+1, n+2\}$$

is open. Moreover, one may see from the description of w_0 that for h in this set 6.0.9 is satisfied. We have an exact sequence of $Q_{n-1}^{\psi_{n-1}}(F)$-modules

$$0 \to I_{w_0}^* \to I_{w_0} \to \bar{I}_{w_0} \to 0,$$

where I_w^* consists of those functions in I_w whose compact support happens to be contained in Q_{n-1}^o, and the third arrow is restriction to the complement of Q_{n-1}^o. This complement is slightly larger than $Q_{n-1}^{\psi_{n-1}}(F)$ in that it contains the full torus of $Q_{n-1}(F)$, but restriction of functions is an isomorphism of $Q_{n-1}^{\psi_{n-1}}(F)$-modules,

$$\bar{I}_{w_0} \to c - ind_{Q_{n-1}^{\psi_{n-1}}(F) \cap w_0^{-1} P_{2^{2n}}(F) w_0}^{Q_{n-1}^{\psi_{n-1}}(F)} \hat{\mu}\delta_{P_{2^{2n}}}^{\frac{1}{2}} \circ Ad(w_0).$$

Clearly $\mathcal{J}_{N_{n-1},\psi_{n-1}}\left(I_{w_0}^*\right) = \{0\}$, and hence

$$\mathcal{J}_{N_{n-1},\psi_{n-1}}\left(Ind_{P_{2^{2n}}(F)}^{G_{4n}(F)}\hat{\mu}\right) \cong \mathcal{J}_{N_{n-1},\psi_{n-1}}\left(c - ind_{Q_{n-1}^{\psi_{n-1}}(F) \cap w_0^{-1} P_{2^{2n}}(F) w_0}^{Q_{n-1}^{\psi_{n-1}}(F)} \hat{\mu}\delta_{P_{2^{2n}}}^{\frac{1}{2}} \circ Ad(w_0)\right).$$

Now let $\mathcal{W}$ denote

$$\left\{ f : Q_{n-1}^{\psi_{n-1}}(F) \to \mathbb{C} \,\middle|\, \begin{array}{l} f(uq) = \psi_{n-1}(u) f(q) \;\forall\, u \in N_{n-1}(F),\; q \in Q_{n-1}^{\psi_{n-1}}(F), \\ f(bm) = \chi(b)\delta_{B(G_{2n+1})}^{\frac{1}{2}} f(m) \;\forall\, b \in B(L_{n-1}^{\psi_{n-1}})(F),\; m \in L_{n-1}^{\psi_{n-1}}(F) \end{array} \right\}.$$

For $f \in c-ind_{Q_{n-1}^{\psi_{n-1}}\cap w_0^{-1}P_{2^{2n}}w_0}^{Q_{n-1}^{\psi_{n-1}}} \hat{\mu}\delta_{P_{2^{2n}}}^{\frac{1}{2}} \circ Ad(w_0)$, let

$$W(f)(q) = \int_{N_{n-1}(F)\cap w_0^{-1}\overline{U_{\max}}(F)w_0} f(uq)\bar{\psi}_{n-1}(u)du.$$

Then W maps $c-ind_{Q_{n-1}^{\psi_{n-1}}(F)\cap w_0^{-1}P_{2^{2n}}(F)w_0}^{Q_{n-1}^{\psi_{n-1}}(F)} \hat{\mu}\delta_{P_{2^{2n}}}^{\frac{1}{2}} \circ Ad(w_0)$ into $\mathcal{W}$. That is, the functions in $c-ind_{Q_{n-1}^{\psi_{n-1}(F)}\cap w_0^{-1}P_{2^{2n}}(F)w_0}^{Q_{n-1}^{\psi_{n-1}}(F)} \hat{\mu}\delta_{P_{2^{2n}}}^{\frac{1}{2}} \circ Ad(w_0)$ are left equivariant with respect to the group $B(G_{2n+1})(F)$, and a quasicharacter of this group that differs from $\chi\delta_{B(G_{2n+1})}^{\frac{1}{2}}$ by the Jacobian of $Ad(b)$, $b \in B(G_{2n+1})(F)$, acting on $N_{n-1}(F)\cap w_0^{-1}\overline{U_{\max}}(F)w_0$.

Let us denote

$$c-ind_{Q_{n-1}^{\psi_{n-1}(F)}\cap w_0^{-1}P_{2^{2n}}(F)w_0}^{Q_{n-1}^{\psi_{n-1}}(F)} \hat{\mu}\delta_{P_{2^{2n}}}^{\frac{1}{2}} \circ Ad(w_0)$$

by V and denote by $V(N_{n-1},\psi_{n-1})$ the kernel of the linear map $V \to \mathcal{J}_{N_{n-1},\psi_{n-1}}(V)$.

It is easy to show that $V(N_{n-1},\psi_{n-1})$ is contained in the kernel of W. In the next lemma, we show that in fact, they are equal. Restriction from $Q_{n-1}^{\psi_{n-1}}(F)$ to $L_{n-1}^{\psi_{n-1}}(F)$ is clearly an isomorphism $\mathcal{W} \to Ind_{B(G_{2n+1})(F)}^{G_{2n+1}(F)}\chi$. □

Lemma 6.0.11. *With notation as in the previous proposition, we have $Ker(W) \subset V(N_{n-1},\psi_{n-1})$.*

Proof. For this proof, we denote the Borel of $L_{n-1}^{\psi_{n-1}}$ by B. Also, let $N^{w_0} = N_{n-1}\cap w_0^{-1}P_{2^{2n}}w_0$, and $N_{w_0} = N_{n-1}\cap w_0^{-1}\overline{U_{\max}}w_0$,

We consider a smooth function $f : Q_{n-1}^{\psi_{n-1}}(F) \to \mathbb{C}$ which is compactly supported modulo $Q_{n-1}^{\psi_{n-1}}(F)\cap w_0^{-1}P_{2^{2n}}(F)w_0$, and satisfies

$$f(bm) = \chi\delta_B^{\frac{1}{2}}(b)f(m) \qquad \forall\, b \in B(F),$$

and

$$f(uq) = f(q) \qquad \forall\, u \in N^{w_0}(F) \text{ and } q \in Q_{n-1}^{\psi_{n-1}}(F).$$

We assume that

$$\int_{N_{w_0}(F)} f(uq)\bar{\psi}_{n-1}(u)du = 0,$$

for all $q \in Q_{n-1}^{\psi_{n-1}}(F)$. What must be shown is that there is a compact subset C of $N_{n-1}(F)$ such that

$$\int_C f(gu)\bar{\psi}_{n-1}(u)du = 0,$$

for all $q \in Q_{n-1}^{\psi_{n-1}}(F)$.

Consider first $m \in L_{n-1}^{\psi_{n-1}}(\mathfrak{o})$. Let $\mathfrak{p}$ denote the unique maximal ideal in $\mathfrak{o}$. If U is a unipotent subgroup and M an integer, we define

$$U(\mathfrak{p}^M) = \{u \in U(F) : u_{ij} \in \mathfrak{p}^M\ \forall i \neq j\}.$$

Observe that for each $M \in \mathbb{N}$, $N_{n-1}(\mathfrak{p}^M)$ is a subgroup of $N_{n-1}(F)$ which is preserved by conjugation by elements of $L_{n-1}^{\psi_{n-1}}(\mathfrak{o})$. We may choose M sufficiently large that $supp(f) \subset N_{w_0}(\mathfrak{p}^{-M})N^{w_0}(F)L_{n-1}^{\psi_{n-1}}(F)$. Then we prove the desired assertion with $C = N_{n-1}(\mathfrak{p}^{-M})$. Indeed, for $m \in L_{n-1}^{\psi_{n-1}}(\mathfrak{o})$, we have

$$\int_{N_{n-1}(\mathfrak{p}^{-M})} f(mu)\psi_{n-1}(u)du = \int_{N_{n-1}(\mathfrak{p}^{-M})} f(um)\psi_{n-1}(u)du,$$

because $Ad(m)$ preserves the subgroup $N_{n-1}(\mathfrak{p}^{-M})$, and has Jacobian 1. Let $c = \mathrm{Vol}(N^{w_0}(\mathfrak{p}^{-M}))$, which is finite. Then by N^{w_0}-invariance of f, the above equals

$$= c\int_{N_{w_0}(\mathfrak{p}^{-M})} f(um)\psi_{n-1}(u)du.$$

This, in turn, is equal to

$$= c\int_{N_{w_0}(F)} f(um)\psi_{n-1}(u)du,$$

since none of the points we have added to the domain of integration are in the support of f, and this last integral is equal to zero by hypothesis.

Next, suppose $q = u_1 m$ with $n \in N_{n-1}(F)$ and $m \in L_{n-1}^{\psi_{n-1}}(\mathfrak{o})$. If $u_1 \in N_{n-1}(F) - N_{n-1}(\mathfrak{p}^{-M})$ then qu is not in the support of f for any $u \in N_{n-1}(\mathfrak{p}^{-M})$. On the other hand, if $u_1 \in N_{n-1}(\mathfrak{p}^{-M})$, then

$$\int_{N_{n-1}(\mathfrak{p}^{-M})} f(u_1 mu)\psi_{n-1}(u)du = \int_{N_{n-1}(\mathfrak{p}^{-M})} f(u_1 um)\psi_{n-1}(u)du$$

$$= \psi_{n-1}(u_1)\int_{N_{n-1}(\mathfrak{p}^{-M})} f(um)\psi_{n-1}(u)du,$$

and now we continue as in the case $u_1 = 1$.

The result for general q now follows from the left-equivariance properties of f and (6.0.6). □

7. Appendix II: Global results

7.1. A Lemma Regarding Unipotent Periods. There is a natural action of $G(F)$ on $\mathcal{U}$ given by $\gamma\cdot(U,\psi) = (\gamma U\gamma^{-1}, \gamma\cdot\psi)$ where $\gamma\cdot\psi(u) = \psi(\gamma^{-1}u\gamma)$. We shall refer to this action as "conjugation." Obviously, unipotent periods which are conjugate are equivalent.

It is convenient to allow ourselves to conjugate our unipotent periods by a slightly larger set of elements. We may allow the involution † to act on unipotent periods by $f^{\dagger(U,\psi_U)}(g) = f^{(U,\psi_U)}({}^\dagger g)$. Denoting the action of $Pin_{4n}(F)$ on $\mathcal{U}$ by $\gamma\cdot(U,\psi_U)$, we have

$$\gamma\cdot(U,\psi_U) \sim \begin{cases} (U,\psi_U) & \text{when } \det \mathrm{pr}\,\gamma = 1, \\ {}^\dagger(U,\psi_U) & \text{when } \det \mathrm{pr}\,\gamma = -1. \end{cases}$$

Observe that in general ${}^\dagger(U,\psi_U)$ is *not* equivalent to (U,ψ_U). For example, it is not difficult to verify that $\dagger(U_{\max},\psi_{LW}) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau,\omega))$.

Lemma 7.1.1. *Suppose $U_1 \supset U_2 \supset (U_1,U_1)$ are unipotent subgroups of a reductive algebraic group G. Suppose H is a subgroup of G and let f be a smooth left $H(F)$-invariant function on $G(\mathbb{A})$. Suppose ψ_2 is a character of U_2 such that $\psi_2|_{(U_1,U_1)} \equiv 0$. Then the set $\mathrm{res}^{-1}(\psi_2)$ of characters of U_1 such that the restriction to U_2 is ψ_2 is nontrivial. (Here "res" is for "restriction" not "residue".) The elements of $\mathrm{res}^{-1}(\psi_2)$ are permuted by the action of $N_H(U_1)(F)$. The following are equivalent.*

1. $f^{(U_2,\psi_2)} \equiv 0$
2. $f^{(U_1,\psi_1)} \equiv 0\ \forall \psi_1 \in \mathrm{res}^{-1}(\psi_2)$
3. *For each $N_H(U_1)(F)$-orbit $\mathcal{O}$ in $\mathrm{res}^{-1}(\psi_2)$ $\exists\psi_1 \in \mathcal{O}$ with $f^{(U_1,\psi_1)} \equiv 0$*

Proof. It is obvious that 1 implies 2 and 3, and that 2 and 3 are equivalent. Consider

$$f^{(U_2,\psi_2)}(u_1 g) = \int_{U_2(F\backslash\mathbb{A})} f(u_2 u_1 g)\psi_2(u_2)du_2,$$

regarded as a function of u_1. It is left u_2 invariant and hence gives rise to a function of the compact abelian group $U_2(\mathbb{A})U_1(F)\backslash U_1(\mathbb{A})$. Denote this function by $\phi(u_1)$. Then

$$\phi(0) = \sum_{\chi} \int_{U_2(\mathbb{A})U_1(F)\backslash U_1(\mathbb{A})} \phi(u_1)\chi(u_1)du_1,$$

where "0" denotes the identity in $U_2(\mathbb{A})U_1(F)\backslash U_1(\mathbb{A})$, and the sum is over characters of $U_2(\mathbb{A})U_1(F)\backslash U_1(\mathbb{A})$. This, in turn, is equal to

$$\sum_{\chi} \int_D \int_{U_2(F\backslash\mathbb{A})} f(u_2u_1g)\psi_2(u_2)du_2\chi(u_1)du_1,$$

for D a fundamental domain for the above quotient in $U_1(\mathbb{A})$. The group $U_1/(U_1,U_1)(F)$ is an F-vector space (cf. section 3.5) which can be decomposed into $U_2/(U_1,U_1)(F)$ and a complement. The F-dual of this vector space is identified, via the choice of ψ_0, with the space of characters of $U_1(\mathbb{A})$ which are trivial on $U_1(F)$. It follows that the sum above is equal to

$$= \sum_{\psi_1 \in \mathrm{res}^{-1}(\psi_2)} \int_{U_1(F\backslash\mathbb{A})} f(u_1g)\psi_1(u_1)du_1.$$

The matter of replacing the sum over χ by one over $\psi_1 \in \mathrm{res}^{-1}(\psi_2)$ is clear from regarding $U_1/(U_1,U_1)(F)$ as a vector space which can be decomposed into $U_2/(U_1,U_1)$ and a complement. Now $2 \Rightarrow 1$ is immediate. □

Corollary 7.1.2. *If $N_G(H)$ permutes the elements of $\mathrm{res}^{-1}(\psi_2)$ transitively, then $(U_2,\psi_2) \sim (U_2,\psi_1)$ for every $\psi_1 \in \mathrm{res}^{-1}(\psi_2)$.*

Definition 7.1.3. *Many of the applications of the above corollary are of a special type, and it will be convenient to introduce a term for them. The special situation is the following: one has three unipotent periods (U_i,ψ_i) for $i=1,2,3$, such that $U_2 = U_1 \cap U_3$ and $\psi_1|_{U_2} = \psi_3|_{U_2} = \psi_2$. Furthermore, U_1 normalizes U_3 and permutes transitively, the set of characters ψ_3' such that $\psi_3'|_{U_2}$, and the same is true with the roles of 1 and 3 reversed. In this situation, the identity*

$$(U_1,\psi_1) \sim (U_2,\psi_2) \sim (U_3,\psi_3),$$

(which follows from Corollary 7.1.2) will be called a **swap,** *and we say that (U_1,ψ_1) "may be swapped for" (U_3,ψ_3), and vice versa.*

7.2. **A lemma regarding the projection, and a remark.**

Lemma 7.2.1. *The action of G_m on itself by conjugation factors through* pr.

Proof. One has only to check that the kernel of pr is in the center of G_m. When we regard G_m as a quotient of $Spin_m \times GL_1$, the quotient of pr is precisely the image of the GL_1 factor in the quotient. □

Corollary 7.2.2. *Let u be a unipotent element of $G_m(\mathbb{A})$ and g any element of $G_m(\mathbb{A})$. Then* $\mathrm{pr}(gug^{-1})$ *is a unipotent element of $SO_m(\mathbb{A})$ and gug^{-1} is the unique unipotent element of its preimage in $G_m(\mathbb{A})$.*

Remark 7.2.3. *This fact, combined with the fact that* pr *is an isomorphism of varieties when restricted to the subvariety of unipotent elements of G_m, means that many statements may be proved for GSpin groups simply by taking the proof of the corresponding statement for special orthogonal groups and inserting the words "any preimage of" here and there.*

7.3. **Relations among Unipotent Periods used in Theorem 5.1.4.** Before we proceed with the proofs it will be convenient to formulate the statements in a slightly different way, making use of the involution †.

We shall let (U_1, ψ_1) and (U_3, ψ_3) be defined as in the proof of 5.1.4. We also keep the definition of the group U_2. However, we now define the character ψ_2 by the formula

$$\psi_2(u) = \psi(u_{13} + \cdots + u_{2n-1,2n+1}),$$

regardless of the parity of n. (This agrees with the previous definition if n is even; if n is odd they differ by an application of †.)

Lemma 7.3.1. *Let (U_1, ψ_1) be defined as in Theorem 5.1.4, and (U_2, ψ_2) defined as just above. Then $(U_1, \psi_1)|(U_2, \psi_2)$ and $(U_1, \psi_1)|\,^{\dagger}(U_2, \psi_2)$.*

Proof. We define some additional unipotent periods which appear at intermediate stages in the argument. Let U_4 be the subgroup defined by $u_{n-1,j} = 0$ for $j = n$ to $2n-2$ and $u_{2n-1,2n} = u_{2n-1,2n+1}$. We define a character ψ_4 of U_4 by the same formula as ψ_1. Then (U_1, ψ_1) may be swapped for (U_4, ψ_4). (See definition 7.1.3.)

Now, for each k from 1 to n, define $(U_5^{(k)}, \psi_5^{(k)})$ as follows. First, for each k, the group $U_5^{(k)}$ is contained in the the subgroup of $U_{\max}$ defined by, $u_{2n-1,2n} = u_{2n-1,2n+1}$. In addition, $u_{n+k-2,j} = 0$ for $j < 2n-1$, and $u_{i,i+1} = 0$ if $n-k \le i < n+k$ and $i \equiv n-k \mod 2$, and $\psi_5^{(k)}(u)$ equals

$$\psi_0\left(\sum_{i=1}^{n-k-1} u_{i,i+1} + \sum_{i=n-k}^{n+k-3} u_{i,i+2} + u_{n+k-2,2n} + u_{n+k-2,2n+1} + \sum_{i=n+k-1}^{2n-1} u_{i,i+1}\right).$$

(Note that one or more of the sums here may be empty.)

Next, let $U_6^{(k)}$ be the subgroup of $U_{\max}$ defined by the conditions $u_{2n-1,2n} = u_{2n-1,2n+1}$, $u_{n+k-2,j} = 0$ for $j < 2n-1$, and $u_{i,i+1} = 0$ if $n-k \le i < n+k-2$ and $i \equiv n-k+1 \mod 2$. The same formula which defines $\psi_5^{(k)}$ also defines a character of $U_6^{(k)}$. We denote this character by $\psi_6^{(k)}$.

We make the following observations:

- $(U_5^{(1)}, \psi_5^{(1)})$ is precisely (U_4, ψ_4).
- For each k, $(U_5^{(k)}, \psi_5^{(k)})$ is conjugate to $(U_6^{(k+1)}, \psi_6^{(k+1)})$. The conjugation is accomplished by any preimage of the permutation matrix which transposes i and $i+1$ for $n-k \le i < n+k$ and $i \equiv n-k \mod 2$.
- $(U_6^{(k)}, \psi_6^{(k)})$ may be swapped for $(U_5^{(k)}, \psi_5^{(k)})$.

Thus $(U_4, \psi_4) \sim (U_5^{(n)}, \psi_5^{(n)})$.

Now, let ψ_2' be the character of U_2 which is defined by

$$\psi_2'(u) = \psi(u_{1,3} + \cdots + u_{2n-2,2n} - u_{2n-2,2n+1} + u_{2n-1,2n+1}).$$

Then $U_5^{(n)}$ is the subgroup of U_2 defined by $u_{2n-1,2n} = u_{2n-1,2n+1}$ and $\psi_5^{(n)}$ is the restriction of ψ_2' to this group. Thus $(U_5^{(n)}, \psi_5^{(n)})|(U_2, \psi_2')$. (It is because of this step that $(U_1, \psi_1) \not\sim (U_2, \psi_2)$.)

Finally, (U_2, ψ_2) and (U_2, ψ_2') are conjugate by the unipotent element which projects to $I_{4n} - \sum_{i=2}^{n} e'_{2i-1,2i-2}$

To obtain $^{\dagger}(U_2, \psi_2)$, we use

$$\psi_2''(u) := \psi(u_{1,3} + \cdots + u_{2n-2,2n} - u_{2n-2,2n+1} + u_{2n-1,2n})$$

instead of ψ_2'. □

Lemma 7.3.2. *Let (U_3,ψ_3) be defined as in Theorem 5.1.4, and let (U_2,ψ_2) be defined as in the previous lemma. Then*

$$(U_3,\psi_3)\in\langle\ {}^{\dagger^n}(U_2,\psi_2),\{(N_\ell,\vartheta):n\le\ell<2n \textit{ and } \vartheta \textit{ in general position.}\}\rangle.$$

Here $\dagger^n$ indicates that we apply $\dagger$ a total of n times, with the effect being $\dagger$ if n is odd and trivial if n is even.

Proof. To prove this assertion we introduce some additional unipotent periods. For $k=1$ to $2n-1$ let $U_7^{(k)}$ denote the subgroup of $U_{\max}$ defined by $u_{i,i+1}=0$ for $i>k$ and $i\equiv k+1 \mod 2$. We use two characters of this group:

$$\tilde{\psi}_7^{(k)}=\psi_0\left(\sum_{1\le i\le k-1}u_{i,i+1}+\sum_{k\le i\le 2n-1}u_{i,i+2}\right),$$

$$\psi_7^{(k)}=\psi_0\left(\sum_{1\le i\le k}u_{i,i+1}+\sum_{k+1\le i\le 2n-1}u_{i,i+2}\right),$$

Then $(U_7,\psi_7^{(k)})$ is conjugate to $(U_7,\tilde{\psi}_7^{(k)})$ by any preimage of the permutation matrix which transposes i and $i+1$ for $k<i<4n-k$ and $i\equiv k+1 \mod 2$. This matrix has determinant -1 iff k is odd.

If k is odd then $(U_7^{(k)},\psi_7^{(k)})$ may be swapped for $(U_7^{(k+1)},\tilde{\psi}_7^{(k+1)})$, while if k is even, it may be swapped for $(U_8^{(k+1)},\tilde{\psi}_8^{(k+1)})$, where $U_8^{(k+1)}$ is the subgroup of $U_7^{(k+1)}$ defined by $u_{2n-1,2n}=0$, and $\tilde{\psi}_8^{(k+1)}$ is the restriction of $\tilde{\psi}_7^{(k+1)}$ to this group.

Now, for $a\in F^\times$ define a character $\tilde{\psi}_7^{(k+1,a)}$ of $U_7^{(k+1)}$ by

$$\tilde{\psi}_7^{(k+1,a)}=\psi(u_{1,2}+\cdots+u_{k-1,k}+u_{k,k+2}+\cdots+u_{2n-1,2n+1}+au_{2n-1,2n}).$$

Then a Fourier expansion along $U_{2n-1,2n}$ shows that

$$(U_8^{(k+1)},\tilde{\psi}_8^{(k+1)})\in\langle(U_7^{(k+1)},\tilde{\psi}_7^{(k+1)}),\{(U_7^{(k+1)},\tilde{\psi}_7^{(k+1,a)}):a\in F^\times\}\rangle.$$

Here $U_{ij}=\{u\in U_{\max}:u_{k,\ell}=0,\ \forall\ (k,\ell)\ne(i,j)\}$.

In Lemma 7.3.3 below we prove that for k even and $a\in F^\times$,

$$(N_{n+\frac{k}{2}},\psi_{n+\frac{k}{2},a})|(U_7^{(k+1)},\tilde{\psi}_7^{(k+1,a)}),$$

where

$$\psi_{\ell,a}(u)=\psi(u_{1,2}+\cdots+u_{\ell-1,\ell}+au_{\ell,2n}+u_{\ell,2n+1}).$$

The present lemma then follows from the following observations:

- $(U_7^{(1)},\tilde{\psi}_7^{(1)})=(U_2,\psi_2)$, (with ψ_2 defined as at the beginning of this section).
- $(U_7^{(2n-1)},\psi_7^{(2n-1)})=(U_3,\psi_3)$
- If one applies $\dagger$ to both sides of a relation among unipotent periods, it remains valid.
- The character $\psi_{n+\frac{k}{2},a}$ of $N_{n+\frac{k}{2}}$ is in general position. (Cf. remarks 5.1.2)
- The set $\{(N_\ell,\vartheta):n\le\ell<2n$ and ϑ in general position.$\}$ is stable under $\dagger$.
- The number of times we conjugate by the preimage of an element of determinant minus 1 in passing from $(U_7^{(k)},\tilde{\psi}_7^{(k)})$ back to $(U_7^{(k)},\psi_7^{(k)})$ is precisely n.

∎

Lemma 7.3.3. *Let* $(N_{n+\frac{k}{2}}, \psi_{n+\frac{k}{2},a})$ *and* $(U_7^{(k+1)}, \tilde{\psi}_7^{(k+1,a)})$ *be defined as in the previous lemma. Then*

$$(N_{n+\frac{k}{2}}, \psi_{n+\frac{k}{2},a})|(U_7^{(k+1)}, \tilde{\psi}_7^{(k+1,a)}).$$

Proof. We regard a as fixed for the duration of this argument, and omit it from the notation. We need still more unipotent periods. Specifically, for each k, ℓ define $U_9^{(k,\ell)}$ to be the subgroup of $U_{\max}$ defined by requiring that $u_{ij} = 0$ under any of the following conditions:

$$k < i \le k + 2\ell,\ i \equiv k + 1 \mod 2 \text{ and } j = i + 1$$

$$i > k + 2\ell$$

$$i = k + 2\ell - 1,\ \text{and } j \ne 4n + 1 - k - 2\ell,$$

$$i = k + 2\ell \text{ and } j < 2n.$$

The formula

$$\psi(u_{1,2} + \cdots + u_{k-1,k} + u_{k,k+2} + u_{k+1,k+3} + \cdots + u_{k+2\ell-2,k_{k+2\ell}} + au_{k+2\ell,2n} + u_{k+2\ell,2n+1})$$

defines a character of this group which we denote $\psi_9^{(k,\ell)}(u)$. Also, let $U_{10}^{(k,\ell)}$ denote the subgroup of $U_{\max}$ defined by requiring that $u_{ij} = 0$ under any of the following conditions:

$$k < i \le k + 2\ell,\ i \equiv k + 1 \mod 2 \text{ and } j = i + 1$$

$$i > k + 2\ell - 1$$

$$i = k + 2\ell - 1 \text{ and } j > 2n, 2n + 1.$$

The formula

$$\psi(u_{1,2} + \cdots + u_{k,k+1} + u_{k+1,k+3} + \cdots + u_{k+2\ell-2,k_{k+2\ell}} + au_{k+2\ell-1,2n} + u_{k+2\ell-1,2n+1})$$

defines a character of this group which we denote $\psi_{10}^{(k,\ell)}(u)$. The period $(U_9, \psi_9^{(k,\ell)})$ is conjugate to $(U_{10}, \psi_{10}^{(k,\ell)})$.

Let $U_{11}^{(k,\ell)}$ denote the subgroup of $U_{\max}$ defined by requiring that $u_{ij} = 0$ under any of the following conditions:

$$k < i \le k + 2\ell,\ i \equiv k \mod 2 \text{ and } j = i + 1$$

$$i > k + 2\ell - 1$$

$$i = k + 2\ell - 1 \text{ and } j > 2n, 2n + 1.$$

Then $(U_{10}, \psi_{10}^{(k,\ell)})$ may be swapped for $(U_{11}, \psi_{11}^{(k,\ell)})$, where $\psi_{11}^{(k,\ell)}$ is defined by the same formula as $\psi_{10}^{(k,\ell)}$.

Also, $(U_{11}, \psi_{11}^{(k,\ell)})$, is clearly divisible by $(U_9, \psi_9^{(k+1,\ell-1)})$: to pass from the former to the latter one simply drops the integration over $u_{k+2\ell-2,j}$, for $j \ne 4n - k - 2\ell + 2$.

To complete the argument: for k even the period $(U_9^{(k+1,n-\frac{k}{2}-1)}, \psi_9^{(k+1,n-\frac{k}{2}-1)})$ divides the period $(U_7^{(k+1)}, \tilde{\psi}_7^{(k+1,a)})$. Indeed the only difference between the two is that in the former, we omit integration over $u_{2n-2,2n}$.

It follows that $(U_9^{(k+1,n-\frac{k}{2}-1)}, \psi_9^{(k+1,n-\frac{k}{2}-1)})$ is divisible by $(U_{10}^{n+\frac{k}{2}-1,1}, \psi_{10}^{n+\frac{k}{2}-1,1})$. Finally, every extension of $\psi_{10}^{n+\frac{k}{2}-1,1}$ to a character of $N_{n+\frac{k}{2}}$ is in the same orbit as $\psi_{n+\frac{k}{2},a}$. (See Remarks 5.1.2.) Hence

$$(U_{10}^{n+\frac{k}{2}-1,1}, \psi_{10}^{n+\frac{k}{2}-1,1}) \sim (N_{n+\frac{k}{2}}, \psi_{n+\frac{k}{2},a}).$$

The result follows. □

Lemma 7.3.4. *As in Theorem 5.1.4, let V_i denote the unipotent radical of the standard parabolic of G_{4n} having Levi isomorphic to $GL_i \times G_{4n-2i}$ (for $1 \le i \le 2n-2$). Let $V_i^{4n-2m-1}$ denote the unipotent radical of the standard maximal parabolic of G_{2n+1} (embedded into G_{4n} as $L_{n-1}^{\psi_{n-1}}$) having Levi isomorphic to $GL_i \times G_{2n-2i+1}$ (for $1 \le i \le n$). Let (N_ℓ, ψ_ℓ) be the period used to define the descent, as usual, and let $(N_\ell, \psi_\ell)^{(4n-2k)}$ denote the analogue for G_{4n-2k}, embedded into G_{4n} inside the Levi of a maximal parabolic.*

Then, $(V_k^{2n+1}, \mathbf{1}) \circ (N_{n-1}, \psi_{n-1})$ is an element of

$$\langle (N_{n+k-1}, \psi_{n+k-1}), \{(N_{n+j-1}, \psi_{n+j-1})^{(4n-2k+2j)} \circ (V_{k-j}, \mathbf{1}) : \quad 1 \le j < k\} \rangle.$$

Proof. In this proof, we shall not need to refer to any of the unipotent periods defined previously. On the other hand we will need to consider several new unipotent periods. For convenience, we start the numbering over from one.

Thus, let $(U_1, \psi_1) = (V_k^{2n+1}, \mathbf{1}) \circ (N_{n-1}, \psi_{n-1})$. To describe this group and character in detail, U_1 is the subgroup defined by $u_{ij} = 0$ if $n-1 < i \le n-1+k < j$, or $n-1+k < i$ and $u_{i,2n} = u_{i,2n+1}$ if $n-1 < i \le n-1+k$, and ψ_1 is given by

$$\psi_1(u) = \psi_0(u_{1,2} + \cdots + u_{n-2,n-1} + u_{n-1,2n} - u_{n-1,2n+1}).$$

Next, let U_2 denote the subgroup of U_1 defined by the additional conditions $u_{ij} = 0$ for $1 \le i \le n-1 < j \le n-1+k$. Let ψ_2 denote the restriction of ψ_1 to this subgroup.

Next, let U_3 denote the subgroup defined by $u_{ij} = 0$ for $i \le k, j \le n-1+k$, and $i > n-1+k$, and $u_{i,2n} = u_{i,2n+1}$ for $i \le k$. Let

$$\psi_3(u) = \psi(u_{k+1,k+2} + \cdots + u_{k+n-2,k+n-1} + u_{k+n-1,2n} - u_{k+n-1,2n+1}).$$

Then (U_2, ψ_2) is conjugate to (U_3, ψ_3), by any element of $G_{4n}(F)$ which projects to

$$\begin{pmatrix} & I_k & & & \\ I_{n-1} & & & & \\ & & I_{4n-2m-2k} & & \\ & & & & I_{n-1} \\ & & & I_k & \end{pmatrix}$$

(cf. subsection 7.2).

Finally, let $U_4 \supset U_3$ denote the subgroup of $U_{\max}$ given by $u_{ij} = 0$ if $j \le k+1$, or $i \ge n+k$. Then take ψ_4 defined by the same formula as ψ_3

Certainly $(U_2, \psi_2)|(U_1, \psi_1)$, and $(U_2, \psi_2) \sim (U_3, \psi_3)$. In Lemma 7.3.5 we prove that $(U_3, \psi_3) \sim (U_4, \psi_4)$. It follows that $(U_4, \psi_4)|(U_1, \psi_1)$. In fact, one may prove by an argument similar to the proof of Lemma 7.3.5 that in fact $(U_2, \psi_2) \sim (U_1, \psi_1)$ and hence $(U_4, \psi_4) \sim (U_1, \psi_1)$. But this is not needed for our purposes.

Next, let $U^{(r)}$ denote the subgroup of $U_{\max}$ defined by $u_{ij} = 0$ for $j \le r$, or $i \ge n+k$. So, $U_4 = U^{(k+1)}$, and $N_{n+k-1} = U^{(1)}$.

Let $\psi^{(r)}$ denote the character of $U^{(r)}$ defined by

$$\psi^{(r)}(u) = \psi_0 \left(\sum_{i=r}^{n-2+k} u_{i,i+1} + u_{n-1+k,2n} + u_{n-1+k,2n+1} \right).$$

Then $(U_4, \psi_4) = (U^{(k+1)}, \psi^{(k+1)})$, and $(N_{n+k-1}, \psi_{n+k-1}) = (U^{(1)}, \psi^{(1)})$. It is an easy consequence of Lemma 7.1.1 that

$$(U^{(r)}, \psi^{(r)}) \in \langle (U^{(r-1)}, \psi^{(r-1)}), (N_{n+k-r}, \psi_{n+k-r})^{(4n-2r+2)} \circ (V_{r-1}, \mathbf{1}) \rangle.$$

The result follows. □

Lemma 7.3.5. *Let (U_3, ψ_3) and (U_4, ψ_4) be defined as in the previous lemma. Then $(U_4, \psi_4) \sim (U_3, \psi_3)$.*

Proof. It's clear that $(U_3, \psi_3)|(U_4, \psi_4)$, so we only need to prove that $(U_4, \psi_4)|(U_3, \psi_3)$. The proof involves a family of groups defining intermediate stages. For ℓ such that $1 \le \ell \le n-1$ we define $U_4^{(\ell)}$ to be the subgroup of U_4 defined by the condition that for $i \le k$ the coordinate u_{ij} must be zero for $j \le k + \ell$. Thus $U_4 = U_4^{(1)} \supset U_4^{(2)} \supset \cdots \supset U_4^{(n-1)} \supset U_3$. For each of these groups we consider the period defined using the restriction of ψ_4.

We must show that $(U_4^{(n-1)}, \psi_4)|(U_3, \psi_3)$ and that $(U_4^{(i)}, \psi_4)|(U_4^{(i-1)}, \psi_4)$. In each case, all that is involved is an invocation of Lemma 7.1.1. For the first application, what must be checked is that the the normalizer of $U_4(F)$ in $G(F)$ permutes $\{\psi_4' : \psi_4'|_{U_3} = \psi_3\}$ transitively. Let $y(\underline{r}) = y(r_1, \ldots, r_k)$ denote the unipotent element in $G_{4n}(F)$ which projects to $I + r_1 e'_{1,2n} + \cdots + r_k e'_{k,2n}$. Then every element of $U_4^{(m)}$ is uniquely expressible as $u_3 y(\underline{r})$, for $u_3 \in U_3$ and $\underline{r} \in \mathbb{G}_a^k$. Hence a map ψ_4' as above is determined by its composition with y, which defines a character of $(F\backslash\mathbb{A})^k$, and hence is of the form

$$(r_1, \ldots, r_k) \mapsto \psi(a_1 r_1 + \cdots + a_k r_k)$$

for some $a_1, \ldots, a_k \in F$. Consider the unipotent element $z(a_1, \ldots, a_k)$ of G_{4n} which projects to $I + a_1 e'_{k+n-1,1} + \cdots + a_k e'_{k+n-1,k}$. We claim first that it normalizes $U_4^{(n-1)}$, and second that $\psi_4(z(a)y(r)z(a)^{-1}) = \psi(a_1 r_1 + \cdots + a_k r_k)$. As noted in 7.2 this may be checked by a matrix multiplication in SO_{4n}.

The proof that $(U_4^{(i)}, \psi_4)|(U_4^{(i-1)}, \psi_4)$ is entirely similar, with the role of $y(\underline{r})$ played by $y^{(i)}(\underline{r})$ which projects to $I + r_1 e'_{1,k+i+1} + \cdots + r_k e'_{k,k+i+1}$ and that of $z(\underline{a})$ played by $z^{(i)}(\underline{a})$ which projects to $I + a_1 e'_{k+i,1} + \cdots + a_k e'_{k+i,k}$. □

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