DESCENT CONSTRUCTION FOR GSPIN GROUPS–EVEN CASE

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Abstract. In this paper we provide an extension of the theory of *descent* of Ginzburg-Rallis-Soudry to the context of "almost orthogonal" representations, that is representations τ with the property that the symmetric square L-function, twisted by some Hecke character ω has a pole. Our theory supplements the recent work of Asgari-Shahidi on the functorial lift from (split and quasisplit forms of) $GSpin_{2n}$ to GL_{2n}.

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1. Introduction

The theory of descent for symplectic cuspidal representations of the general linear group $GL_{2n}(\mathbb{A})$ was developed in a sequence of remarkable works [GRS1]-[GRS5]. In these works the authors constructed in an explicit way a space $\sigma(\pi)$ of cuspidal automorphic functions on $SO_{2n+1}(\mathbb{A})$ which weakly lifts to a cuspidal self-dual representation π of $GL_{2n}(\mathbb{A})$ with the property that $L(\pi, \wedge^2, s)$ has a pole at $s = 1$. In [C-K-PS-S2] the method the of converse theorem is used to show the existence of a weak functorial lift from generic cuspidal automorphic representations of classical groups to automorphic representations of the general linear group. The combination of these methods allows the authors of [GRS4] to describe the image of the functorial lift of [C-K-PS-S1].

Thus, the conjunction of the descent method with the method of the converse theorem provides a very detailed description of the image of functoriality corresponding to the standard embedding of ${}^LG \to GL_N(\mathbb{C})$ with G a classical group. For an excellent survey we refer the reader to [So1].

Recently, Asgari and Shahidi proved in [Asg-Sha1] the existence of weak functorial lift from GSpin groups to the general linear group. Later, in the special case of $GSp(4)$ they were able to show in [Asg-Sha2] that this weak functorial lift is in fact strong in an appropriate sense.

In this paper we extend the descent method of Ginzburg, Rallis, and Soudry to GSpin groups. As a bonus,for $n \geq 2$ we can provide a "lower bound" on the image of the functorial lift from any quasisplit form of $GSpin_{2n}$ to GL_{2n} constructed by Asgari and Shahidi.

Let us briefly review the method. For simplicity of the exposition we assume that we are trying to construct a descent for a cuspidal representation, τ.

We first relate the property of essential self-duality to the existence of a pole of an L-function of τ, and then construct an Eisenstein series with the L-function appearing in the constant term. In fact there are two possibilities for what the L-function is, and hence two possibilities for the structure of the Eisenstein series, and we only consider one in these notes. Our Eisenstein series will be defined on the group $GSpin_{4n+1}$ induced from a Levi M isomorphic to $GL_{2n} \times GL_1$. Now a pole of the relevant L-function allows us to construct a residue representation $\mathcal{E}_{-1}(\tau, \omega)$ of $GSpin_{4n+1}(\mathbb{A})$. Next, we give an embedding of each quasisplit form of $GSpin_{2n}$ into $GSpin_{4n+1}$, and construct, using formation of a Fourier coefficient, a space of functions on this subgroup of $GSpin_{4n+1}$.

Now, quasisplit forms of $GSpin_{2n}$ are in natural one-to-one correspondence with quadratic characters $\chi : \mathbb{A}^\times/F^\times \to \pm 1$. To discern the form of $GSpin_{2n}$ to which a given representation τ will descend, we observe that $\tau \cong \tilde{\tau} \otimes \omega$ implies $\omega_\tau^2 = \omega^{2n}$. Here ω_τ denotes the central character of τ. Hence ω_τ/ω^n is some quadratic character χ.

Let $DC_\omega^\chi(\tau)$ denote the space of functions constructed on the quasisplit form of $GSpin_{2n}$ corresponding to the character χ, which we denote $GSpin_{2n}^\chi$. Then we prove that $DC_\omega^\chi(\tau)$ is zero, except when χ is the quadratic character obtained from τ and ω, in which case it is nonzero, and all of the functions in it are cuspidal. It follows that it decomposes as a direct sum of irreducible automorphic cuspidal representations of $GSpin_{2n}^\chi$. We then show that each of these irreducible constituents lifts weakly to τ by the functorial lifting associated to the map

$$ {}^L(GSpin_{2n+1}^\chi) = GSO_{2n}(\mathbb{C}) \rtimes \mathrm{Gal}(E/F) \to GL_{2n}(\mathbb{C}) = {}^LGL_{2n}. $$

sending the nontrivial element of $\mathrm{Gal}(E/F)$ to

$$ \begin{pmatrix} I_{n-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & I_{n-1} \end{pmatrix}. $$

(Here E is the quadratic extension of F corresponding to χ.) In fact in these notes the representation τ may be an isobaric sum of several cuspidal representations $\tau_1, \ldots, \tau_r$. The main differences are that the residue is a multi-residue, and the notation is more complicated.

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Without David Ginzburg and David Soudry's many careful and patient explanations of the "classical" case– $\omega = 1$– this work would not have been completed. It is important to point out that not all of the arguments shown to us have appeared in print. We mention in particular the computation of Jacquet modules in Appendix II, and the nonvanishing argument in Appendix III. Nevertheless, in each case the specialization of our arguments to the case $\omega = 1$ may be correctly attributed to Ginzburg, Rallis, Soudry (with any errors or stylistic blemishes introduced being our own responsibility).

This work was undertaken while both authors were in Bonn at the Hausdorff Research Institute for Mathematics, in connection with a series of lectures of Professor Soudry's. They wish to thank the Hausdorff Institute and Michael Rapoport for the opportunity. Finally, the second author wishes to thank Prof. Erez Lapid for many enlightening discussions on the subject matter of these notes.

2. The main result

Let $G = GSpin_{2n}$ and let $H = GL_{2n}$. Consider the inclusion

$$^LG =^L (GSpin_{2n}) = GSO_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}) = \, ^LGL_{2n} =^L H.$$

We denote this map r. Also, if $\pi \cong \otimes_v' \pi_v$ is an automorphic representation of a group $G'(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of a number field F, then the semisimple conjugacy class in the L-group $^LG'$ associated to the local representation π_v at an unramified place v will be denoted t_{π_v}. We say that an automorphic representation σ of $G(\mathbb{A})$ is a weak lift of the automorphic representation τ of $H(\mathbb{A})$ if for almost all places, $r(t_{\sigma_v}) \subset t_{\tau_v}$.

To formulate our main result we introduce the notion of η-orthogonal representations. Let τ be an irreducible automorphic cuspidal representation of GL_{2n}. Suppose that τ is essentially self-dual, i.e. that the contragredient $\tilde{\tau}$ of τ is isomorphic to $\tau \otimes \eta$ for some Hecke character η. It follows from [Ja-Sh2] (see remark (4.11) pp. 553-54) that $L(s, \tau \times \tau \otimes \eta)$ has a simple pole at $s = 1$. Now, $L(s, \tau \times \tau \otimes \eta)$ is the Langlands L function of the representation $\tau \boxtimes \eta$ (exterior tensor product) of the group $GL_{2n}(\mathbb{A}) \times GL_1(\mathbb{A})$ associated to the representation of the L group $GL_{2n}(\mathbb{C}) \times GL_1(\mathbb{C})$ (finite Galois form) on $M_{2n\times 2n}(\mathbb{C})$ in which $GL_{2n}(\mathbb{C})$ acts by $g \cdot X = gX\ {}^tg$ and $GL_1(\mathbb{C})$ acts by scalar multiplication. But this representation is reducible, decomposing into the subspaces of skew-symmetric and symmetric matrices. We denote the associated L functions $L(s, \tau, \wedge^2 \times \eta)$ and $L(s, \tau, sym^2 \times \eta)$ respectively. The local factors at finite ramified places may be defined using the local Langlands classification ([L2],[H-T],[Henn1]) and the definition of an Artin L function attached to a finite dimensional representation of the Weil group [Tate1], or they may be defined as in [Sha2]. By [Henn2] these two definitions agree. Then we have

$$L(s, \tau \times \tau \otimes \eta) = L(s, \tau, \wedge^2 \times \eta) L(s, \tau, sym^2 \times \eta).$$

As both of the L functions on the right-hand side are obtainable via the Langlands-Shahidi method, neither may vanish at $s = 1$ (see [Gel-Sha] §2.6 p. 84). Thus, exactly one of these two L functions has a simple pole at $s = 1$ while the other is holomorphic and nonvanishing. Similarly, if $\tilde{\tau}$ is not isomorphic to $\tau \otimes \eta$ then they are both holomorphic at $s = 1$. (This requires the extension of [Ja-Sh2] remark (4.11) to *completed* L functions– i.e., the statement that none of the local L functions has a pole at $s = 1$. The requisite facts about local L functions are well-known and a proof is reviewed at the end of Theorem 4.0.4.) One may prove the second assertion using results of Langlands via the method explained on p. 840 of [Kim1].

We will say that τ is η-symplectic in case $L(s, \tau, \wedge^2 \times \eta)$ has a pole at $s = 1$ and η-orthogonal otherwise. We also define "almost symplectic" to mean "η-symplectic for some η," and "almost orthogonal" similarly.

Remarks 2.0.1. (1) *There is another natural notion of "orthogonal/symplectic representation." Specifically, one could say that an automorphic representation is orthogonal/symplectic if the space it acts on supports an invariant symmetric/skew-symmetric form. The two notions appear to be related, but do not coincide. See* [PraRam].

(2) *There is a third approach to defining a local factor for $L(s, \tau, \wedge^2 \times \eta)$, which is to apply the "gcd" construction described in* [Gel-Sha] *section I.1.6, p. 17, to the integrals in* [Ja-Sh1]. *As far as we know this is not written down anywhere.*

(3) *An integral representation for $L(s, \tau, sym^2)$ was given in* [BG]. *The problem of extending this to $L(s, \tau, sym^2 \times \eta)$ has been considered by Banks* [Banks1, Banks2]. *Nontrivial technical difficulties arise, particularly in the case we consider, when τ is defined on GL_{2n}* [Banks3].

(4) *Let AS denote the functorial lift constructed in* [Asg-Sha1]. *It is shown in* [Asg-Sha1] *that $AS(\pi)$ is nearly equivalent to $\widetilde{AS(\pi)} \otimes \omega_\pi$, where ω_π denotes the central character of the representation π. (Of course, this means that they are the same space of functions when $AS(\pi)$ is cuspidal.) Thus, in practice it turns out to make sense to use $\eta = \omega^{-1} (= \bar{\omega})$.*

By proposition 2 of [L3], every irreducible automorphic representation of $GL_n(\mathbb{A})$ is isomorphic to a subquotient of $Ind_{P(\mathbb{A})}^{GL_n(\mathbb{A})} \tau_1 |\det_1|^{s_1} \otimes \cdots \otimes \tau_r |\det_r|^{s_r}$ for some real numbers $s_1, \ldots s_r$ and irreducible unitary automorphic cuspidal representations $\tau_1, \ldots, \tau_r$ of $GL_{n_1}(\mathbb{A}), \ldots, GL_{n_r}(\mathbb{A})$ respectively, such that $n_1 + \cdots + n_r = n$. Here P is the standard parabolic of GL_n corresponding to the ordered partition $(n_1, \ldots, n_r)$ of n. In the case when $s_i = 0$ for all i, this induced representation is irreducible. (This follows from the irreducibility of all the local induced representations, which is Theorem 3.2 of [Ja].) Also, the representations obtained by numbering a given set of cuspidal representations in different ways are isomorphic. (This follows from the fact that the standard intertwining operator between them does not vanish, which follows from [MW1], II.1.8 (meromorphically continued in IV.1.9(e)), and IV.1.10(b). In IV.3.12 these elements are combined to prove that the intertwining operator does not have a pole. The proof that it does not have a zero is an easy adaptation.) Furthermore, if two such induced representations are isomorphic, then they are obtained from two numberings of the same set of cuspidal representations ([Ja-Sh3], Theorem 4.4, p.809). An irreducible unitary representation τ of $GL_n(\mathbb{A})$ which is obtained from irreducible unitary cuspidal representations $\tau_1, \ldots, \tau_r$ in this manner is sometimes called the *isobaric sum* of the cuspidals, and denoted $\tau_1 \boxplus \cdots \boxplus \tau_r$. (A more general notion of "isobaric representation" was introduced in [L4], but we don't need it.)

Theorem 2.1. *For $r \in \mathbb{N}$, take $\tau_1, \ldots, \tau_r$ to be irreducible unitary automorphic cuspidal representations of $GL_{2n_1}(\mathbb{A}), \ldots, GL_{2n_r}(\mathbb{A})$, respectively, and let $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$. Let $n = n_1 + \cdots + n_r$, and assume that $n \geq 2$. Let ω denote a Hecke character, which is not the square of another Hecke character. Suppose that*

- *τ_i is ω^{-1}-orthogonal for each i, and*
- *$\tau_i \cong \tau_j \Rightarrow i = j$.*

For each i, let $\chi_i = \omega_{\tau_i}/\omega^{n_i}$ (which is quadratic), and let $\chi = \prod_{i=1}^r \chi_i$. Then there exists an irreducible generic cuspidal automorphic representation σ of $GSpin_{2n}^{\chi}(\mathbb{A})$ such that

- *σ weakly lifts to τ, and*
- *the central character ω_σ of σ is ω.*

Remark 2.0.2. *As was helpfully explained to us by H. Jacquet, the $n = 1$ case of our theorem follows from earlier work of Labesse-Langlands* [L-L]. *See also* [Kaz]. *Indeed, when $n = 1$, the function $L(s, \tau, \mathrm{sym}^2 \times \omega^{-1})$ has a pole iff χ is nontrivial, because $L(s, \tau, \wedge^2 \times \omega^{-1}) = L(s, \chi)$). In this case the representation τ that we consider is a cuspidal automorphic representation of $GL(2, \mathbb{A})$. It is known that in this case $\tilde{\tau} = \tau \otimes \omega_\tau^{-1}$ (see, e.g., [?], Theorem 3.3.5, p. 305). It follows that our original L-function on τ is, in this case, equivalent to requiring that $\tau = \tau \otimes \chi$ for some nontrivial quadratic character τ. The automorphic representation obtained from the descent construction in this case is simply a character of $\mathrm{Res}_F^E GL_1(\mathbb{A})$, where E is the quadratic extension of F corresponding to χ. Thus, we have recovered proposition 6.5, p. 771 of* [L-L].

Corollary 2.2. *The image of the functorial lift AS described in Theorem 1.1 (p. 140) of* [Asg-Sha1] *contains the set of all representations $\tau_1 \boxplus \cdots \boxplus \tau_r$ such that*

- *$\tau_i \cong \tau_j \Rightarrow i = j$,*
- *there is a Hecke character ω such that τ_i is ω^{-1}- orthogonal for each i.*

3. Notation

3.1. **General.** Throughout most of the paper, F will denote a number field. In Appendix II, it will be a non-Archimedean local field of characteristic zero.

We denote by J the matrix, of any size, with ones on the diagonal running from upper right to lower left, and by J' the matrix $\left(\begin{smallmatrix} & J \\ -J & \end{smallmatrix} \right)$ of any even size. We also employ the notation ${}^t g$ for the transpose of g and ${}_t g$ for the "other transpose" $J\, {}^t g J$. We employ the shorthand $G(F\backslash\mathbb{A}) = G(F)\backslash G(\mathbb{A})$, where G is any F-group.

We denote the Weyl group of the reductive group G by W_G or by W, when the meaning is clear from context.

If π is an automorphic or local representation, then $\tilde{\pi}$ is the contragredient, and ω_π the central character.

3.2. **Various Products.** Most tensor products will be denoted $\otimes$. However $\boxtimes$ will sometimes be used to distinguish the "outer" tensor product from the "inner" tensor products and "twisting." Let us recall these notions.

If (π_1, V_1) and (π_2, V_2) are representations of groups G_1 and G_2, then one may consider the representation of $G_1 \times G_2$ on $V_1 \otimes V_2$ given on pure tensors by

$$(\pi_1 \otimes \pi_2)(g_1, g_2) v_1 \otimes v_2 = \pi_1(g_1) v_1 \otimes \pi_2(g_2) v_2.$$

If (π_1, V_1) and (π_2, V_2) happen to be two representations of the *same* group G, then this construction yields a representation of $G \times G$. The space $V_1 \otimes V_2$ also supports a natural "tensor product representation" of the group G itself with the action given on pure tensors by

$$(\pi_1 \otimes \pi_2)(g) v_1 \otimes v_2 = \pi_1(g) v_1 \otimes \pi_2(g) v_2.$$

The representation of $G \times G$ on $V_1 \otimes V_2$ is sometimes called the outer tensor product and denoted $\boxtimes$ to avoid ambiguity.

Adding to the mix, the twist of a representation π of $GL_n(\mathbb{A})$ by a Hecke character χ is often denoted $\pi \otimes \chi$. In terms of the constructions above, it is the inner tensor product of π and the representation of $GL_n(\mathbb{A})$ obtained by composing χ with $\det$. We shall keep to this notation. We shall also need to consider the (outer) tensor product representation of $GL_n(\mathbb{A}) \times GL_1(\mathbb{A})$, for which we employ $\boxtimes$.

Let us mention that $\boxtimes$ will not be used in the sense of [L4].

In addition to $\otimes$ and $\boxtimes$, we use $\boxplus$ for "isobaric sum" as described above. We use $\times$ for Cartesian product of sets, groups, etc., and in the notation for various L functions (e.g., $sym^2 \times \omega^{-1}$).

3.3. **Notions of "genericity".** Let G be a quasisplit reductive group over the number field F, and $U_{\max}$ a maximal unipotent subgroup. First let ψ_v be a generic character (cf. [Kim2], p. 147, and also [Sha1], p.304) of $U_{\max}(F_v)$ for some place v of F, and (π_v, V) a representation of $G(F_v)$. We say that π_v is ψ_v-generic if it supports a nontrivial ψ_v-Whittaker functional (i.e., a $U_{\max}(F_v)$-equivariant linear map $V \to \mathbb{C}_{\psi_v}$, which is continuous in an appropriate topology, see [Sha1], p. 304. Here $\mathbb{C}_{\psi_v}$ denotes the one-dimensional $U_{\max}(F_v)$-module with action via the character ψ_v.) Now let $\psi = \prod_v \psi_v$ and $\pi \cong \otimes_v{}' \pi_v$ be a character of $U_{\max}(F\backslash\mathbb{A})$ and an automorphic representation of $G(\mathbb{A})$ respectively.

Ignoring topological considerations, it is easy to see that the space $\mathrm{Hom}_{U_{\max}(\mathbb{A})}(V_\pi, \mathbb{C}_\psi)$ is nontrivial iff each of the spaces $\mathrm{Hom}_{U_{\max}(F_v)}(V_{\pi_v}, \mathbb{C}_{\psi_v})$ is. However, it turns out that the more important issue is not whether there exists *some* nontrivial ψ-Whittaker functional, but whether the specific ψ-Whittaker functional given by

$$\varphi \mapsto \int_{U_{\max}(F\backslash\mathbb{A})} \varphi(u)\bar{\psi}(u)\, du$$

is nonvanishing. We refer to this Whittaker functional as the ψ-Whittaker *integral.* (See [Gel-So] for an example where the Whittaker integral vanishes, but a nonzero Whittaker functional exists.)

We would like to take this opportunity to draw attention to the subtle fact that there are two slightly different notions of global genericity for automorphic representations in common usage. The first states that a representation is globally ψ-generic if it supports a nonzero ψ-Whittaker integral. The second– which was the notion originally introduced in [PS]– requires that a cuspidal representation be *orthogonal to the kernel* of the ψ-Whittaker integral in $L^2_{\mathrm{cusp}}(G(F\backslash\mathbb{A}))$, in order to be called "generic." Clearly, the latter condition implies the former (except for the zero representation).

A nice feature of the stronger formulation is that the condition defines a subspace of $L^2_{\mathrm{cusp}}(G(F\backslash\mathbb{A}))$, which one may term the ψ-generic spectrum. Furthermore, this subspace satisfies multiplicity one, even if $L^2_{\mathrm{cusp}}(G(F\backslash\mathbb{A}))$ does not. (Cf. [PS]) A nice feature of the weaker formulation is that it does not rely on the L^2-pairing, and hence no technicalities arise in applying the notion to non-cuspidal forms and representations.

Throughout most of this paper, we shall say that a representation "is ψ-generic" if it supports a nonzero ψ-Whittaker integral, and "is generic" if it satisfies this condition for *some* ψ. We shall say that a cuspidal representation is "in the ψ-generic spectrum" if it is orthogonal to the kernel of the ψ-Whittaker integral.

Let $P_0 = N_G(U_{\max})$. If $P_0(F_v)$ permutes the characters of $U(F_v)$ transitively, then we may refer to a representation as "generic" or "non-generic" without reference to a specific ψ_v, and without ambiguity. The same applies to both notions of global genericity, in the case when $P_0(F)$ permutes the characters of $U_{\max}(F\backslash\mathbb{A})$ transitively. This condition is satisfied by GL_n and $GSpin_{2n+1}$, but *not* by $GSpin_{2n}$.

3.4. **Similitude groups and GSpin groups.** We first define the *similitude orthogonal and symplectic groups* to be

$$GO_m = \{g \in GL_m : gJ\,{}^tg = \lambda(g)J \text{ for some } \lambda(g) \in \mathbb{G}_m\},$$

$$GSp_{2n} = \{g \in GL_{2n} : gJ'\,{}^tg = \lambda(g)J' \text{ for some } \lambda(g) \in \mathbb{G}_m\}.$$

For each of these groups the map $g \mapsto \lambda(g)$ is a rational character called the *similitude factor.* If m is odd then GO_m is in fact isomorphic to $SO_m \times GL_1$. This case will play no further role. The group GO_{2n} is disconnected; indeed the subgroup generated by SO_{2n} and $\left\{ \begin{pmatrix} \lambda I_n & \\ & I_n \end{pmatrix} : \lambda \in \mathbb{G}_m \right\}$ is a connected index two subgroup, which we denote GSO_{2n}.

We shall now define GSpin groups as the groups whose duals are the similitude classical groups $GSp_{2n}(\mathbb{C}), GSO_{2n}(\mathbb{C})$. Thus we write down the based root data, but employ notation appropriate to the application in which what we write down will arise as the dual of something.

The groups GSp_{2n} and GSO_{2n} share a maximal torus, consisting of matrices of the form

$$\mathrm{diag}(t_1, \ldots, t_n, \lambda t_n^{-1}, \ldots, \lambda t_1^{-1}).$$

The coordinates used just above correspond to a choice of $\mathbb{Z}$-bases for the lattices of characters and cocharacters. For $i = 1$ to n, let e_i^* denote the character that sends this torus element to t_i for $i = 1$ to n and e_0^* being the map that sends it to the similitude factor, λ. Let $\{e_i : i = 0 \text{ to } n\}$ denote the dual basis for the cocharacter lattice. Let $X^\vee$ denote the character lattice and X the cocharacter lattice. Each similitude classical group has a Borel subgroup equal to the set of upper triangular matrices which are in it. In each case we employ this choice of Borel, and let $\Delta^\vee$ denote the set of simple roots and Δ the set of simple coroots. Then we easily compute that for GSp_{2n}

$$\Delta^\vee = \{e_i^* - e_{i+1}^*,\ i = 1 \text{ to } n-1\} \cup \{2e_n^* - e_0^*\}.$$

$$\Delta = \{e_i - e_{i+1},\ i = 1 \text{ to } n-1\} \cup \{e_n\}.$$

and for GSO_{2n}

$$\Delta^\vee = \{e_i^* - e_{i+1}^*,\ i = 1 \text{ to } n-1\} \cup \{e_{n-1}^* + e_n^* - e_0^*\}.$$

$$\Delta = \{e_i - e_{i+1},\ i = 1 \text{ to } n-1\} \cup \{e_{n-1} + e_n\}.$$

We now define $GSpin_{2n+1}$ to be the $F-$split connected reductive algebraic group having based root datum dual to that of GSp_{2n}, and $GSpin_{2n}$ to be the one having based root datum dual to that of GSO_{2n}. We have here used the fact that F-split connected reductive algebraic groups are classified by based root data, for which see p.274 of [Spr].

By the classification results in Chapter 16 of [Spr] (especially 16.3.2, 16.3.3 16.4.2), one finds that $GSpin_{2n+1}$ is in fact the unique *quasisplit* F-group having based root datum dual to that of GSp_{2n}, and that there is a 1-1 correspondence between quasisplit F groups G such that ${}^LG^0 = GSO_{2n}(\mathbb{C})$ and homomorphisms from $\mathrm{Gal}(\bar{F}/F)$ to the group of automorphisms of the lattice $X(T)$ which preserve the set Δ of simple roots. This group has two elements: the identity and and element which reverses the roots $e_{n-1} - e_n$ and $e_{n-1} + e_n$ while fixing the other simple roots. The effect of this automorphism on the $\mathbb{Z}$-bases $\{e_i : 0 \le i \le n\}$, and $\{e_i^* : 0 \le i \le n\}$ is as follows:

$$e_i \mapsto \begin{cases} e_i & i \neq 0, n \\ -e_n & i = n \\ e_0 + e_n & i = 0 \end{cases} \qquad e_i^* \mapsto \begin{cases} e_i^* & i \neq n \\ e_0^* - e_n^* & i = n \end{cases}$$

It follows that the lattices of F-rational characters and cocharacters are spanned by

$$\{e_i : 0 < i < n\} \cup \{2e_0 + e_n\}, \qquad \text{and} \qquad \{e_i : 0 \le i < n\},$$

respectively.

By class field theory homomorphisms from $\mathrm{Gal}(\bar{F}/F)$ to a group with two elements are in one-to-one correspondence with quadratic characters $\chi : \mathbb{A}_F^\times / F^\times \to \{\pm 1\}$. We denote the F-group corresponding to the character χ by $GSpin_{2n}^\chi$. The F-group corresponding to the trivial character is the unique split F-group having the specified root datum, and is also denoted simply by $GSpin_{2n}$.

To save space, the group $GSpin_m$ will usually be denoted by G_m, and $GSpin_{2n}^\chi$ by G_{2n}^χ.

Observe that in either the odd or even case e_0^* is a generator for the lattice of cocharacters of the center of G_m.

Because we define G_m in the manner we do, it comes equipped with a choice of Borel subgroup and maximal torus, as do various reductive subgroups we shall consider below. In each case, we denote the Borel subgroup of the reductive group G by $B(G)$, and the maximal torus by $T(G)$.

A straightforward adaptation of the proof of Theorem 16.3.2 of [Spr] shows that there exist surjections $G_m \to SO_m$ defined over F. We fix one such and denote it pr. We require that pr is such that $\mathrm{pr}(B(G_m))$ consists of upper triangular matrices.

An alternative description of the same group as a quotient of $Spin_m \times GL_1$ is given in [Asg]. Proposition 2.4 on p. 678 of [Asg] shows that the two definitions are equivalent.

For those familiar with the construction of $Spin_m$ as a subgroup of the multiplicative group of a Clifford algebra, we remark that there is a third construction of $GSpin_m$ as the slightly larger group obtained by including the nonzero scalars in the Clifford algebra as well. In this guise, it is sometimes referred to as the "Clifford group." (See, e.g., [I] p.999.) This description will not play a role for us.

We will construct an Eisenstein series on G_{2m+1} induced from a standard parabolic $P = MU$ such that M is isomorphic to $GL_m \times GL_1$. There is a unique such parabolic. We shall refer to this parabolic as the "Siegel."

Remark 3.4.1.
- *We can identify the based root datum of the Levi M with that of $GL_m \times GL_1$ in such a fashion that e_0 corresponds to GL_1 and does not appear at all in GL_m. We can then identify M itself with $GL_m \times GL_1$ via a particular choice of isomorphism which is compatible this and with the usual usage of e_i, e_i^* for characters, cocharacters of the standard torus of GL_m.*
- *Having made this identification, a Levi M' which is contained in M will be identified with $GL_1 \times GL_{m_1} \times \dots GL_{m_k}$, (for some $m_1, \dots, m_k \in \mathbb{N}$ that add up to m) in the natural way: GL_1 is identified with the GL_1 factor of M, and then $GL_{m_1} \times \dots GL_{m_k}$ is identified with the subgroup of M corresponding to block diagonal elements with the specified block sizes, in the specified order.*
- *The lattice of rational characters of M is spanned by the maps $(g, \alpha) \mapsto \alpha$ and $(g, \alpha) \mapsto \det g$. Restriction defines an embedding $X(M) \to X(T(G_{2m+1}))$, which sends these maps to e_0 and $(e_1 + \dots + e_m)$, respectively. By abuse of notation, we shall refer to the rational character of M corresponding to e_0 as e_0 as well.*
- *The modulus of P is $(g, \alpha) \to \det g^m$.*

We also fix a maximal compact subgroup K_m of $G_m(\mathbb{A})$. Any which satisfies the conditions required in [MW1] (see pages 1 and 4) will do.

3.5. Weyl group of $GSpin_{2m+1}$; it's action on standard Levis and their representations.

Lemma 3.5.1. *The Weyl group of G_m is canonically identified with that of SO_m.*

Proof. For this lemma only, let T denote the torus of SO_m and $\tilde{T}$ that of G_m. Then the following diagram commutes:

$$\begin{tikzcd}
Z_{G_m}(\tilde{T}) \arrow[r] \arrow[d] & N_{G_m}(\tilde{T}) \arrow[d] \\
Z_{SO_m}(T) \arrow[r] & N_{SO_m}(T).
\end{tikzcd}$$

Both horizontal arrows are inclusions and both vertical arrows are pr. $\square$

One easily checks that every element of the Weyl group of SO_{2n+1} is represented by a matrix of the form $w = \det w_0 w_0$, where w_0 is a permutation matrix. We denote the permutation associated to w_0 also by w_0. The set of permutations w_0 obtained is precisely the set of permutations $w_0 \in \mathfrak{S}_{2n}$ satisfying, $w_0(2n + 2 - i) = 2n + 2 - w_0(i)$ It is well known that the Weyl group of SO_{2n+1} (or

G_{2n+1}) is isomorphic to $\mathfrak{S}_n \rtimes \{\pm 1\}^n$. To fix a concrete isomorphism, we identify $p \in \mathfrak{S}_n$ with an $n \times n$ matrix in the usual way, and then with

$$\begin{pmatrix} p & & \\ & 1 & \\ & & {}^tp^{-1} \end{pmatrix} \in SO_{2n}.$$

We identify $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$ with the permutation p of $\{1, \dots, 2n+1\}$ such that

$$p(i) = \begin{cases} i & \text{if } \epsilon_i = 1, \\ 2n+2-i & \text{if } \epsilon_i = -1. \end{cases}$$

We then identify $(p, \underline{\epsilon}) \in \mathfrak{S}_n \times \{\pm 1\}^n$ (direct product of sets) with $p \cdot \underline{\epsilon} \in W_{SO_{2n+1}}$.

With this identification made,

(3.5.2)

$$(p,\underline{\epsilon})\cdot \begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_n & & & & \\ & & & 1 & & & \\ & & & & t_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_1^{-1} \end{pmatrix} \cdot (p,\underline{\epsilon})^{-1} = \begin{pmatrix} t_{p^{-1}(1)}^{\epsilon_{p^{-1}(1)}} & & & & & & \\ & \ddots & & & & & \\ & & t_{p^{-1}(n)}^{\epsilon_{p^{-1}(n)}} & & & & \\ & & & 1 & & & \\ & & & & t_{p^{-1}(n)}^{-\epsilon_{p^{-1}(n)}} & & \\ & & & & & \ddots & \\ & & & & & & t_{p^{-1}(1)}^{-\epsilon_{p^{-1}(1)}} \end{pmatrix}.$$

Lemma 3.5.3. *Let $(p, \underline{\epsilon}) \in \mathfrak{S}_n \rtimes \{\pm 1\}^{n-1}$ be idenified with an element of $W_{SO_{2m}} = W_{G_{2m}}$ as above. Then the action on the character and cocharacter lattices of G_{2m} is given as follows:*

$$(p,\underline{\epsilon}) \cdot e_i = \begin{cases} e_{p(i)} & i > 0, \epsilon_{p(i)} = 1, \\ -e_{p(i)} & i > 0, \epsilon_{p(i)} = -1, \\ e_0 + \sum_{\epsilon_{p(i)}=-1} e_{p(i)} & i = 0. \end{cases}$$

$$(p,\underline{\epsilon}) \cdot e_i^* = \begin{cases} e_{p(i)}^* & i > 0, \epsilon_{p(i)} = 1, \\ e_0^* - e_{p(i)}^* & i > 0, \epsilon_{p(i)} = -1, \\ e_0^* & i = 0. \end{cases}$$

Remark 3.5.4. *Much of this can be deduced from* (3.5.2)*, keeping in mind that $w \in W_G$ acts on cocharacters by $(w \cdot \varphi)(t) = w\varphi(t)w^{-1}$ and on characters by $(w \cdot \chi)(t) = \chi(w^{-1}tw)$. However, it is more convenient to give a different proof.*

Proof. Let $\alpha_i = e_i - e_{i+1}, i = 1$ to $n-1$ and $\alpha_n = e_n$. Let s_i denote the elementary reflection in $W_{G_{2n}}$ corresponding to α_i. Then it is easily verified that $s_1, \dots, s_{n-1}$ generate a group isomorphic to $\mathfrak{S}_n$ which acts on $\{e_1, \dots, e_n\} \in X(T)$ and $\{e_1^*, \dots, e_n^*\} \in X^\vee(T)$ by permuting the indices and acts trivially on e_0 and e_0^*. Also

$$s_n \cdot e_i = \begin{cases} e_i & i \neq n, 0 \\ e_0 + e_n & i = 0 \\ -e_n & i = n \end{cases}$$

$$s_n \cdot e_i^* = \begin{cases} e_i^* & i \neq n \\ e_0^* - e_n^* & i = n \end{cases}$$

If $\underline{\epsilon} \in \{\pm 1\}^{n-1}$ is such that $\#\{i : \epsilon_i = -1\} = 1$, then $\underline{\epsilon}$ is conjugate to s_n by an element of the subgroup isomorphic to $\mathfrak{S}_n$ generated by $s_1, \dots, s_{n-1}$. An arbitrary element of $\{\pm 1\}^{n-1}$ is a product of elements of this form, so one is able to deduce the assertion for general $(p, \underline{\epsilon})$. $\square$

Observe that the $\mathfrak{S}_n$ factor in the semidirect product is precisely the Weyl group of the Siegel Levi.

In the study of intertwining operators and Eisenstein series (e.g., section 4 below), one encounters a certain subset of the Weyl group associated to a standard Levi, M. Specifically,

$$W(M) := \left\{ w \in W_{G_{2n+1}} \;\middle|\; \begin{array}{l} w \text{ is of minimal length in } w \cdot W_M \\ wMw^{-1} \text{ is a standard Levi of } G_{2n+1} \end{array} \right\}.$$

For our purposes, it is enough to consider the case when M is a subgroup of the Siegel Levi. In this case it is isomorphic to $GL_{m_1} \times \dots \times GL_{m_r} \times GL_1$ for some integers $m_1, \dots, m_r$ which add up to n, and we shall only need to consider the case when m_i is even for each i. (This, of course, forces n to be even as well.)

Lemma 3.5.5. *For each $w \in W(M)$ with M as above, there exist a permutation $p \in \mathfrak{S}_r$ and and element $\underline{\epsilon} \in \{\pm 1\}^r$ such that, if $m \in M = (g, \alpha)$ with $\alpha \in GL_1$ and*

$$g = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_r \end{pmatrix} \in GL_n,$$

then

$$wmw^{-1} = (g', \alpha \cdot \prod_{\epsilon_i = -1} \det g_i) \quad g' = \begin{pmatrix} g_1' & & \\ & \ddots & \\ & & g_r' \end{pmatrix},$$

where

$$g_i' \approx \begin{cases} g_{p^{-1}(i)} & \text{if } \epsilon_{p^{-1}(i)} = 1, \\ {}_t g_{p^{-1}(i)}^{-1} & \text{if } \epsilon_{p^{-1}(i)} = -1. \end{cases}$$

Here $\approx$ has been used to denote equality up to an inner automorphism. The map $(p, \underline{\epsilon}) \mapsto w$ is a bijection between $W(M)$ and $\mathfrak{S}_r \times \{\pm 1\}^r$. (Direct product of sets: $W(M)$ is not, in general, a group.)

Proof. Since wMw^{-1} is a standard Levi which does not contain any short roots, it is again contained in the Siegel Levi.

The Levi M determines an equivalence relation $\sim$ on the set of indices, $\{1, \dots, n\}$ defined by the condition that $i \sim i+1$ iff $e_i - e_{i+1}$ is an root of M. When viewed as elements of $\mathfrak{S}_n \rtimes \{\pm 1\}^{n-1}$, the elements of $W(M)$ are those pairs $(p, \underline{\epsilon})$ such that $i \sim i+1 \Rightarrow p(i+1) = p(i)+1$, and $i \sim j \Rightarrow \epsilon_i = \epsilon_j$. This gives the identification with $\mathfrak{S}_r \times \{\pm 1\}^r$.

It is clear that the precise value of g_i' is determined only up to conjugacy by an element of the torus (because we do not specify a particular representative for our Weyl group element). By Theorem 16.3.2 of [Spr], it may be discerned, to this level of precision, by looking at the effect of w on the based root datum of M. The result now follows from Lemma 3.5.3. $\square$

Corollary 3.5.6. *Let $w \in W(M)$ be associated to $(p, \underline{\epsilon}) \in \mathfrak{S}_r \times \{\pm 1\}^r$ as above. Let $\tau_1, \dots, \tau_r$ be irreducible cuspidal representations of $GL_{m_1}(\mathbb{A}), \dots, GL_{m_r}(\mathbb{A})$, respectively, and let ω be a Hecke character. Then our identification of M with $GL_{m_1} \times \dots \times GL_{m_r} \times GL_1$ determines an identification of $\bigotimes_{i=1}^r \tau_i \boxtimes \omega$ with a representation of $M(\mathbb{A})$. Let $M' = wMw^{-1}$. Then M' is also identified, via*

3.4.1 with $GL_{m_{p^{-1}(1)}} \times \cdots \times GL_{m_{p^{-1}(r)}} \times GL_1$, and we have

$$\bigotimes_{i=1}^{r} \tau_i \boxtimes \omega \circ Ad(w^{-1}) = \bigotimes_{i=1}^{r} \tau_i' \boxtimes \omega,$$

where

$$\tau_i' = \begin{cases} \tau_{p^{-1}(i)} & \text{if } \epsilon_{p^{-1}(i)} = 1, \\ \tilde{\tau}_{p^{-1}(i)} \otimes \omega & \text{if } \epsilon_{p^{-1}(i)} = -1. \end{cases}$$

Proof. The contragredient $\tilde{\tau}_i$ of τ_i may be realized as an action on the same space of functions as τ_i via $g \cdot \varphi(g_1) = \varphi(g_1\ {}_tg^{-1})$. This follows from strong multiplicity one and the analogous statement for local representations, for which see [GK75] page 96, or [BZ1] page 57. Combining this fact with the Lemma, we obtain the Corollary. □

3.6. **Unramified Correspondence.**

Lemma 3.6.1. *Suppose that $\tau \cong \otimes_v' \tau_v$ is an ω^{-1}-orthogonal irreducible cuspidal automorphic representation of $GL_{2n}(\mathbb{A})$. Let v be a place such that τ_v is unramified. Let $t_{\tau,v}$ denote the semisimple conjugacy class in $GL_{2n}(\mathbb{C})$ associated to τ_v. Let $r : GO_{2n}(\mathbb{C}) \to GL_{2n}(\mathbb{C})$ be the natural inclusion. Then $t_{\tau,v}$ contains elements of the image of r.*

Proof. For convenience in the application, we take GL_{2n} to be identified with a subgroup of the Levi of the Siegel parabolic as in section 3.4. Since τ_v is both unramified and generic, it is isomorphic to $\mathrm{Ind}_{B(GL_{2n})(F_v)}^{GL_{2n}(F_v)} \mu$ for some unramified character μ of the maximal torus $T(GL_{2n})(F_v)$ such that this induced representation is irreducible. (See [Car], section 4, [Z] Theorem 8.1, p. 195.) Let $\mu_i = \mu \circ e_i^*$.

Since $\tau \cong \tilde{\tau} \otimes \omega$, it follows that $\tau_v \cong \tilde{\tau}_v \otimes \omega_v$ and from this we deduce that $\{\mu_i : 1 \leq i \leq 2n\}$ and $\{\mu_i^{-1}\omega : 1 \leq i \leq 2n\}$ are the same set. Hence $\prod_{i=1}^{2n} \mu_i = \chi\omega^n$, where χ is quadratic.

Now, what we need to prove is the following: if S is a set of $2n$ unramified characters of F_v, such that for each i there exists j such that $\mu_i = \mu_j^{-1}\omega$, then there is a permutation $\sigma : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ such that $\mu_{\sigma(i)} = \omega\mu_{2n-\sigma(i)}^{-1}$ for $i = 1$ to $n-1$. This we prove by induction on n. If $n = 1$, there is nothing to be proved.

If $n > 1$ it is sufficient to show that there exist $i \neq j$ such that $\mu_i = \mu_j^{-1}\omega$. If there exists i such that $\mu_i \neq \mu_i^{-1}\omega$ then this is clear. On the other hand, there are exactly two unramified characters μ such that $\mu = \mu^{-1}\omega$.

Now, suppose that $\mu_1, \ldots, \mu_{2n}$ have been renumbered according to σ as above. Then $\mu_{n+1}\mu_n = \omega\chi$. If χ is trivial, it follows that $\mu_i = \omega\mu_{2n-i}^{-1}$ for all i, and hence that the conjugacy class $t_{\tau,v}$ contains elements of the maximal torus of $GSO_{2n}(\mathbb{C})$.

On the other hand, if χ is nontrivial, then $\mu_n \neq \omega\mu_{n+1}^{-1}$, from which it follows that $\mu_n^2\mu_{n+1} = \omega$ and $\mu_{n+1} = \chi\mu_n$. It follows that $t_{\tau,v}$ contains a diagonal element which is conjugate, in $GL_{2n}(\mathbb{C})$, to an element of the connected component of $GO_{2n}(\mathbb{C})$ which does not contain the identity. □

Corollary 3.6.2. *Suppose $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ with τ_i an ω^{-1}-orthogonal irreducible cuspidal automorphic representation of $GL_{2n_i}(\mathbb{A})$, for each i. Then the same conclusion holds.*

Corollary-to-the-Proof 3.6.3. *Let τ be as in corollary 3.6.2, and let v be a place at which τ and ω are unramified. Let η be one of the two unramified characters such that $\eta^2 = \omega_v$. Let χ_{un} denote the unique nontrivial unramified quadratic character of $F_v^\times$. Then $\tau_v \cong \mathrm{Ind}_{B(GL_{2n})(F_v)}^{GL_{2n}(F_v)} \mu$ (normalized induction), for an unramified character μ of the torus of $GL_{2n}(F_v)$ which satisfies either*

$$\mu \circ e_{2n+1-i}^* = \omega_v \cdot (\mu \circ e_i^*)^{-1} \quad \forall i = 1 \text{ to } n,$$

or

$$\mu \circ e^*_{2n+1-i} = \omega_v \cdot (\mu \circ e^*_i)^{-1} \quad \forall i = 1 \text{ to } n-1, \quad \mu \circ e^*_n = \eta, \quad \mu \circ e^*_{n+1} = \chi_{un}\eta.$$

3.7. **Unipotent subgroups and their characters.** The kernel of pr consists of semisimple elements. In particular, the number of unipotent elements of a fiber is zero or one, and it's one if and only if the element of SO_m is unipotent. In other words, pr yields a bijection of unipotent elements (indeed, an isomorphism of unipotent subvarieties), and we may specify unipotent elements or subgroups by their images under pr . This also defines coordinates for any unipotent element or subgroup, which we use when defining characters. Thus, we write u_{ij} for the i, j entry of $\mathrm{pr}(u)$.

Above we fixed a specific isomorphism of a subgroup of G_{2m} with GL_m. If u is a unipotent element of of this subgroup this identification with an $m \times m$ matrix gives a second definition of u_{ij} This is not a problem, however, as the two definitions agree.

Most of the unipotent groups we consider are subgroups of the maximal unipotent of G_m consisting of elements u with $\mathrm{pr}(u)$ upper triangular. We denote this group $U_{\max}$. A complete set of coordinates is $\{u_{ij} : 1 \le i < j \le m - i\}$. We denote the opposite maximal unipotent by $\overline{U_{\max}}$. It consists of all unipotent elements of G_m such that $\mathrm{pr}(u)$ is *lower* triangular.

We fix once and for all a character ψ_0 of $\mathbb{A}/F$. We use this character together with the coordinates just above to specify characters of our unipotent subgroups.

When specifying subgroups of $U_{\max}$ and their characters, the restriction to $\{(i, j) : 1 \le i < j \le m - i\}$ is implicit.

It will also sometimes be necessary to describe unipotent subgroups such that only a few of the entries in the corresponding elements of SO_m are nonzero. For this purpose we introduce the notation $e'_{ij} = e_{ij} - e_{m+1-j,m+1-i}$. One may check that for all $i \ne j$ and $a \in F$, the matrix $I + ae'_{ij}$ is an element of $SO_m(F)$.

3.8. **"Unipotent periods".** We now introduce the framework within which, we believe, certain of the computations involved in the descent construction can be most easily understood.

Let G be a reductive algebraic group defined over a number field F. If U is a unipotent subgroup of G and ψ_U is a character of $U(F\backslash\mathbb{A})$, we define the *unipotent period* (U, ψ_U) associated to this pair to be given by the formula

$$\varphi^{(U,\psi_U)}(g) := \int_{U(F\backslash\mathbb{A})} \varphi(ug)\psi_U(u)du.$$

Clearly, φ must be restricted to a space of left $U(F)$-invariant functions such that the integral is defined (for example, because φ is smooth).

Let $\mathcal{U}$ denote the set of unipotent periods. For V a space of functions defined on $G(\mathbb{A})$, put

$$\mathcal{U}^\perp(V) = \{(U, \psi) \in \mathcal{U} : \varphi^{(U,\psi)} \equiv 0 \; \forall \varphi \in V\}.$$

When V is the space of a representation π we will employ also the notation $\mathcal{U}^\perp(\pi)$. We also employ the notation $(U, \psi) \perp V$ for $(U, \psi) \in \mathcal{U}^\perp(V)$ and similarly $(U, \psi) \perp \pi$.

We also require a vocabulary to express relationships among unipotent periods. We shall say that

$$(U, \psi_U) \in \langle (U_1, \psi_{U_1}), \dots, (U_n, \psi_{U_n}), \dots \rangle$$

if $V \perp (U_i, \psi_{U_i}) \forall i \Rightarrow V \perp (U, \psi_U)$. Clearly, if $(U_1, \psi_{U_1}) \in \langle (U_2, \psi_2), (U_3, \psi_3) \rangle$, and $(U_2, \psi_2) \in \langle (U_4, \psi_4), (U_5, \psi_5) \rangle$ then $(U_1, \psi_1) \in \langle (U_3, \psi_3), (U_4, \psi_4), (U_5, \psi_5) \rangle$.

We also use notation $(U_1, \psi_1)|(U_2, \psi_2)$, or the language "$(U_1, \psi_1)$ divides (U_2, ψ_2)," " (U_2, ψ_2) is divisible by (U_1, ψ_1) " for $(U_2, \psi_2) \in \langle (U_1, \psi_1) \rangle$. Finally, $(U_1, \psi_1) \sim (U_2, \psi_2)$ means $(U_1, \psi_1)|(U_2, \psi_2)$ and $(U_2, \psi_2)|(U_1, \psi_1)$. This is an equivalence relation and we shall refer to unipotent periods which are related in this way as "equivalent."

It is *sometimes* possible to compose unipotent periods. Specifically, if $f^{(U_1,\psi_1)}$ is left-invariant by $U_2(F)$, then one may consider $(f^{(U_1,\psi_1)})^{(U_2,\psi_2)}$. We denote the composite by $(U_2,\psi_2)\circ(U_1,\psi_1)$.

Now, suppose that U is the unipotent radical of a parabolic P of G with Levi M. The choice of ψ_0 gives rise to an identification of the space of characters of $U(F)\backslash U(\mathbb{A})$ with the F points of $\overline{U}/(\overline{U},\overline{U})$ which is compatible with the action of $M(F)$. Here $\overline{U}$ denotes the unipotent radical of the parabolic $\overline{P}$ of G opposite to P. For ϑ a character, let M^ϑ denote the stabilizer of ϑ (regarded as a point in $\overline{U}/(\overline{U},\overline{U})(F)$) in M. So M^ϑ is an algebraic subgroup of M defined over F.

Definition 3.8.1. *Then we define $FC^\vartheta : C^\infty(G(F\backslash\mathbb{A}))\to C^\infty(M^\vartheta(F\backslash\mathbb{A}))$ by*
$$FC^\vartheta(\varphi)(m)=\varphi^{(U,\vartheta)}(m)=\int_{U(F\backslash\mathbb{A})}\varphi(um)\vartheta(u)du.$$
This is certainly an $M^\vartheta(\mathbb{A})$-equivariant map.

4. Eisenstein series

The main purpose of this section is to construct, for each integer $n\ge 2$ and Hecke character ω, a map from the set of all isobaric representations τ satisfying the hypotheses of theorem 2.1 into the residual spectrum of G_{4n+1}. We use the same notation $\mathcal{E}_{-1}(\tau,\omega)$ for all n. The construction is given by a multi-residue of an Eisenstein series in several complex variables, induced from the cuspidal representations $\tau_1,\dots,\tau_r$ used to form τ. (Note that by [Ja-Sh3], Theorem 4.4, p.809, this data is recoverable from τ.)

Let ω be a Hecke character. Let $\tau_1,\dots,\tau_r$ be a irreducible cuspidal automorphic representations of $GL_{2n_1},\dots,GL_{2n_r}$, respectively.

For each i, let V_{τ_i} denote the space of cuspforms on which τ_i acts. Then pointwise multipication
$$\varphi_1\otimes\cdots\otimes\varphi_r\mapsto\prod_{i=1}^r\varphi_i$$
extends to an isomorphism between the abstract tensor product $\bigotimes_{i=1}^r V_{\tau_i}$ and the space of all functions
$$\Phi(g_1,\dots,g_r)=\sum_{i=1}^N c_i\prod_{j=1}^r\varphi_{i,j}(g_j)\quad c_i\in\mathbb{C},\ \varphi_{i,j}\in V_{\tau_j}\ \forall i,j.$$
(This is an elementary exercise.) We consider the representation $\tau_1\otimes\cdots\otimes\tau_r$ of $GL_{2n_1}\times\cdots\times GL_{2n_r}$, realized on this latter space, which we denote $V_{\otimes\tau_i}$.

Let $n=n_1+\cdots+n_r$.

We will construct an Eisenstein series on G_{4n+1} induced from the subgroup $P=MU$ of the Siegel parabolic such that $M\cong GL_{2n_1}\times\cdots\times GL_{2n_r}\times GL_1$. Let $s_1,\dots s_r$ be a complex variables. Using the identification of M with $GL_{2n_1}\times\cdots\times GL_{2n_r}\times GL_1$ fixed in section 3.4 above, we define an action of $M(\mathbb{A})$ on the space of $\tau_1\otimes\cdots\otimes\tau_r$ by
$$(g_1,\dots,g_r,\alpha)\cdot\prod_{j=1}^r\varphi_j(h_j)=\left(\prod_{j=1}^r\varphi(h_jg_j)|\det g_j|^{s_j}\right)\omega(\alpha).\tag{4.0.2}$$
We denote this representation of $M(\mathbb{A})$, by $(\bigotimes_{i=1}^r\tau_i\otimes|\det{}_i|^{s_i})\boxtimes\omega$.

To shorten the notation, we write $\underline{g}=(g_1,\dots,g_r)$. Then (4.0.2) may be shortened to
$$\underline{g}\cdot\Phi(\underline{h})=\Phi(\underline{h}\cdot\underline{g})\left(\prod_{j=1}^r|\det g_j|^{s_j}\right)\omega(\alpha).$$
We shall also employ the shorthand $\underline{s}=(s_1,\dots,s_r)$, and $\underline{\tau}=(\tau_1,\dots,\tau_r)$.

For each $\underline{s}$ we have the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G_{4n+1}(\mathbb{A})}(\bigotimes_{i=1}^r \tau_i \otimes |\det{}_i|^{s_i}) \boxtimes \omega$, (normalized induction) of $G_{4n+1}(\mathbb{A})$. The standard realization of this representation is action by right translation on the space $V^{(1)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ given by

$$\left\{ \tilde{F} : G_{4n+1}(\mathbb{A}) \to V_\tau, \text{ smooth } \middle| \tilde{F}((\underline{g}, \alpha)h)(\underline{g}') = \tilde{F}(h)(\underline{g}'\underline{g})\omega(\alpha) \prod_{i=1}^r |\det g_i|^{s_i + n + \sum_{j=i+1}^r n_i - \sum_{j=1}^{i-1} n_i} \right\}.$$

(The factor

$$\prod_{i=1}^r |\det g_i|^{n + \sum_{j=i+1}^r n_i - \sum_{j=1}^{i-1} n_i}$$

is equal to $|\delta_P|^{\frac{1}{2}}$, and makes the induction normalized.) A second useful realization is action by right translation on

$$V^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega) = \left\{ f : G_{4n+1}(\mathbb{A}) \to \mathbb{C}, \middle| f(h) = \tilde{F}(h)(id), \tilde{F} \in V^{(1)}(\underline{s}, \underline{\tau}, \omega) \right\}.$$

(Here id denotes the identity element of $GL_{2n}(\mathbb{A})$.)

These representations fit together into a fiber bundle over $\mathbb{C}^r$. So a section of this bundle is a function f defined on $\mathbb{C}^r$ such that $f(\underline{s}) \in V^{(i)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ ($i = 1$ or 2) for each $\underline{s}$. We shall only require the use of flat, K-finite sections, which are defined as follows. Take $f_0 \in V^{(i)}(\underline{0}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ K-finite, and define $f(\underline{s})(h)$ by

$$f(\underline{s})(u(\underline{g}, \alpha)k) = f_0(u(\underline{g}, \alpha)k) \prod_{i=1}^r |\det g_i|^{s_i}$$

for $u \in U(\mathbb{A}), \underline{g} \in GL_{2n_1}(\mathbb{A}) \times \cdots \times GL_{2n_r}(\mathbb{A}), \alpha \in \mathbb{A}^\times, k \in K$. This is well defined. (I.e., although g_i is not uniquely determined in the decomposition, $|\det g_i|$ is. Cf. the definition of m_P on p.7 of [MW1].)

We begin with a flat K finite section of the bundle of representations realized on the spaces $V^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$.

Remark 4.0.3. *Clearly, the function f is determined by $f(\underline{s}^*)$ for any choice of base point $\underline{s}^*$. In particular, any function of f may be regarded as a function of $f_{\underline{s}^*} \in V^{(2)}(\underline{s}^*, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$, for any particular value of $\underline{s}^*$. We have exploited this fact with $\underline{s}^* = 0$ to streamline the definitions.* A posteriori *it will become clear that the point $\underline{s}^* = \underline{\mathbf{\frac{1}{2}}} := (\frac{1}{2}, \ldots, \frac{1}{2})$ is of particular importance, and we shall then switch to $\underline{s}^* = \underline{\mathbf{\frac{1}{2}}}$.*

For such f the sum

$$E(f)(g)(\underline{s}) := \sum_{\gamma \in P(F) \backslash G(F)} f(\underline{s})(\gamma g)$$

converges for all $\underline{s}$ such that $\mathrm{Re}(s_r), \mathrm{Re}(s_i - s_{i+1}), i = 1$ to $r-1$ are all sufficiently large. ([MW1], §II.1.5, pp.85-86). It has meromorphic continuation to $\mathbb{C}^r$ ([MW1] §IV.1.8(a), IV.1.9(c),p.140). These are our Eisenstein series. We collect some of their well-known properties in the following theorem.

Theorem 4.0.4. (1) *The function*

$$\prod_{i \neq j} (s_i + s_j - 1) \prod_{i=1}^r (s_i - \frac{1}{2}) E(f)(g)(\underline{s}) \tag{4.0.5}$$

is holomorphic at $s = \underline{\mathbf{\frac{1}{2}}}$. (More precisely, while $E(f)(g)$ may have singularities, there is a holomorphic function defined on an open neighborhood of $\underline{s} = \underline{\mathbf{\frac{1}{2}}}$ which agrees with (4.0.7) on the complement of the hyperplanes $s_i = \frac{1}{2}$, and $s_i + s_j = 1$.)

(2) *The function (4.0.5) remains holomorphic (in the same sense) when $s_i + s_j - 1$ is omitted, provided $\tau_i \not\cong \omega \otimes \tilde{\tau}_j$. It remains holomorphic when $s_i - \frac{1}{2}$ is omitted, provided τ_i is not ω^{-1}- orthogonal. Furthermore, each of these sufficient conditions is also necessary, in that the holomorphicity conclusion will fail, for some f and g, if any of the factors is omitted without the corresponding condition on $\underline{\tau}$ being satisfied. From this we deduce that if*

(4.0.6) *the representations $\tau_1, \ldots, \tau_r$ are all distinct and ω^{-1}-orthogonal,*

then the function

$$\prod_{i=1}^{r}(s_i - \frac{1}{2})E(f)(g)(\underline{s}) \tag{4.0.7}$$

is holomorphic at $s = \underline{\mathbf{\frac{1}{2}}}$ for all f, g and nonvanishing at $s = \underline{\mathbf{\frac{1}{2}}}$ for some f, g.

(3) *Let us now assume condition (4.0.6) holds, and regard f as a function of $f_{\underline{\mathbf{\frac{1}{2}}}} \in V^{(2)}(\underline{\mathbf{\frac{1}{2}}}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$. Let $E_{-1}(f_{\underline{\mathbf{\frac{1}{2}}}})(g)$ denote the value of the function (4.0.7) at $\underline{s} = \underline{\mathbf{\frac{1}{2}}}$ (defined by analytic continuation). Then $E_{-1}(f)$ is an L^2 function for all $f_{\underline{\mathbf{\frac{1}{2}}}} \in V^{(2)}(\underline{\mathbf{\frac{1}{2}}}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$.*

(4) *The function E_{-1} is an intertwining operator from $\mathrm{Ind}_{P(\mathbb{A})}^{G_{4n+1}(\mathbb{A})}(\bigotimes_{i=1}^r \tau_i \otimes |\det{}_i|^{\frac{1}{2}}) \boxtimes \omega$ into the space of L^2 automorphic forms.*

(5) *If $\mathcal{E}_{-1}(\tau, \omega)$ is the image of E_{-1}, and ψ_{LW} is the character of $U_{\max}$ given by $\psi_{LW}(u) = \psi_0(\sum_{i=1}^{2n-1} u_{i,i+1})$, then $(U_{\max}, \psi_{LW}) \notin \mathcal{U}^{\perp}(\mathcal{E}_{-1}(\tau, \omega))$.*

(6) *The space of functions $\mathcal{E}_{-1}(\tau, \omega)$ does not depend on the order chosen on the cuspidal representations $\tau_1, \ldots, \tau_r$. Thus it is well-defined as a function of the isobaric representation τ.*

Remark 4.0.8. *By induction in stages, the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})}(\bigotimes_{i=1}^r \tau_i \otimes |\det{}_i|^{\frac{1}{2}}) \boxtimes \omega$, which comes up in part (4) of the theorem can also be written as $\mathrm{Ind}_{P_{Sieg}(\mathbb{A})}^{G_{4n}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$, where $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ as before, and P_{Sieg} is the Siegel parabolic. (Cf. section 2.) Here, we also exploit the identification of the Levi M_{Sieg} of P_{Sieg} with $GL_{2n} \times GL_1$ fixed in 3.4.1.*

Proof. We first review the standard arguments by which the presence or absence of a singularity of an Eisenstein series reduces to the presence or absence of a singularity of a relative rank one intertwining operator. To do so, we recall the set

$$W(M) := \left\{ w \in W_{G_{4n+1}} \;\middle|\; \begin{array}{l} w \text{ is of minimal length in } w \cdot W_M \\ wMw^{-1} \text{ is a standard Levi of } G_{4n+1} \end{array} \right\}.$$

It will be convenient and harmless to treat the elements of $W(M)$ as though they were elements of $G_{4n+1}(F)$, rather than repeatedly choose representatives and remark the independence of the choice. For each $w \in W(M)$, $\underline{s} \in \mathbb{C}^r$, we define P^w to be the standard parabolic with Levi wMw^{-1}. For $\underline{s}$ such that s_r and $s_i - s_{i+1}, i = 1$ to $r-1$ are all sufficiently large, the integral

$$M(w, \underline{s})f(g) := \int_{U_{\max} \cap w\overline{U_{\max}}w^{-1}(F\backslash\mathbb{A})} f(\underline{s})(w^{-1}ug)\, du$$

converges ([MW1], II.1.6), defining an operator $M(w, \underline{s})$ from $V^{(2)}(\underline{s}, \bigotimes_{i=1}^{r} \tau_i \boxtimes \omega)$ to a space of functions which is easily verified to afford a realization of

$$\mathrm{Ind}_{P^w(\mathbb{A})}^{G_{4n+1}(\mathbb{A})} \left((\bigotimes_{i=1}^{r} \tau_i \otimes |\det{}_i|^{s_i}) \boxtimes \omega \right) \circ Ad(w^{-1}).$$

Here, $((\bigotimes_{i=1}^{r} \tau_i \otimes |\det{}_i|^{s_i}) \boxtimes \omega) \circ Ad(w^{-1})$, denotes the representation of wMw^{-1} obtained by composing the representation $(\bigotimes_{i=1}^{r} \tau_i \otimes |\det{}_i|^{s_i}) \boxtimes \omega)$ of M with conjugation by w^{-1}. We denote this latter space of functions by $V_w^{(2)}(\underline{s}, \bigotimes_{i=1}^{r} \tau_i \boxtimes \omega)$. Then $M(w, \underline{s})f(g)$ has meromorphic continuation to $\mathbb{C}^r$. (IV.1.8(b).)

It may be helpful also to review the sorts of singularities which Eisenstein series and intertwining operators have– lying along so-called "root hyperplanes." (cf. IV.1.6) We defer the notion of "*root* hyperplane" until later. For now, we allow arbitrary hyperplanes in $\mathbb{C}^r$, defined by equations of the form $l(\underline{s}) = c$, with l a linear functional $\mathbb{C}^r \to \mathbb{C}$ and c a constant. Then for any bounded open set $U \subset \mathbb{C}^r$, there exist a finite number of distinct hyperplanes $H_1, \dots, H_N$, which "carry" the singularities of the Eisenstein series and intertwining operators in U, in the following sense. For each i fix l_i, c_i such that $H_i = \{\underline{s} \in \mathbb{C}^r \mid l_i(\underline{s}) = c_i\}$. Then for each i there is a non-negative integer $\nu(H_i)$ such that

$$\prod_{i=1}^{N} (l_i(\underline{s}) - c_i)^{\nu(H_i)} E(f)(g)(\underline{s}) \tag{4.0.9}$$

continues to a function holomorphic on all of U. Covering $\mathbb{C}^r$ with bounded open sets and taking a union, we obtain an infinite, but *locally* finite, set of hyperplanes which carry all the singularities of the Eisenstein series and intertwining operators. The same hyperplane H will of course occur more than once. It is easily verified that the minimal exponent $\nu(H)$ appearing in (4.0.9) is the same each time. Thus we may speak of whether an Eisenstein series or intertwining operator does or does not have a pole along H, and of the order of the pole.

One may define "analytic/meromorphic continuation" for functions taking values in Fréchet spaces of locally L^2 functions and the like ([MW1] I.4.9, IV.1.3) of functions and operators. In this case, outside of the domain of convergence, one's functions are defined only up to L^2 equivalence. However, in view of I.4.10, one has a unique smooth representative for the class. For us it will be more convenient simply to adopt the convention that when we say the Eisenstein series has a pole along H, we mean for some f, g.

Now let us state the relationship between poles of Eisenstein series and intertwining operators, which we prove in an appendix.

Proposition 4.0.10. *For $f \in V^{(2)}(\underline{s}, \bigotimes_{i=1}^{r} \tau_i \boxtimes \omega)$, there exists $g \in G_{4n+1}(\mathbb{A})$ such that $E(f)(g)$ has a pole along H if and only if there exist $w \in W(M), g' \in G_{4n+1}(\mathbb{A})$ such that $M(w, \underline{s})f(g')$ has a pole along H.*

The same construction can be performed with the Levi M replaced by wMw^{-1}, yielding an operator

$$M_w(w', w \cdot \underline{s}) : V_w^{(2)}(\underline{s}, \bigotimes_{i=1}^{r} \tau_i \boxtimes \omega) \to V_{w'w}^{(2)}(\underline{s}, \bigotimes_{i=1}^{r} \tau_i \boxtimes \omega),$$

for each $w' \in W(wMw^{-1})$. Furthermore, one has for all f, g, the equality of meromorphic functions

$$M_w(w', w \cdot \underline{s}) \circ M(w, \underline{s})f(g) = M(w'w, \underline{s})f(g)$$

([MW1], II.1.6, IV.4.1). (For now, the reader may think of "$w \cdot \underline{s}$" simply as a notational contrivance. We shall give it a precise meaning below.)

Next we wish to describe the decomposition of $w \in W(M)$ as a product of elementary symmetries, as in [MW1] I.1.8. The lattice $X(Z_M)$ of rational characters of the center of M has a unique basis $\{e_0, \varepsilon_1, \dots, \varepsilon_r\}$, with the property that for each $i = 1, \dots m$, there exists $j \in \{1, \dots, r\}$ such that the restriction of e_i as in 3.4 to Z_M is ε_j. The set of restrictions of positive roots of G_{4n+1} to Z_M is

$$\{0\} \cup \{\varepsilon_i - \varepsilon_j : 1 \le i < j \le r\} \cup \{\varepsilon_i : 1 \le i \le r\} \cup \{\varepsilon_i + \varepsilon_j : 1 \le i < j \le r\} \cup \{2\varepsilon_i : 1 \le i \le r\}.$$

We denote the set obtained by excluding zero by $\Phi^+(Z_M)$. For $\alpha \in \Phi^+(Z_M)$, and $w \in W(M)$, one may say "$w\alpha > 0$" or "$w\alpha < 0$" without ambiguity. We say an element of $\Phi^+(Z_M)$ is indivisible if it is not of the form $2\varepsilon_i$.

Each element $w \in W(M)$ can be decomposed as a product $s_{\alpha_1} \dots s_{\alpha_\ell}$ of elementary symmetries as in [MW1] I.1.8. The element s_{α_ℓ} will be in $W(M)$, while $s_{\alpha_{\ell-1}}$ will be in $W(s_{\alpha_\ell} M s_{\alpha_\ell}^{-1})$ and so on. Each is labeled with the unique indivisible restricted root (for the operative Levi) which it reverses. That is $\{\alpha \in \Phi^+(Z_M) : s_{\alpha_\ell}\alpha < 0\} = \{\alpha_\ell\}$, or $\{\alpha_\ell, 2\alpha_\ell\}$ and in the latter case $\alpha_\ell = \varepsilon_r$. (Cf. [MW1] I.1.8.)

Let $w = s_{\alpha_1} \dots s_{\alpha_\ell}$ be a minimal-length decomposition into elementary symmetries, and put $w_i = s_{\alpha_{i+1}} \dots s_{\alpha_\ell}$. Then

$$\{\alpha \in \Phi^+(Z_M),\ \text{indivisible } \mid w\alpha < 0\} = \{w_i^{-1}\alpha_i \mid 1 \le i \le \ell\}$$

and ℓ is the cardinality of this set (i.e., there is no repetition). Combining this discussion with that of the previous paragraphs, we obtain a decomposition of $M(w, \underline{s})$ as a composite of intertwining operators $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})$, each corresponding naturally to one of the elements of $\{\alpha \in \Phi^+(Z_M),\ \text{indivisible } \mid w\alpha < 0\}$.

Let $\det_i$ denote the rational character $(\underline{g}, \alpha) \mapsto \det g_i$ of M. Then $\{e_0, \det_1, \dots, \det_r\}$ is a basis for the lattice $X(M)$ of rational characters of M. Here, the character e_0 of T introduced in 3.4 has been identified with a character of M as in 3.4.1. Let $\{e_0^*, \det_1^*, \dots, \det_r^*\}$ be the dual basis of the dual lattice. Again, e_0^* is the same as in 3.4. Elements of $X(M)$ may be paired with elements of $X^\vee(T)$ defining a projection from $X^\vee(T)$ onto the dual lattice. For each $i = 1, \dots m$, there exists unique $j \in \{1, \dots, r\}$ such that e_i^* maps to $\det_j^*$. If α is a root, then the projection of the coroot $\alpha^\vee$ to the dual lattice of $X(M)$ depends only on the restriction of α to Z_M, and the correspondence is as follows:

$$\begin{gathered} 0 \leftrightarrow 0, \\ \varepsilon_i - \varepsilon_j \leftrightarrow \det{}_i^* - \det{}_j^*, \\ \varepsilon_i + \varepsilon_j \leftrightarrow \det{}_i^* + \det{}_j^* - e_0^*, \\ \varepsilon_i \leftrightarrow 2\det{}_i^* - e_0^* \\ 2\varepsilon_i \leftrightarrow 2\det{}_i^* - e_0^*. \end{gathered}$$

We denote the element corresponding to $\alpha \in \Phi^+(Z_M)$ by $\alpha^\vee$ (in agreement with [MW1], I.1.11).

We may identify $\underline{s} \in \mathbb{C}^r$ with

$$\sum_{i=1}^r \det{}_i \otimes s_i \in X(M) \otimes_{\mathbb{Z}} \mathbb{C}.$$

This is compatible with [MW1], I.1.4. Restriction of functions gives a natural injective map $X(M) \to X(T)$, and hence $X(M) \otimes_{\mathbb{Z}} \mathbb{C} \to X(T) \otimes_{\mathbb{Z}} \mathbb{C}$, which we use to identify the first space with a subspace of the second. This gives the notation $w \cdot \underline{s}$ a precise meaning, as an element of $X(wMw^{-1}) \otimes_{\mathbb{Z}} \mathbb{C}$, which is compatible with the usage above. In addition, it gives a "meaning" to the set

$$\{s_i - s_j\} \cup \{s_i + s_j\} \cup \{2s_i\},$$

of linear functionals on $\mathbb{C}^r$, identifying each with an element of $\Phi^+(Z_M)$. Formally,

Definition 4.0.11. *A root hyperplane (relative to the Levi M) is a hyperplane of the form*

$$H = \{s \in \mathbb{C}^r \mid \langle \alpha^\vee, \underline{s} \rangle = c\}$$

for some $\alpha \in \Phi^+(Z_M)$ which is indivisible, and some $c \in \mathbb{C}$. We say that the hyperplane H is associated to the root α, which is uniquely determined.

The next main statement is

Lemma 4.0.12. *Let $w = s_{\alpha_1} \dots s_{\alpha_\ell}$ be any decomposition of minimal length, and for each i let $w_i = s_{\alpha_{i+1}} \dots s_{\alpha_\ell}$. Then the set of poles of $M(w, \underline{s})$ is the disjoint union of the sets of poles of the operators $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})$. A pole of $M(w, \underline{s})$ comes from $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})$ if and only if it is associated to $w_i^{-1}\alpha_i$. Furthermore, if $\{\underline{s} \in \mathbb{C}^r | \langle \alpha^\vee, \underline{s} \rangle = c\}$ is a pole of $M(w, \underline{s})$, then $c \neq 0$.*

We now prove (1). A root hyperplane passing through $\underline{\mathbf{\frac{1}{2}}}$ is defined by an equation of one of three forms: $s_i = \frac{1}{2}, s_i + s_j = 1$, or $s_i - s_j = 0$. The third kind can not support singularities of the Eisenstein series. The first two can, but by [MW1]IV.1.11 (c), they will be without multiplicity, and so the factor of

$$\prod_{i \neq j} (s_i + s_j - 1) \prod_{i=1}^{r} (s_i - \frac{1}{2})$$

will take care of them.

The operators corresponding to elementary symmetries are called relative rank one because they could be defined without reference G_{4n+1}, considering M instead as a maximal Levi of another Levi subgroup M_α of G_{4n+1}, having semisimple rank one greater than that of M. Furthermore, in a suitable sense, the relative rank one operator only "lives on one component of M_α," which will allow us to deduce the general case of (2) from the case $r = 1$ and a similar fact about intertwining operators on GL_n. Let us make this more precise.

Fix $\alpha \in \Phi^+(Z_M)$. There is a minimal Levi subgroup M_α of G_{4n+1} containing M such that α is the restriction of a root of M_α. (It is standard iff α is the restriction of a simple root.) Fix $w \in W(M)$ such that $w\alpha < 0$, and a decomposition $w = s_{\alpha_1} \dots s_{\alpha_\ell}$ of w as into elementary symmetries, which is of minimal length. For some unique i, we have $\alpha = w_i^{-1}\alpha_i$, where w_i is as above. Then $w_i M_\alpha w_i^{-1}$ is a standard Levi of G_{4n+1}. Different choices of decomposition give different (even conjugate) embeddings of the same reductive group into G_{4n+1} as a standard Levi.

If $\alpha = \varepsilon_j - \varepsilon_k$, or $\varepsilon_j + \varepsilon_k$, then M_{α_i} is isomorphic to $GL_{2(n_j+n_k)} \times \prod_{l \neq j,k} GL_{2n_l} \times GL_1$. while if $\alpha = \varepsilon_j$, it is isomorphic to $G_{4n_j+1} \times \prod_{k \neq j} GL_{2n_k}$. Let G' denote $GL_{2(n_j+n_k)}$ or G_{4n_j+1} as appropriate and let ι be a choice of isomorphism with the "new" factor. Then $\iota^{-1}(\iota(G') \cap P^{w_i})$ is a maximal parabolic subgroup $P' = M'U'$ of G', and $\sigma := (\bigotimes_{i=1}^r \tau \otimes \omega) \circ Ad(w_i) \circ \iota$, is an irreducible unitary cuspidal automorphic representation of $M'(\mathbb{A})$. The map ι also induces a linear projection

$$\iota_* : X(w_i M w_i^{-1}) \otimes_{\mathbb{Z}} \mathbb{C} \to X(M') \otimes_{\mathbb{Z}} \mathbb{C}.$$

(Recall that we have agreed to think of $w_i \cdot \underline{s}$ as an element of the former space.)

Following, [MW1] I.1.4, define m^μ for $m \in M'(\mathbb{A})$ and μ in $X(M') \otimes_{\mathbb{Z}} \mathbb{C}$, by stipulating that $m^\mu = |\chi(m)|^s$ if $\mu = \chi \otimes s$ and $m^{\mu_1+\mu_2} = m^{\mu_1} m^{\mu_2}$.

The set $W'_G(M')$, defined analogously to $W(M)$ above, contains a unique nontrivial element. It is the elementary symmetry s_β associated to the restriction to Z'_M of any of the unique simple root of G' which is not a root of M'. The map ι identifies s_β with s_{α_i}.

For $\mu \in X(M') \otimes_{\mathbb{Z}} \mathbb{C}$, let $V^{(1)}(\mu, \sigma)$ denote

$$\{h : G'(\mathbb{A}) \to V_\sigma, \text{ smooth} \mid h(mg')(m') = h'(g')(m'm)m^{\mu+\rho_{P'}} \quad m, m' \in M'(\mathbb{A}), g' \in G'(\mathbb{A})\},$$

$$V^{(2)}(\mu, \sigma) = \{h : G'(\mathbb{A}) \to \mathbb{C}, \text{ smooth} | h(g')(e) \in V^{(1)}(\mu, \sigma)\}.$$

There is a standard intertwining operator $M(s_\beta, \mu) : V^{(2)}(\mu, \sigma) \to V^{(2)}_{s_\beta}(\mu, \sigma)$. One has the identity

$$M_{w_{i-1}}(s_{\alpha_i}, w_i \cdot \underline{s}) f(\iota(h)g) = M(s_\beta, \mu) f(\iota(h)g).$$

That is, if p_g denotes the map

$$V^{(2)}_{w_i}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega) \to V^{(2)}(\mu, \sigma)$$

corresponding to evaluation at $\iota(h)g$ for a fixed g, then, for all g, the following diagram commutes:

$$\begin{tikzcd}
V^{(2)}_{w_i}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega) \arrow[r, "M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})"] \arrow[d, "p_g"'] & V^{(2)}_{w_{i-1}}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega) \arrow[d, "p_g"'] \\
V^{(2)}(\iota_*(w_i \cdot \underline{s} + \rho_{P_{\alpha_i}}), \sigma) \arrow[r, "M(s_\beta, \iota_*(w_i \cdot \underline{s} + \rho_{P_{\alpha_i}}))"] & V^{(2)}_{s_\beta}(\iota_*(w_i \cdot \underline{s} + \rho_{P_{\alpha_i}}), \sigma).
\end{tikzcd}$$

Hence $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})$ has a pole along a root hyperplane associated to α iff $M(\iota_*(w_i \cdot s + \rho_{P_{\alpha_i}}), \sigma)$ does.

Since the set of poles of $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})$ is equal to the set of poles of $M(w, \underline{s})$ along hyperplanes associated to α, it is independent of the choice of decomposition $w = s_{\alpha_1} \dots s_{\alpha_\ell}$. Hence, for each $\alpha \in \Phi^+(Z_M)$, we may use a decomposition tailored to that α.

First suppose $\alpha = \varepsilon_j - \varepsilon_k$. One may choose a decomposition so that w_i corresponds to the permutation matrix in GL_{2n} (identified with a subgroup of the Siegel Levi) which moves the jth block of M up so that it is immediately after the ith, and otherwise preserves order. It is then easily verified that $\sigma = \tau_i \otimes \tau_j$ and

$$\begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}^{\iota_*(w_i \cdot s) + \rho_{P_{\alpha_i}}} = |\det h_1|^{s_i + \kappa} |\det h_2|^{s_j + \kappa},$$

where $\kappa = \sum_{k > i, k \neq j} n_k - \sum_{k < i} n_k + n$.

Next suppose $\alpha = 2\varepsilon_j$. Then we choose a decomposition so that w_i is in the Weyl group of GL_{2n}, and moves the jth block to be last, otherwise preserving order. Then one easily verifies that σ is the representation $\tau_j \boxtimes \omega$ of the Siegel Levi of G_{4n_j}, and that, for (g', α) in the Siegel Levi of G_{4n_j},

$$(g', \alpha)^{\iota_*(w_i \cdot \underline{s} + \rho_{P_{\alpha_i}})} = |\det g'|^{s_j}.$$

Finally, suppose $\alpha = \varepsilon_j + \varepsilon_k$. Then we choose a decomposition so that w_i that projects to a permutation matrix in SO_{4n+1} of the form

$$\begin{pmatrix} I & & & & \\ & & & I & \\ & & I & & \\ & I & & & \\ & & & & 1 \end{pmatrix},$$

with the off-diagonal blocks being $2n_j \times 2n_j$, and the first block being $\sum_{k=1}^i 2n_k$. We deduce from Corollary 3.5.6 that $\sigma = \tau_i \otimes (\tilde{\tau}_j \otimes \omega)$, and from Lemma 3.5.5 that

$$\begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}^{\iota_*(w_i \cdot s) + \rho_{P_{\alpha_i}}} = |\det h_1|^{s_i + \kappa} |\det h_2|^{-s_j + \kappa},$$

where κ is as before.

Thus (2) follows from

Proposition 4.0.13. *Let w denote the unique nontrivial element of $W(M)$, in the case when M is the Levi of the Siegel parabolic of G_{2m+1}. Let τ be a cuspidal representation of GL_m. Then $M(w,s)f(g)$ has a pole at $s=\frac{1}{2}$ for some $f \in \mathrm{Ind}_{P(\mathbb{A})}^{G_{2m+1}(\mathbb{A})}(\tau \otimes |\det|^s) \boxtimes \omega$, and $g \in G_{2m+1}(\mathbb{A})$ if and only if τ is ω^{-1}-orthogonal*

Remarks 4.0.14. *Of course we are only interested in the case $m=2n$. Furthermore, since we assume ω is not the square of another Hecke character, it follows that τ can be ω^{-1}-orthogonal only if m is even. However, the proof of this proposition is "blind to" the parity of m.*

Proposition 4.0.15. *Let $P=MU$ be a maximal standard parabolic of GL_n such that $M \cong GL_k \times GL_{n-k}$. Let f be an element of $\mathrm{Ind}_{P(\mathbb{A})}^{GL_n(\mathbb{A})}(\tau_1 \otimes |\det|^{s_1}) \bigotimes (\tau_2 \otimes |\det|^{s_2})$. Let w be the unique nontrivial element of $W(M)$. Then $M(w,\underline{s})f(g)$ is singular along the hyperplane $s_1-s_2=1$ for some f,g iff $n=2k$ and $\tau_2 \cong \tau_1$.*

We defer the proofs to the appendix.

Now, we assume (4.0.6) holds and prove the remaining part of the theorem. Let $N(\underline{s})=\prod_{i=1}^r(s_i-\frac{1}{2})$.

Item (3) follows from [MW1] I.4.11. The constant term of $E(f)$ along a parabolic $P'=M'U'$ has nontrivial cuspidal component iff M' is conjugate to M. (IV.1.9 (b)(ii)). For such P' it is equal to

$$\sum_{w\in W(M),\, wMw^{-1}=M'} M(w,\underline{s})f(g).$$

Take $w \in W(M)$, such that $wMw^{-1}=M'$. If $w\cdot\varepsilon_i>0$ for some i, then $M(w,\underline{s})f(g)$ does not have a pole at $s_i-\frac{1}{2}$, and hence $N(\underline{s})M(w,\underline{s})f(g)$ vanishes at $\underline{\mathbf{\tfrac{1}{2}}}$. On the other hand, if $w\cdot\varepsilon_i<0$ for all i, then $M(w,\underline{s})f(g)$ satisfies the criterion of I.4.11.

It follows from [MW1] IV.1.9 (b)(i) applied to $N(\underline{s})E(f)$ (which is valid by IV.1.9 (d)) that the residue is an automorphic form. To complete the proof of (4), let $\rho(g)$ denote right translation. It is clear that for values of s in the domain of convergence, $N(\underline{s})E(\rho(g)f)(\underline{s})=N(\underline{s})\rho(g)(E(f)(\underline{s}))$. By uniqueness of analytic continuation, the equality also holds at values of s where both sides are defined by analytic continuation, including $\underline{\mathbf{\tfrac{1}{2}}}$. The action of the Lie algebra at the infinite places is handled similarly.

Next we consider the constant term of $E(f)$ along the Siegel parabolic. By [MW1] II.1.7(ii) it may be expressed in terms of GL_{2n} Eisenstein series, formed using the functions $M(w,\underline{s})f$, corresponding to those $w\in W(M)$ such that $w^{-1}(e_i-e_{i+1})>0$ for all i. (Note: we proved in Lemma 3.5.3 that wMw^{-1} is contained in the Siegel Levi for every $w\in W(M)$.) When we pass to $E_{-1}(f)$, the term corresponding to w only survives if $w\cdot\varepsilon_i<0$ for all i. This condition picks out a unique element, w_0. It is the shortest element of $W_{GL_{2n}}\cdot w_\ell \cdot W_{GL_{2n}}$, where w_ℓ is the longest element of $W_{G_{4n+1}}$, and we have identified GL_{2n} with a subgroup of the Siegel Levi as usual. Via corollary 3.5.6 one finds that

$$(\bigotimes_{i=1}^r \tau_i \boxtimes \omega)\circ Ad(w_0) = (\bigotimes_{i=1}^r(\tilde{\tau}_{r+1-i}\otimes\omega)\boxtimes\omega) = (\bigotimes_{i=1}^r \tau_{r+1-i}\boxtimes\omega).$$

For $f\in V^{(2)}(\bigotimes_{i=1}^r \tau_i\boxtimes\omega, \underline{\mathbf{\tfrac{1}{2}}})$, $M(w_0,\underline{\mathbf{\tfrac{1}{2}}})f|_{GL_{2n}(\mathbb{A})}$ is an element of the analogue of $V^{(2)}(\bigotimes_{i=1}^r \tau_i\boxtimes\omega,\underline{s})$, for the induced representation

$$\mathrm{Ind}_{\bar{P}^0(\mathbb{A})}^{GL_{2n}(\mathbb{A})}(\bigotimes \tau_{r+1-i}\otimes|\det{}_i|^{n-\frac{1}{2}}) = |\det|^{n-\frac{1}{2}}\otimes\tau$$

of GL_{2n}. Here $\bar{P}^0=GL_{2n}\cap P^{w_0}$, and $\tau=\tau_1\boxplus\cdots\boxplus\tau_r$. Furthermore, since this representation is irreducible, it may be regarded as an arbitrary element. Also, we may regard this representation as induced from $\tau_1,\ldots,\tau_r$ in the usual order. Let $\bar{P}$ denote the relevant parabolic of GL_{2n}.

The representation τ sits inside a fiber bundle of induced representations $\mathrm{Ind}_{\bar{P}(\mathbb{A})}^{GL_{2n}(\mathbb{A})}(\bigotimes_{i=1}^r \tau_i \otimes |\det{}_i|^{s_i})$. For a flat, K-finite section f let $E^{GL_{2n}}(f)(g)(\underline{s})$ be the GL_{2n} Eisenstein series defined by

$$\sum_{\bar{P}(F)\backslash GL_{2n}(F)} f(\underline{s})(\gamma g)$$

when $s_i - s_{i+1}$ is sufficiently large for each i, and by meromorphic continuation elsewhere.

Let $U_{\max}^{GL_{2n}}$ denote the usual maximal unipotent subgroup of GL_{2n}, consisting of all upper triangular unipotent matrices. Let $\psi_W(u) = \psi_0(u_{1,2} + \cdots + u_{m-1,m})$ be the usual generic character.

To complete the proof of (5), we must prove that

$$\int_{U_{\max}^{GL_{2n}}(F\backslash\mathbb{A})} E^{GL_{2n}}(f)(ug)(\underline{0})\psi_W(u)\, du \neq 0 \tag{4.0.16}$$

for some $f \in \mathrm{Ind}_{\bar{P}(\mathbb{A})}^{GL_{2n}(\mathbb{A})} \bigotimes_{i=1}^r \tau_{r+1-i}, g \in GL_{2n}(\mathbb{A})$, i.e., that the space of GL_{2n} Eisenstein series $E^{GL_{2n}}(f)$ is globally ψ_W-generic. Granted this, (5) follows from [MW1]II.1.7(ii) and the discussion just above.

The following proposition follows from work of Shahidi.

Proposition 4.0.17.

$$\int_{U_{\max}^{GL_{2n}}(F\backslash\mathbb{A})} E^{GL_{2n}}(f)(ug)(\underline{s})\psi_W(u)\, du = \prod_{v\in S} W_v(g_v) \cdot \prod_{v\notin S} W_v^\circ(g_v) \cdot \prod_{i<j} L^S(s_i - s_j + 1, \tau_i \times \tilde{\tau}_j)^{-1},$$

where, for each v, W_v is a Whittaker function in the $\psi_{W,v}$-Whittaker model of $\mathrm{Ind}_{\bar{P}(F_v)}^{GL_{2n}(F_v)}(\bigotimes_{i=1}^r \tau_{i,v} \otimes |\det{}_i|_v^{s_i})$*, S is a finite set of places, depending on f, outside of which τ_v is unramified and W_v° is the normalized spherical vector in the the $\psi_{W,v}$-Whittaker model of* $\mathrm{Ind}_{\bar{P}(F_v)}^{GL_{2n}(F_v)}(\bigotimes_{i=1}^r \tau_{i,v} \otimes |\det{}_i|_v^{s_i})$*. A flat, K-finite section f may be chosen so that, for all $v \in S$, the function W_v is not identically zero at $\underline{s} = \underline{0}$.*

We briefly review the steps of the proof in the appendix.

It follows from [Ja-Sh3] Propositions 3.3 and 3.6 that the product of partial L functions appearing in Proposition 4.0.17 does not have a pole at $\underline{s} = \underline{0}$ provided the representations $\tau_1, \dots, \tau_r$ are distinct. This completes the proof of (5).

Finally, (6) follows from the functional equation of the Eisenstein series ([MW1]IV.1.10(a)), and the fact that τ is equal to an irreducible full induced representation (as opposed to a constituent of a reducible one). □

5. Main Results

5.1. **Descent Construction.** In this section, we shall make use of remark 4.0.8, and regard $\mathcal{E}_{-1}(\tau, \omega)$ as affording an automorphic realization of the representation induced from the representation $\tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$ of the Siegel Levi. Thus we may dispense with the smaller Levi denoted by P in the previous section, and in this section we denote the Siegel parabolic more briefly by $P = MU$.

Next we describe certain unipotent periods of G_{2m} which play a key role in the argument. For $1 \leq \ell < m$, let N_ℓ be the subgroup of $U_{\max}$ defined by $u_{ij} = 0$ for $i > \ell$. (Recall that according to the convention above, this refers only to those i, j with $i < j \leq m - i$.) This is the unipotent radical of a standard parabolic Q_ℓ having Levi L_ℓ isomorphic to $GL_1^\ell \times G_{2m-2\ell}$.

Let ϑ be a character of N_ℓ then we may define

$$DC^\ell(\tau, \omega, \vartheta) = FC^\vartheta \mathcal{E}_{-1}(\tau, \omega).$$

Theorem 5.1.1. *Let ω be a Hecke character. Let $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ be an isobaric sum of ω^{-1}-orthogonal irreducible cuspidal automorphic representations $\tau_1, \dots, \tau_r$, of $GL_{2n_1}(\mathbb{A}), \dots GL_{2n_r}(\mathbb{A})$, respectively. If $\ell > n$, and ϑ is in general position, then*

$$DC^{\ell}(\tau, \omega, \vartheta) = \{0\}.$$

Proof. By Theorem 4.0.4, (3) the representation $\mathcal{E}_{-1}(\tau, \omega)$ decomposes discretely. Let $\pi \cong \otimes'_v \pi_v$ be one of the irreducible components, and $p_\pi : \mathcal{E}_{-1}(\tau, \omega) \to \pi$ the natural projection.

Fix a place v_0 such which τ_{v_0} and π_{v_0} are unramified. For any $\xi^{v_0} \in \otimes'_{v \neq v_0} Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|_v^{\frac{1}{2}} \boxtimes \omega_v$ we define a map

$$i_{\xi^{v_0}} : Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0} \to Ind_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$$

by $i_{\xi^{v_0}}(\xi_v) = \iota(\xi_{v_0} \otimes \xi^{v_0})$, where ι is an isomorphism of the restricted product $\otimes'_v Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|_v^{\frac{1}{2}} \boxtimes \omega_v$ with the global induced representation $Ind_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$. Clearly

$$\mathcal{E}_{-1}(\tau, \omega) = E_{-1} \circ \iota(\otimes'_v Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|_v^{\frac{1}{2}} \boxtimes \omega_v).$$

For any decomposable vector $\xi = \xi_{v_0} \otimes \xi^{v_0}$,

$$p_\pi \circ E_{-1} \circ \iota(\xi) = p_\pi \circ E_{-1} \circ i_{\xi^{v_0}}(\xi_{v_0}).$$

Thus, π_{v_0} is a quotient of $Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0}$, and hence (since we took v_0 such that π_{v_0} is unramified) it is isomorphic to the unramified constituent ${}^{un}Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0}$.

Denote the isomorphism of π with $\otimes'_v \pi_v$ by the same symbol ι. This time, fix $\zeta^{v_0} \in \otimes'_{v \neq v_0} \pi_v$, and define $i_{\zeta^{v_0}} :{}^{un} Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0} \to \pi$. It follows easily from the definitions that

$$FC^{\vartheta} \circ i_{\zeta^{v_0}}$$

factors through the Jacquet module $\mathcal{J}_{N_\ell, \vartheta}({}^{un}Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0})$. Propositions 7.0.16 and 7.0.18 below each show that this Jacquet module vanishes at approximately half of all places. Inasmuch as vanishing at a single place would suffice to prove global vanishing, the result follows. $\square$

A general character of N_ℓ is of the form

$$\psi_0(c_1 u_{1,2} + \cdots + c_{\ell-1} u_{\ell-1,\ell} + d_1 u_{\ell,\ell+1} + \cdots + d_{4n+1-2\ell} u_{\ell,4n+1-\ell}). \tag{5.1.2}$$

As described in section 3.8, the Levi L_ℓ acts on the space of characters of $N_\ell(F \backslash \mathbb{A})$. In order to define embeddings of the various forms of G_{2n} into G_{4n+1}, we need to make this more explicit.

First, we fix a specific isomorphism of $GL_1^\ell \times G_{4n-2\ell+1}$ with L_ℓ as follows. As in section 3.4, let $e_0, \dots, e_{2n}$ and $e_0^*, \dots, e_{2n}^*$ denote the $\mathbb{Z}$-bases of $X(T(G_{4n+1}))$ and $X^\vee(T(G_{4n+1}))$, respectively. Let $\hat{e}_0, \dots \hat{e}_{2n-\ell}$, and $\hat{e}_0^*, \dots, \hat{e}_{2n-\ell}^*$ denote the analogues for $G_{4n-2\ell+1}$. We identify $(\alpha_1, \dots, \alpha_\ell, \prod_{i=1}^{2n-\ell} \hat{e}_i^*(t_i)) \in GL_1^\ell \times T(G_{4n-2\ell+1})$ with $\prod_{i=1}^{\ell} e_i^*(\alpha_i) \cdot \prod_{i=1}^{2n-\ell} e_{i+\ell}^*(t_i) \in T(G_{4n+1})$. In addition, we require that $g \in G_{4n-2\ell+1}$ be identified with an element of G_{4n+1} which projects to

$$\begin{pmatrix} I_\ell & & \\ & \mathrm{pr}(g) & \\ & & I_\ell \end{pmatrix} \in SO_{4n+1}.$$

Together, these requirements determine a unique identification.

Let $\underline{d}$ denote the column vector ${}^t(d_1, \dots, d_{4n+1-2\ell})$. Suppose $\vartheta(u)$ is the character of N_ℓ given by (5.1.2), and, for $h \in L_\ell$, let

(5.1.3) $h \cdot \vartheta(u) = \vartheta(h^{-1}uh) = \psi_0({}^h c_1 u_{1,2} + \dots + {}^h c_{\ell-1} u_{\ell-1,\ell} + {}^h d_1 u_{\ell,\ell+1} + \dots + {}^h d_{4n+1-2\ell} u_{\ell,4n+1-\ell})$.

This is an action of L_ℓ on the space of characters, and it is easily verified that for h identified with $(\alpha_1, \dots, \alpha_\ell, g)$, with $\alpha_1, \dots, \alpha_\ell \in GL_1(F)$ and $g \in G_{4n-2\ell+1}(F)$, we have

$$ {}^h c_i = \frac{\alpha_{i+1}}{\alpha_i} \cdot c_i, \ i = 1, \dots, \ell - 1, \quad \text{and} \quad {}^h \underline{d} = \alpha_\ell^{-1} \cdot \mathrm{pr}(g) \cdot \underline{d}. $$

The above discussion amounts to an identification of the action of $L_\ell(F)$ on the space of characters of $N_\ell(F \backslash \mathbb{A})$ with a certain rational representation of L_ℓ defined over F, consisting of the direct sum of $\ell - 1$ one dimensional representations and a $(4n - 2\ell + 1)$-dimensional representation on which the $G_{4n-2\ell+1}$ factor in L_ℓ acts via its "standard" representation. We may consider this rational representation over any field. Over an algebraically closed field there is an open orbit, which consists of all those elements such that $c_i \neq 0$ for all i and ${}^t\underline{d}J\underline{d} \neq 0$. Here, J is defined as in 3.1. Over a general field two such elements are in the same F-orbit iff the two values of ${}^t\underline{d}J\underline{d}$ are in the same square class. Thus, this square class is an important invariant of the character ϑ.

Definition 5.1.4. *Let ϑ be the character of $N_\ell(F \backslash \mathbb{A})$ given by*

$$ \vartheta(u) = \psi_0(c_1 u_{1,2} + \dots + c_{\ell-1} u_{\ell-1,\ell} + d_1 u_{\ell,\ell+1} + \dots + d_{4n+1-2\ell} u_{\ell,4n+1-\ell}). $$

We denote the square class of ${}^t\underline{d}J\underline{d}$ by $\mathrm{Invt}(\vartheta)$. *We say that ϑ is in general position if $c_i \neq 0$ for $1 \le i \le \ell - 1$ and* $\mathrm{Invt}(\vartheta) \neq 0$. *We denote the square class consisting of the nonzero squares by $\square$.*

Clearly, a nonzero square class in F may also be used to determine a quasi-split form of G_{2n}. Indeed, the natural datum for determining a quasi-split group with G such that ${}^L G^0 = GSO_{2n}(\mathbb{C})$ is a homomorphism $\mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}(GSO_{2n}(\mathbb{C}))/\mathrm{Inn}(GSO_{2n}(\mathbb{C}))$, which has two elements. Such homomorphisms are in one-to-one correspondence with quadratic characters by class field theory, and this has been exploited in defining G_{2n}^χ above. But they are also in natural one-to-one correspondence with square classes in $F^\times$, and this parametrization will be more convenient for the next part of the discussion.

Definition 5.1.5. *Let $\mathbf{a}$ be a square class in $F^\times$. Let $F(\sqrt{\mathbf{a}})$ denote the smallest extension of F in which the elements of $\mathbf{a}$ are squares, and for $a \in F^\times$, let $F(\sqrt{a}) = F(\sqrt{\{a\}})$. Let $G_{2n}^{\mathbf{a}}$ denote the quasi-split form of G_{2n} such that the associated map* $\mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}(GSO_{2n}(\mathbb{C}))/\mathrm{Inn}(GSO_{2n}(\mathbb{C}))$ *factors through* $\mathrm{Gal}(F(\sqrt{\mathbf{a}})/F)$.

Remark 5.1.6. *Of course, if $\mathbf{a} = \square$, then $F(\sqrt{\mathbf{a}}) = F$ and $G_{2n}^{\mathbf{a}}$ is just the split group G_{2n}.*

Lemma 5.1.7.
(1) *If ϑ is a character of N_ℓ in general position, then the stabilizer L_ℓ^ϑ (cf. M^ϑ in definition 3.8.1) has two connected components*
(2) *The identity component $(L_\ell^\vartheta)^0$ is isomorphic over F to $G_{4n-2\ell}^{\mathrm{Invt}(\vartheta)}$.*

Proof. Identify $(\alpha_1, \dots, \alpha_\ell, g) \in GL_1^\ell \times G_{4n-2\ell+1}$ with an element of L_ℓ as above.

The identity component of L_ℓ^ϑ consists of those $(\alpha_1, \dots, \alpha_\ell, g)$ such that $\alpha_i = 1$ for all i and g fixes the vector in the standard representation obtained from ϑ. The other consists of those such that $\alpha_i = -1$ for all i, and g maps the vector in the standard representation obtained from ϑ to its negative (which is the only scalar multiple of the same length). This proves (1).

We turn to (2). First suppose $\mathrm{Invt}(\vartheta) = \square$. It suffices to consider the specific character ψ_ℓ defined by

$$ \psi_\ell(u) = \psi_0(u_{12} + \dots + u_{\ell-1,\ell} + u_{\ell,2n+1}). $$

For this character, the column vector $\underline{d}$ is $v_1 :=^t (0, \dots, 0, 1, 0, \dots, 0)$. It is easily checked that the stabilizer of this point in $SO_{4n-2\ell+1}$ is isomorphic to the split form of $SO_{4n-2\ell}$. In addition, the stabilizer in $G_{4n-2\ell+1}$ contains a split torus of rank $2n-\ell+1$, and hence is a split group. An element of $U_{\max}$ fixes v_1, if and only if it satisfies $u_{i,2n-\ell+1}=0$ for $i=1$ to $2n-\ell$. From this we easily compute the based root datum of the stabilizer of v_1 and find that it is the same as that of $G_{4n-2\ell}$.

To complete the proof of (2), let a be a non-square in $F^\times$, and let $v_a={}^t(0,\dots,0,1,0,\frac{a}{2},0,\dots,0)\in F^{4n-2\ell+1}$ (nonzero entries in positions $2n-\ell$ and $2n-\ell+2$ only). Let ψ_ℓ^a be the character of $N_\ell(F\backslash\mathbb{A})$ corresponding to $c_i=1\forall i$ and $\underline{d}=v_a$. The stabilizers of ψ_ℓ^a and ψ_ℓ are conjugate over the quadratic extension E of F obtained by adjoining a square root of a. Indeed, let $\sqrt{a}$ be an element of E such that $(\sqrt{a})^2=a$. Suppose

$$\mathrm{pr}(h_a)=\begin{pmatrix}\sqrt{a}^{-1}I_{2n-1}&&\\&M_{\sqrt{a}}&\\&&\sqrt{a}I_{2n-1}\end{pmatrix},\quad\text{where}\quad M_{\sqrt{a}}=\begin{pmatrix}-\frac{1}{2\sqrt{a}}&\sqrt{a}^{-1}&\sqrt{a}^{-1}\\ \frac{1}{2}&0&1\\ \frac{\sqrt{a}}{4}&\frac{\sqrt{a}}{2}&-\frac{\sqrt{a}}{2}\end{pmatrix}.$$

Then $h_a\cdot\psi_\ell=\psi_\ell^a$. For each a, fix an element h_a as above for use throughout.

Clearly $(L_\ell^{\psi_\ell^a})^0=h_a(L_\ell^{\psi_\ell})^0h_a^{-1}$. The image of this group under pr is isomorphic over F to the non-split quasisplit form of $SO_{4n-2\ell}$ corresponding the square class of a. It follows that $(L_n^{\psi_\ell^a})^0$ is isomorphic over F to the non-split quasisplit form of $G_{4n-2\ell}$ associated to the square class of a. $\square$

In the course of the preceding proof, we have seen that it is enough to consider one conveniently chosen representative from each F-orbit of characters in general position. However, it is generally more convenient to make definitions for general $a\in F^\times$ than it is to choose representatives for the square classes in $F^\times$.

Definition 5.1.8. *Take $a\in F^\times$, and let ψ_ℓ^a be the character of N_ℓ defined by*

$$\psi_\ell^a(u)=\psi_0(u_{12}+\cdots+u_{\ell-1,\ell}+u_{\ell,2n}+\frac{a}{2}u_{\ell,2n+2}).$$

We also keep the notation

$$\psi_\ell(u)=\psi_0(u_{12}+\cdots+u_{\ell-1,\ell}+u_{\ell,2n+1}).$$

Then the orbit of ψ_ℓ^a is determined by the square class of a. The character ψ_ℓ is in the same orbit as ψ_ℓ^1.

Note that for any given square class $\mathbf{a}$ we have many conjugate embeddings of $G_{2n}^{\mathbf{a}}$ into G_{4n+1}: one for each element a of $\mathbf{a}$.

Definition 5.1.9. *For each element a of $F^\times$, we let G_{2n}^a denote $(L_n^{\psi_n^a})^0$. It is a subgroup of G_{4n+1}, which is isomorphic over F to $G_{2n}^{\{a\}}$, where $\{a\}$ is the square class of a.*

Lemma 5.1.10. *Assume $\{a\}\neq\square$. Then,*

1. *An element u of $U_{\max}$ is in G_{2n}^a iff it satisfies $u_{ij}=0$ for $i\le n$ or $i=2n$, and $u_{i,2n}=-\frac{a}{2}u_{i,2n+2}$ for $n<i<2n$. The set of such elements u is equal to $h_a(U_{\max}\cap(L_n^{\psi_n})^0)h_a^{-1}$, and is a maximal unipotent subgroup of G_{2n}^a.*
2. *And element $t=\prod_{i=0}^{2n}e_i^*(t_i)$ of $T(G_{4n+1})$ is in G_{2n}^a iff it satisfies $t_i=1$ for $0<i\le n$, and $i=2n$. The set of such t is a maximal F-split torus of G_{2n}^a.*
3. *There is a maximal torus of G_{2n}^a which contains the above maximal F-split torus and is contained in the standard Levi of G_{4n+1} whose unique positive root is the short simple root e_n. Its set of F points is equal to*

$$\left\{h_ath_a^{-1}:t=\prod_{i=1}^{n-1}e_{n+i}^*(t_i)e_{2n}^*(x\cdot\bar{x}^{-1})e_0^*(\bar{x}),\ t_1,\dots,t_{n-1}\in F^\times,\ x\in F(\sqrt{a})^\times\right\},$$

where $^-$ *denotes the action of the nontrivial element of* $\mathrm{Gal}(F(\sqrt{a})/F)$.

If $\{a\} = \square$, *then* (1) *remains true, while*

$$\left\{ h_a t h_a^{-1} : t = \prod_{i=0}^{n} e_{n+i}^*(t_i) \right\},$$

is a maximal torus, and is F*-split, since* h_a *has entries in* F.

Remark 5.1.11. *We may write an element of our maximal torus as*

$$\left\{ h_a \prod_{i=1}^{n-1} e_{n+i}^*(t_i) \cdot e_{2n}^* \left((x+y\sqrt{a}) \cdot (x-y\sqrt{a})^{-1} \right) e_0(x-y\sqrt{a}) h_a^{-1} : t_i \in F,\ x,y \in F, x^2-ay^2 \neq 0 \right\},$$

regardless of $\{a\}$.

Proof. Item (1) is easily checked. (Recall that pr is an isomorphism on $U_{\max}$.) Similarly, it is easily checked that an element t of $T(G_{4n+1})$ stabilizes the specified character iff $t_1 = \cdots = t_n = t_{2n} = \pm 1$. As noted in the proof of Lemma 5.1.7, if they are all minus 1, then this element is in the other connected component of $L_n^{\psi_n^a}$.

Recall that $(L_n^{\psi_n})^0$, with ψ_n as in Definition 5.1.8 is isomorphic to G_{2n}. There is an "obvious" choice of isomorphism $\mathrm{inc} : G_{2n} \to (L_n^{\psi_n})^0$, such that

$$\mathrm{inc} \circ \bar{e}_i^* = \begin{cases} e_0^* & i = 0, \\ e_{n+i}^* & 1 \leq i \leq n, \end{cases} \qquad \text{and} \qquad \mathrm{inc}(u)_{ij} = \begin{cases} 0 & i \leq n, \text{ or } j = 2n+1, \\ u_{i-n,j-n} & i > n,\ j < 2n+1, \\ u_{i-n,j-n-1} & i > n,\ j > 2n+1. \end{cases}$$

Here, we have used e_i^* for elements of the $\mathbb{Z}$-basis of the cocharacter lattice of G_{4n+1} and $\bar{e}_i^*$ for elements of that of G_{2n}. It follows from the definitions that conjugation by h_a is an isomorphism of G_{2n}^a with $(L_n^{\psi_n})^0$, which is defined over $F(\sqrt{a})$. This yields an identification of the maximal F-split torus of $G_{2n}^{\{a\}}$ as computed in section 3.4 with the F-split torus in item (2).

Clearly $h_a \cdot \mathrm{inc}(T(G_{2n})) \cdot h_a^{-1}$ is a maximal torus of G_{2n}^a. The fact that an element is of the form specified in item (3) of the present lemma follows from the action of $\mathrm{Gal}(\bar{F}/F)$ on the lattice of cocharacters computed in section 3.4. $\square$

Definition 5.1.12. *Let*

$$DC_\omega^a(\tau) = FC^{\psi_n^a} \mathcal{E}_{-1}(\tau, \omega).$$

It is a space of smooth functions $G_{2n}^a(F\backslash\mathbb{A}) \to \mathbb{C}$, *and affords a representation of the group* $G_{2n}^a(\mathbb{A})$ *acting by right translation, where we have identified* G_{2n}^a *with the identity component of* $L_n^{\psi_n^a}$.

Definition 5.1.13. *We say that a square class* **a** *in* $F^\times$ *and a character* χ *are* compatible *if they correspond to the same homomorphism from* $\mathrm{Gal}(\bar{F}/F)$ *to the group with two elements. We say that an element* a *of* $F^\times$ *and a character* χ *are compatible if* χ *is compatible with the square class of* a.

Theorem 5.1.14. *Let* ω *be a Hecke character. Let* $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ *be the isobaric sum of distinct* ω^{-1}*-orthogonal unitary cuspidal automorphic representations of* $GL_{2n_1}(\mathbb{A}), \ldots, GL_{2n_r}(\mathbb{A})$, *respectively. For* $i = 1$ *to* r *let* ω_{τ_i} *denote the central character of* τ_i *and let* $\chi_i = \omega_{\tau_i}/\omega^{n_i}$, *which is quadratic. Let* $\chi = \prod_{i=1}^r \chi_i$. *Suppose that* χ *and* a *are not compatible. Then* $DC_\omega^a(\tau) = \{0\}$.

Proof. As in Theorem 5.1.1, it suffices to prove the vanishing of the corresponding twisted Jacquet module of $\mathrm{Ind}_{P(F_v)}^{G_{4n+1}(F_v)} \tau_v \otimes |\det|^{\frac{1}{2}} \boxtimes \omega_v$ at a single unramified place v. The vanishing follows from Proposition 7.0.16, if there is an unramified place v such that χ_v is trivial and a is not a square, and from Proposition 7.0.18 if there is an unramified place v such that χ_v is nontrivial and a is a square. If χ and a are incompatible, then there is at least one unramified place at which one of these cases occurs. $\square$

Theorem 5.1.15. *Let ω be a Hecke character. Let $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ be the isobaric sum of distinct ω^{-1}-orthogonal unitary cuspidal automorphic representations of $GL_{2n_1}(\mathbb{A}), \ldots, GL_{2n_r}(\mathbb{A})$, respectively. For $i = 1$ to r let ω_{τ_i} denote the central character of τ_i and let $\chi_i = \omega_{\tau_i}/\omega^{n_i}$, which is quadratic. Let $\chi = \prod_{i=1}^r \chi_i$. Then $DC_\omega^a(\tau)$ is nontrivial if and only if χ and a are compatible, in which case the space $DC_\omega^a(\tau)$ is a nonzero, cuspidal representation of $G_{2n}^a(\mathbb{A})$, with central character ω, which supports a nonzero Whittaker integral for the generic character of $U_{\max}(\mathbb{A}) \cap G_{2n}^a(\mathbb{A})$ given by*

$$u \mapsto \psi_0\left(\sum_{i=1}^{2n-2} u_{i,i+1} + u_{2n-1,2n+2}\right).$$

If σ is any irreducible automorphic representation contained in $DC_\omega^a(\tau)$, then σ lifts weakly to τ under the map r.

Remark 5.1.16. *Since $DC_\omega(\tau)$ is nonzero and cuspidal, there exists at least one irreducible component σ. In the case of special orthogonal groups, one may show (*[So1]*, pp. 8-9, item 4) that the descent module is in the ψ-generic spectrum for a suitable choice of ψ (cf. section 3.3). It that all of the irreducible components are globally ψ-generic. This is done using the Rankin-Selberg integrals of* [Gi-PS-R]*,*[So2]*. In the odd case, one may also show (*[GRS4]*, Theorem 8, p. 757, or* [So1] *page 9, item 6) using the results of* [Ji-So] *that the descent module is irreducible. This does* **not** *extend to the even case, even for special orthogonal groups, because the construction actually yields a representation of the full stabilizer– which is isomorphic to the full orthogonal group. (Cf. Proposition 7.0.20.)*

Proof. The statements are proved by combining relationships between unipotent periods and knowledge about $\mathcal{E}_{-1}(\tau, \omega)$.

For $a \in F^\times$, we let (U_1^a, ψ_1^a) denote the unipotent period obtained by composing the period (N_n, ψ_n^a), used in defining the descent to G_{2n}^a, (embedded into G_{4n+1} as the stabilizer of ψ_n^a) with a period which defines a Whittaker integral on this group. Specifically, U_1 is the subgroup of the standard maximal unipotent defined by the relations $u_{i,2n} = -\frac{a}{2} u_{i,2n+2}$ for $i = n+1$ to $2n-1$, as well as $u_{2n,2n+1} = 0$, and

$$\psi_1(u) = \psi_0(u_{1,2} + \cdots + u_{n-2,n-1} + u_{n-1,2n} + \frac{a}{2} u_{n-1,2n+1} + u_{n,n+1} + \cdots + u_{2n-1,2n}).$$

The definitions of U_1^a and ψ_1^a make sense also in the case when $a = 0$, although in that case there is no interpretation in terms of a descent. We use this period in that case also.

Next, let U_2 denote the subgroup of the standard maximal unipotent defined by $u_{2n,2n+1} = 0$, and $u_{12} = u_{34} = \cdots = u_{2n-1,2n}$. For all $a \in F$, we may define a character of this group by the formula

$$\psi_2^a(u) = \psi_0\left(\sum_{i=1}^{2n-2} u_{i,i+2} + u_{2n-1,2n+2} + \frac{a}{2} u_{2n-1,2n}\right).$$

Finally, let U_3 denote the maximal unipotent, and ψ_3 denote

$$\psi_3(u) = \psi(u_{1,2} + \cdots + u_{2n-1,2n}).$$

Thus (U_3, ψ_3) is the composite of the unipotent period defining the constant term along the Siegel parabolic, and one which defines a Whittaker integral on the Levi of this parabolic. By Theorem 4.0.4 (5) this period is *not* in $\mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$.

In the appendices, we show

(1) $(U_1^a, \psi_1^a) \sim (U_2, \psi_2^a)$, for all $a \in F$, in Lemma 8.3.1,

(2) $(U_2, \psi_2^0) \in \langle\{(U_2, \psi_2^a) : a \in F^\times\}\rangle$, in Lemma 8.4.2, and
(3) $(U_3, \psi_3) \in \langle (U_2, \psi_2^0), \{(N_\ell, \vartheta) : n < \ell < 2n$ and ϑ in general position.$\}\rangle$ in Lemma 8.3.2.

By Theorem 5.1.1 $(N_\ell, \vartheta) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$ for all $n < \ell < 2n$ and ϑ in general position. It follows that at least one of the periods (U_1^a, ψ_1^a) is not in $\mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$. This establishes genericity (and hence nontriviality) of the corresponding descent module $DC_\omega^a(\tau)$.

Turning to cuspidality, we prove in the appendices an identity relating:

- Constant terms on G_{2n}^a,
- Descent periods in G_{4n+1},
- Constant terms on G_{4n+1},
- Descent periods on $G_{4n-2k+1}$, embedded in G_{4n} as a subgroup of a Levi.

To formulate the exact relationship we introduce some notation for the maximal parabolics of GSpin groups.

The group G_{4n+1} has one standard maximal parabolic having Levi $GL_i \times G_{4n-2i+1}$ for each value of i from 1 to $2n$. Let us denote the unipotent radical of this parabolic by V_i. We denote the trivial character of any unipotent group by $\mathbf{1}$.

For any square class $\mathbf{a}$, the group $G_{2n}^{\mathbf{a}}$ has one standard maximal parabolic having Levi $GL_k \times G_{2n-2k}^{\mathbf{a}}$ for each value of k from 1 to $n-2$. We denote the unipotent radical of this parabolic by V_k^{2n}. The split group $G_{2n} = G_{2n}^{\square}$ also has two parabolics with Levi isomorphic to $GL_n \times GL_1$. One has the property that $e_{n-1} - e_n$ is a root of the Levi, and the other does not. Let us denote the unipotent radical of this first parabolic by V_n^{2n}. Then the unipotent radical of the other is ${}^\dagger V_n^{2n}$, where $\dagger$ is the outer automorphism of G_{2n} which reverses the last two simple roots while fixing the others. In a nonsplit quasisplit form of G_{2n}, there is a parabolic subgroup with Levi isomorphic to the product of GL_{n-1} and a nonsplit torus which is maximal. (The corresponding parabolic in the split case is not maximal.) We denote its unipotent radical by V_{n-1}^{2n}.

We prove in Lemma 8.3.3 that, for $1 \le k \le n-1$, $(V_k^{2n}, \mathbf{1}) \circ (N_n, \psi_n^a)$ is contained in

$$\langle (N_{n+k}, \psi_{n+k}), \{(N_{n+j}, \psi_{n+j}^a)^{(4n-2k+2j+1)} \circ (V_{k-j}, \mathbf{1}) : \quad 1 \le j < k\}\rangle,$$

where $(N_{n+j}, \psi_{n+j}^a)^{(4n-2k+2j+1)}$ denotes the descent period, defined as above, but on the group $G_{4n-2k+2j+1}$, embedded into G_{4n+1} as a component of the Levi with unipotent radical V_{k-j}.

Now suppose that a is a square. Then both $(V_n^{2n}, \mathbf{1}) \circ (N_n, \psi_n^a)$ and $({}^\dagger V_n^{2n}, \mathbf{1}) \circ (N_n, \psi_n^a)$ are in

$$\langle (N_{2n}, \psi_{2n}), \{(N_{n+j}, \psi_{n+j}^a)^{(2n+2j+1)} \circ (V_{n-j}, \mathbf{1}) : \quad 1 \le j < n\}\rangle.$$

Indeed, the two periods are actually conjugate in G_{4n+1}, so it suffices to consider only one of them.

By Theorem 5.1.1 $(N_{n+k}, \psi_{n+k}^a) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$ for $k = 1$ to n. Furthermore, for k, j such that $1 \le j < k \le n$ the function $E(f)(s)^{(V_{k-j}, \mathbf{1})}$ may be expressed in terms of Eisenstein series on GL_{k-j} and $G_{4n-2k+2j}$, using Proposition II.1.7 (ii) of [MW1]. What we require is the following:

Lemma 5.1.17. *For all $f \in V^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau \boxtimes \omega)$*

$$E_{-1}(f)^{(V_{k-j}, \mathbf{1})}\Big|_{G_{4n-2k+2j+1}(\mathbb{A})} \in \bigoplus_S \mathcal{E}_{-1}(\tau_S, \omega),$$

where the sum is over subsets S of $\{1, \dots, r\}$ such that $\sum_{i \in S} 2n_i = 2n - k + j$, and, for each such S, $\mathcal{E}_{-1}(\tau_S, \omega)$ is the space of functions on $G_{4n-2k+2j+1}(\mathbb{A})$ obtained by applying the construction of $\mathcal{E}_{-1}(\tau, \omega)$ to $\{\tau_i : i \in S\}$, instead of $\{\tau_i : 1 \le i \le r\}$.

Once again, this is immediate from [MW1] Proposition II.1.7 (ii).

Applying Theorem 5.1.1, with τ replaced by τ_S and $2n$ by $2n - k + j$, we deduce

$$(N_{n+j}, \psi_{n+j})^{(4n-2k+2j+1)} \in \mathcal{U}^\perp\left(\mathcal{E}_{-1}(\tau_S, \omega)\right) \quad \forall S,$$

and hence $(N_{n+j-1}, \psi_{n+j-1})^{(4n-2k+2j)} \circ (V_{k-j}, \mathbf{1}) \in \mathcal{U}^{\perp}(\mathcal{E}_{-1}(\tau, \omega))$. This shows that any nonzero function appearing in any of the spaces $DC^a_\omega(\tau)$ must be cuspidal. Such a function is also easily seen to be of uniformly moderate growth, being the integral of an automorphic form over a compact domain. In addition, such a function is easily seen to have central character ω, and any function with these properties is necessarily square integrable modulo the center ([MW1] I.2.12). It follows that each of the spaces $DC^a_\omega(\tau)$ decomposes discretely.

Now, suppose $\sigma \cong \otimes'_v \sigma_v$ is an irreducible representation which is contained in $DC^a_\omega(\tau)$. Let p_σ denote the natural projection $DC^a_\omega(\tau) \to \sigma$. Once again, by Theorem 4.0.4 (3), the representation $\mathcal{E}_{-1}(\tau, \omega)$ decomposes discretely. Let π be an irreducible component of $\mathcal{E}_{-1}(\tau, \omega)$ such that the restriction of $p_\sigma \circ FC^{\psi^a_n}$ to π is nontrivial. As discussed previously in the proof of Theorem 5.1.1, at all but finitely many v, τ is unramified at v and furthermore, π_v is the unramified constituent ${}^{un}Ind_{P(F_v)}^{G_{4n+1}(F_v)} \tau_v \boxtimes \omega_v \otimes |\det|_v^{\frac{1}{2}}$ of $Ind_{P(F_v)}^{G_{4n+1}(F_v)} \tau_v \boxtimes \omega_v \otimes |\det|_v^{\frac{1}{2}}$. If v_0 is such a place, the map $p_\sigma \circ FC^{\psi^a_n} \circ i_{\zeta^{v_0}}$, with $i_{\zeta^{v_0}}$ defined as in Theorem 5.1.1, factors through $\mathcal{J}_{N_n, \psi^a_n}\left({}^{un}Ind_{P(F_{v_0})}^{G_{4n+1}(F_{v_0})} \tau_v \otimes |\det|_v^{\frac{1}{2}} \boxtimes \omega_v\right)$, and gives rise to a $G^a_{2n}(F_{v_0})$-equivariant map from this Jacquet-module onto σ_{v_0}.

To pin things down precisely, assume that τ_v is the unramified component of $Ind_{B(GL_{2n})(F_v)}^{GL_{2n}(F_v)} \mu$, and let $\mu_1, \dots, \mu_{2n}$ be defined as in the proof of Lemma 3.6.1. By Lemma 3.6.1, we may assume without loss of generality that $\mu_{2n+1-i} = \omega \mu_i^{-1}$ for $i = 1$ to $n-1$, and that either $\mu_n = \omega \mu_{n+1}^{-1}$, or $\mu_n^2 = \mu_{n+1}^2 = \mu_n \mu_{n+1} \chi_{un} = \omega$ (with χ_{un} defined as in the lemma). Furthermore, suppose that χ_v is the local component at v of the global quadratic character obtained from τ and ω as in the statement of the theorem. Then either χ_v is trivial and $\mu_n = \omega \mu_{n+1}^{-1}$, or $\chi_v = \chi_{un}$ and $\mu_n^2 = \mu_{n+1}^2 = \mu_n \mu_{n+1} \chi_{un} = \omega$.

Recall that a basis for the lattice of F-rational cocharacters of the maximal torus of G^a_{2n} fixed in Lemma 5.1.10 is given by

$$\{e^*_{n+i} : 1 \leq i < n\} \cup \{e^*_0\} \cup \{e^*_n, \textit{if } a \text{ is a square}\}.$$

Observe that when a is not a square in F, it is a square in F_v for many unramified v, and that the cocharacter e^*_n is F_v-rational at such v.

In the appendices, we show that in the nonsplit case

$$\mathcal{J}_{N_n, \psi_n}\left({}^{un}Ind_{P(F_v)}^{G_{4n+1}(F_v)} \tau_v \boxtimes \omega_v \otimes |\det|_v^{\frac{1}{2}}\right)$$

is isomorphic as a $G^a_{2n}(F_v)$-module to a subquotient of a principal series representation π_v of $G^a_{2n}(F_v)$ such that the corresponding parameter $t_{\pi,v}$ maps to the parameter $t_{\tau,v}$ under r. In the split case, we obtain instead a direct sum of two principal series representations, but *both* have parameters which map to $t_{\tau,v}$. It follows that τ is the weak lift of σ associated to the map r. $\square$

6. Appendix I: Eisenstein series

In this appendix we complete the proofs of several intermediate statements used in the proof of Theorem 4.0.4. As far as we know, all of these results are well-known to the experts, but do not appear in the literature in the precise form we need.

6.1. **Proof of Proposition 4.0.10.** First, suppose that a set D of hyperplanes carries all the singularities of all the intertwining operators $M(w, \underline{s})f$. Then it follows from [MW1] II.1.7, IV.1.9 (b) that it carries all the singularities of the cuspidal components of all the constant terms of $E(f)(g)(\underline{s})$. By I.4.10, it therefore carries the singularities of the Eisenstein series itself.

On the other hand, it is clear that a set which carries the singularities of the Eisenstein series carries those of all of its constant terms. Thus, what we need to prove is:

Lemma 6.1.1. *Fix M' a standard Levi which is conjugate to M and $\alpha \in \Phi^+(Z_M)$. Let H be the root hyperplane given by $\langle \alpha^\vee, \underline{s}\rangle = c$, $c \neq 0$. Consider the family of functions $M(w, \underline{s})f$ corresponding to $\{w \in W(M) | wMw^{-1} = M'\}$. If any one or them has a pole along H, then the constant term of the Eisenstein series along P' does as well. In other words, it is not possible for two poles to cancel one another.*

Proof. Clearly, it is enough to prove this under the additional hypothesis that $M' = M$.

Let A_M^+ denote the group isomorphic to $(\mathbb{R}_+^\times)^{r+1}$, embedded diagonally at the infinite places, which is inside the center of M.

The Lie algebra of A_M^+ is naturally identified with the real dual of $X(M)\otimes_{\mathbb{Z}}\mathbb{R}$. Recall that above we identified $\underline{s}$ with an element of $X(M)\otimes_{\mathbb{Z}}\mathbb{C}$. So, there is a natural pairing $\langle X, \underline{s}\rangle$, $X \in \mathfrak{a}_M^+$, given as follows. Write $\det_i$ for the determinant of the ith block of an element of M, regarded as a $2n\times 2n$ matrix via the identification with $GL_m \times GL_1$ fixed above. Then we have

$$\prod_{i=1}^{r} |\det{}_i \exp(\log y \cdot X)|^{s_i} = y^{\langle X, \underline{s}\rangle}.$$

It follows that

$$|M(w, \underline{s})f(\exp(\log y \cdot X)g)| = y^{\mathrm{Re}(\langle w^{-1}X, \underline{s}\rangle)} \cdot \delta_P^{\frac{1}{2}}(w^{-1}\exp(\log y \cdot X)w) \cdot |M(w, \underline{s})f(g)|.$$

Here δ_P is the modular quasicharacter of P.

Let

$$W_{sing}(M, H) = \{w \in W(M), wMw^{-1} = M, M(w, \underline{s}) \text{ has a pole along } H\}.$$

Suppose that this set is nonzero. Choose $w_0 \in W_{sing}(M, H)$ such that the order of the pole of $M(w_0, \underline{s})$ is of maximal order. Let $\nu(H)$ denote the order. Choose $X \in \mathfrak{a}_M^+$ such that the points $w^{-1}\cdot X, w \in W_{sing}(M, H)$ are all distinct. Consider the family of functions

$$(\langle \alpha^\vee, \underline{s}\rangle - c)^{\nu(H)} M(w, \underline{s})f(\exp(\log y \cdot X)g), \quad w \in W_{sing}(M, H).$$

They have singularities carried by a locally finite set of root hyperplanes not containing H. Assume g has been chosen so that $(\langle \alpha^\vee, \underline{s}\rangle - c)^{\nu(H)} M(w_0, \underline{s})f(g) \neq 0$. For $\underline{s}$ restricted to an open subset of H not intersecting any of the singular hyperplanes we obtain a family of holomorphic functions, at least one of which is nonzero. If we further exclude the intersection of H with the hyperplanes

$$\langle w_1^{-1}X - w_2^{-1}X, \underline{s}\rangle = 0, \quad w_1, w_2 \in W_{sing}(M, H),$$

(which can not coincide with H because $c \neq 0$), then at every point $\underline{s}$, those functions which are nonzero all have distinct orders of magnitude as functions of y. Hence they can not possibly cancel one another. □

6.2. **Proof of Lemma 4.0.12.** Regarding $w_i \cdot \underline{s} + \rho_{P_{\alpha_i}}$ as an element of $X(w_iMw_i^{-1})\otimes_{\mathbb{Z}}\mathbb{C}$, we may decompose it as $\mu_1 + \langle \alpha_i^\vee, w_i\cdot \underline{s}\rangle\tilde{\alpha}_i$, where $\tilde{\alpha}_i$ is defined by the property that

$$\langle \alpha^\vee, \tilde{\alpha}_i\rangle = \delta_{\alpha,\alpha_i}, \quad \text{for } \alpha \in \Phi^+(Z_{w_iMw_i^{-1}}).$$

Then it follows easily from the definitions that μ_1 is in the image of the natural projection $X(M_{\alpha_i})\otimes_{\mathbb{Z}}\mathbb{C} \to X(w_iMw_i^{-1})\otimes_{\mathbb{Z}}\mathbb{C}$ corresponding to restriction of characters of $M_{\alpha_i}(\mathbb{A})$ to $w_iMw_i^{-1}(\mathbb{A})$.

Take f a K-finite flat section of $\mathrm{Ind}_{P^{w_i}(\mathbb{A})}^{G_{4n+1}(\mathbb{A})}(\bigotimes_{j=1}^r \tau_j \otimes |\det{}_j|^{s_j} \boxtimes \omega)\circ Ad(w_i^{-1})$. Then $M_{w_i}(s_{\alpha_i}, w_i\cdot \underline{s})f$ resides in a finite dimensional subspace of $\mathrm{Ind}_{P^{w_{i-1}}(\mathbb{A})}^{G_{4n+1}(\mathbb{A})}(\bigotimes_{j=1}^r \tau_j \otimes |\det{}_j|^{s_j} \boxtimes \omega)\circ Ad(w_{i-1}^{-1})$, corresponding to a finite set of K-types determined by f. Write $M_{w_i}(s_{\alpha_i}, w_i\cdot\underline{s})f$ in terms of a basis of flat K-finite sections. The coefficients are functions of $\underline{s}$, but it follows easily from the integral definition where this is valid, and by meromorphic continuation elsewhere, that in fact they are independent of μ_1 (which corresponds to a character of $M_{\alpha_1}(\mathbb{A})$ and may be pulled out of the integration). Thus, they depend only on $\langle w_i \cdot \underline{s}, \alpha_i^\vee \rangle = \langle \underline{s}, w_i^{-1}\alpha_i^\vee \rangle$.

The first two assertions are now clear. A proof that $c \neq 0$ is obtained by a straightforward modification of the opening paragraph of [MW1], IV.3.12.

6.3. **Proof of Proposition 4.0.13.** In this section, we denote by $V^{(i)}(s, \tau, \omega)$, $i = 1, 2$, the spaces of functions previously introduced in section 4 as $V^{(i)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$, in the special case when $r = 1$.

Let $\tilde{M}(s)$ denote the analogue of $M(w, s)$ defined using $V^{(1)}(s, \tau, \omega)$. It maps into the space $V^{(3)}(-s, \tilde{\tau} \otimes \omega, \omega)$ given by

$$\left\{ \tilde{F} : G_{2m+1}(\mathbb{A}) \to V_\tau, \text{ smooth } \middle| \tilde{F}((g, \alpha)h)(g_1) = \omega(\alpha \det g) |\det g|^{-s+\frac{m}{2}} \tilde{F}(h)(g_1 \, {}_t g^{-1}) \right\}.$$

Fix realizations of the local induced representations τ_v and an isomorphism $\iota : \otimes'_v \tau_v \to \tau$. Define, for each v, $V^{(1)}(s, \tau_v, \omega_v)$ to be

$$\left\{ \tilde{F}_v : G_{2m+1}(F_v) \to V_{\tau_v}, \text{ smooth } \middle| \tilde{F}_v((g, \alpha)h) = \omega_v(\alpha) |\det g|_v^{s+\frac{m}{2}} \tau_v(g) \tilde{F}_v(h) \right\},$$

and $V^{(3)}(s, \tilde{\tau}_v \otimes \omega_v, \omega_v)$ to be

$$\left\{ \tilde{F}_v : G_{2m+1}(F_v) \to V_{\tau_v}, \text{ smooth } \middle| \tilde{F}_v((g, \alpha)h) = \omega_v(\alpha \det g) |\det g|_v^{s+\frac{m}{2}} \tau_v({}_t g^{-1}) \tilde{F}_v(h) \right\}.$$

Then the formula

$$\tilde{\iota}(\otimes_v \tilde{F}_v)(g) = \iota(\otimes'_v \tilde{F}_v(g_v))$$

defines maps

$$\otimes'_v V^{(1)}(s, \tau_v, \omega_v) \to V^{(1)}(s, \tau, \omega),$$
$$\otimes'_v V^{(3)}(s, \tilde{\tau}_v \otimes \omega_v, \omega_v) \to V^{(3)}(s, \tilde{\tau} \otimes \omega_v, \omega),$$

both of which we denote by $\tilde{\iota}$.

It is known that each map is, in fact, an isomorphism. For the benefit of the reader we sketch an argument. On pp. 307 of [Sha1] certain explicit elements of (a generalization of) $V^{(1)}(s, \tau, \omega)$ are constructed as integrals involving matrix coefficients. Using Schur orthogonality, one may check that $\tilde{F}$ is expressible in this form iff both the K-module it generates and the $K \cap M(\mathbb{A})$-module it generates are irreducible. It is clear that such vectors span the space of all K-finite vectors. On the other hand the (finite dimensional) space of matrix coefficients of this irreducible representation of K is spanned by those that factor as a product of matrix coefficients of local representations, and these are clearly in the image of $\tilde{\iota}$.

For $\tilde{F}_v \in V^{(1)}(s, \tau_v, \omega_v)$, let

$$A_v(s)\tilde{F}_v(g) = \int_{U_w(F_v)} \tilde{F}_v(\dot{w}ug) du.$$

Then the following diagram commutes

$$\begin{array}{ccc} \otimes'_v V^{(1)}(s, \tau_v, \omega_v) & \xrightarrow{A(s)} & \otimes'_v V^{(1)}(-s, \tau_v, \omega_v) \\ \tilde{\iota}\big\downarrow & & \tilde{\iota}\big\downarrow \\ V^{(1)}(s, \tau, \omega) & \xrightarrow{\tilde{M}(s)} & V^{(1)}(-s, \tau, \omega) \end{array}$$

with $A(s) := \otimes_v A_v(s)$.

Now, $M(w, s)f(s)$ has a pole (i.e., there exists $g \in G_{2m+1}(\mathbb{A})$ such that $M(w, s)f(s)(g)$ has a pole) if and only if $\tilde{M}(s)\tilde{F}(s)$ has a pole (i.e., there exist $g \in G_{2m+1}(\mathbb{A})$ and $m \in M(\mathbb{A})$ such that $\tilde{M}(s)\tilde{F}(s)(g)(m)$ has a pole), where $\tilde{F}$ is the element of $V^{(1)}(s, \tau, \omega)$ such that $f(g) = \tilde{F}(g)(id)$.

We wish to show that there exists $\tilde{F}$ such that this is the case iff τ is ω^{-1}-orthogonal. Clearly, we may restrict attention to $\tilde{F}$ of the form $\tilde{\iota}(\otimes_v \tilde{F}_v)$.

Recall that for all but finitely many non-archimedean v, the space V_{τ_v} comes equipped with a choice of $GL_m(\mathfrak{o}_v)$-fixed vector ξ_v° used to define the restricted tensor product.

If $\tilde{F} = \tilde{\iota}(\otimes_v \tilde{F}_v) \in V^{(1)}(s,\tau,\omega)$, then there is a finite set S of places, such that if $v \notin S$ then v is non-archimedean, τ_v is unramified, and $\tilde{F}_v(s) = \tilde{F}^\circ_{(s,\tau_v,\omega_v)}$ is the unique element of $V^{(1)}(s,\tau_v,\omega_v)$ satisfying $\tilde{F}_{(s,\tau_v,\omega_v)}(k) = \xi_v^\circ$ for all $k \in G_{2m+1}(\mathfrak{o}_v)$.

Now

$$A_v(s)\tilde{F}^\circ_{(s,\tau_v,\omega_v)} = \frac{L_v(2s,\tau_v, sym^2 \times \omega_v^{-1})}{L_v(2s+1,\tau_v, sym^2\times\omega_v^{-1})}\tilde{F}^\circ_{(-s,\tilde{\tau}_v\otimes\omega_v,\omega_v)}.$$

(A proof of this appears in [L1], albeit not in this precise language. See especially pp. 25-27.) Thus,

$$A(s)\tilde{\iota}(\otimes_v\tilde{F}_v) = \frac{L^S(2s,\tau,sym^2\times\omega^{-1})}{L^S(2s+1,\tau,sym^2\times\omega^{-1})}\tilde{\iota}\left(\left(\bigotimes_{v\in S} A_v(s)\tilde{F}_v(s)\right)\otimes\left(\bigotimes_{v\notin S}\tilde{F}_{-s,\tilde{\tau}_v\otimes\omega_v,\omega_v}\right)\right).$$

To complete the proof we must show:

- **(i):** $A_v(s)$ is holomorphic and nonvanishing (i.e., not the zero operator) on $Ind_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})}\tau\otimes|\det|^s\boxtimes\omega$ at $s=\frac{1}{2}$, for all τ.
- **(ii):** $L_v(s,\tau_v,sym^2\times\omega_v^{-1})$ is holomorphic and nonvanishing at $s=1$, for all τ_v.
- **(iii):** $L^S(s,\tau,sym^2\times\omega^{-1})$ is holomorphic and nonvanishing at $s=2$.

Item (iii) is covered by Proposition 7.3 of [Kim-Sh]. Items (i) and (ii) are essentially contained in Proposition 3.6, p. 153 of [Asg-Sha1]. Since what we need is *part* of the same information, presented differently, we repeat the part of the arguments we are using.

The nonvanishing part of (i) is a completely general fact (i.e., holds at least for any Levi of any split reductive group). For example, the only element of the arguments made on p. 813 of [GRS3] which is particular to the situation they consider there (the Siegel of Sp_{4n}) is the precise ratio of L functions appearing in the constant term.

Similarly, local L functions never vanish. At a finite prime the local L function is $P(q_v^{-s})^{-1}$ for some polynomial P, while at an infinite prime it is given in terms of the Γ function and functions of exponential type.

We turn to holomorphicity.

Lemma 6.3.1. *Let π_v be any representation of $GL_m(F_v)$, which is irreducible, generic, and unitary. Then there exist*

- *integers $k_1,\dots,k_r$ of such that $k_1+\dots+k_r=m$,*
- *real numbers $\alpha_1,\dots,\alpha_r\in(-\frac{1}{2},\frac{1}{2})$,*
- *discrete series representations δ_i of $GL_{k_i}(F_v)$ for $i=1$ to r*

such that

$$\pi_v \cong Ind_{P_{(k)}(F_v)}^{GL_m(F_v)}\bigotimes_{i=1}^r(\delta_i\otimes|\det{}_i|^{\alpha_i}).$$

Here $P_{(k)}$ denotes the standard parabolic of GL_m with Levi consisting of block diagonal matrices with the block sizes $k_1,\dots,k_r$ (in that order), and $\det_i$ *denotes the determinant of the ith block.*

Remark 6.3.2. *In fact, one may prove a much more precise statement, but the above is what is needed for our purposes.*

Proof. This follows from the main theorem of [Tad2] (see p. 3) together with the fact that the representation denoted $u(\delta,m)$ in that paper is only generic if $m=1$. For this latter statement see the “Proof of (a)⇒(f)” on p. 93 of [Vog] in the Archimedean case (see also the very last remark of the paper, on p. 98) and Theorem 8.1 on p. 195 of [Z] in the non-Archimedean case. (For the notion of “highest derivative” see p. 452 of [BZ2]: a representation is generic iff its “highest derivative” is the trivial representation of the trivial group, which corresponds to the empty multiset under the Zelevinsky classification.) □

Continuing with the proof of Proposition 4.0.13, let $(k) = (k_1, \dots, k_r)$, $\delta = (\delta_1, \dots, \delta_r)$ and $\alpha = (\alpha_1, \dots, \alpha_r)$ be obtained from τ_v as just above, and let $\tilde{P}_{(k)}$ denote the standard parabolic of G_{2m} which is contained in the Siegel parabolic P such that $\tilde{P}_{(k)} \cap M = P_{(k)}$.

Then

$$Ind_{P(F_v)}^{G_{2m}(F_v)} \tau_v \otimes |\det|_v^s \boxtimes \omega_v^s \cong Ind_{\tilde{P}_{(k)}(F_v)}^{G_{2m}(F_v)} \bigotimes_{i=1}^r (\delta_i \otimes |\det{}_i|_v^{s+\alpha_i}) \boxtimes \omega_v.$$

This family (as s varies) of representations lies inside the larger family,

$$Ind_{\tilde{P}_{(k)}(F_v)}^{G_{2m}(F_v)} \bigotimes_{i=1}^r (\delta_i \otimes |\det{}_i|^{s_i}) \boxtimes \omega_v \qquad s = (s_1, \dots, s_r) \in \mathbb{C}^r,$$

and our intertwining operator $A_v(s)$ is the restriction, to the line $s_i = s + \alpha_i$ of the standard intertwining operator for this induced representation, which we denote $A_v(\underline{s})$. This operator is defined, for all $\mathrm{Re}(s_i)$ sufficiently large, by the same integral as $A_v(s)$.

A result of Harish-Chandra says that “$\mathrm{Re}(s_i)$ sufficiently large” can be sharpened to “$\mathrm{Re}(s_i) > 0$.” (This is because all δ_i are discrete series, although tempered would be enough.) This result is given in the p-adic case as [Sil] Theorem 5.3.5.4, and in the Archimedean case, [Kn] Theorem 7.22, p. 196.

Hence, the integral defining $A_v(s)$ converges for $s > \max_i(-\alpha_i)$, and in particular converges at $\frac{1}{2}$.

From the relationship between the local L functions and the so-called local coefficients, it follows that the local L functions are also holomorphic in the same region. For this relationship see [Sha3] for the Archimedean case and [Sha2], p. 289 and p. 308 for the non-Archimedean case.

This completes the proof of (i) and (ii).

6.4. **Proof of Proposition 4.0.15.** The proof is the same as the previous proposition, except that the ratio of partial L function which emerges from the intertwining operators at the unramified places is

$$\frac{L^S(s_1 - s_2, \tau_1 \times \tilde{\tau}_2)}{L^S(s_1 - s_2 + 1, \tau_1 \times \tilde{\tau}_2)}.$$

Convergence of local L functions and intertwining operators at $s_1 - s_2 = 1$ follows again from Lemma 6.3.1. The only difference is the reference for (iii), which in this case is Theorem 5.3 on p. 555 of [Ja-Sh2].

6.5. **Proof of 4.0.17.** As noted, this material is mostly due to Shahidi.

Since the statement is true (with the same proof) for general m, not only $m = 2n$, we prove it in that setting.

In this subsection only, we write τ for the irreducible unitary cuspidal representation $\bigotimes_{i=1}^r \tau_i$ of $M(\mathbb{A})$ (as opposed to the isobaric representation $\tau_1 \boxplus \cdots \boxplus \tau_r$).

First, observe that the integral in question is clearly absolutely and uniformly convergent, and as such defines a meromorphic function of $\underline{s}$ for each g with poles contained in the set of poles of the Eisenstein series itself.

For $\underline{s}$ in the domain of convergence

$$\int_{U_{\max}^{GL_m}(F\backslash\mathbb{A})} E^{GL_m}(f)(ug)(\underline{s})\psi_W(u)\,du = \int_{U_{w_1}(\mathbb{A})\cdot U^{w_1}(F\backslash\mathbb{A})} f(\underline{s})(w_1^{-1}ug)\psi_W(u)\,du, \tag{6.5.1}$$

where w_1 is the longest element of $W_{GL_m}(\bar{M})$ (defined analogously to $W(M)$ above), $U_{w_1} = U_{\max}^{GL_m} \cap w_1 \overline{U_{\max}}^{GL_m} w_1^{-1}$ and $U^{w_1} = U_{\max}^{GL_m} \cap w_1 U_{\max}^{GL_m} w_1^{-1}$.

Indeed,

$$\bar{P}(F)\backslash GL_m(F) = \coprod_w w^{-1}U_w(F),$$

where the union is over w of minimal length in $wW_{\bar{M}}$. Telescoping, we obtain a sum of terms similar to the right hand side of (6.5.1) for these w. Let $U_{\max}^M = M \cap U_{\max}$. Observe that $wU_{\max}^M w^{-1} \subset U_{\max}$ for all such w. The restriction of ψ_W to $wU_{\max}^M w^{-1}$ is a generic character iff wMw^{-1} is a standard Levi. If it is not, the term corresponding to w vanishes by cuspidality of τ.

On the other hand, $f(w^{-1}ug)$ vanishes if $w^{-1}U_\alpha w$ is contained in the unipotent radical of $\bar{P}$ (which we denote $U_{\bar{P}}$) for any simple root α. Here U_α denotes the one-dimensional unipotent subgroup corresponding to the root α. The element w_1 is the only element of $W_{GL_m}(\bar{M})$ such that this does not hold for any α.

Let λ denote the Whittaker functional on V_τ given by

$$\varphi \mapsto \int_{U_{\max}^M(F\backslash\mathbb{A})} \varphi(u)\ \psi_W(w_1uw_1^{-1})\,du.$$

Then (6.5.1) equals

$$\int_{U_{w_1}(\mathbb{A})} \lambda(\tilde{f}(\underline{s})(ug))\psi_W(u)\,du, \tag{6.5.2}$$

where $\tilde{f} : GL_m(\mathbb{A}) \to V_{\otimes\tau_i}$ is given by $\tilde{f}(g)(m) = f(mg)\delta_{\bar{P}}^{-\frac{1}{2}}$. (I.e., $\tilde{f}$ is the element of the analogue of $V^{(1)}(\bigotimes_{i=1}^r \tau_i \boxtimes \omega, \underline{s})$, corresponding to f.)

For each place v there exists a Whittaker functional λ_v on the local representation τ_v such that $\lambda(\otimes_v \xi_v) = \prod_v \lambda_v(\xi_v)$. (A finite product because $\lambda_v(\xi_v^\circ) = 1$ for almost all v. Cf. [Sha1], §1.2.) The induced representation $\mathrm{Ind}_{\bar{P}(\mathbb{A})}^{GL_m(\mathbb{A})}(\bigotimes_{i=1}^r \tau_i|\det{}_i|^{s_i}$ is isomorphic to a restricted tensor product of local induced representations $\otimes_v{}' \,\mathrm{Ind}_{\bar{P}(F_v)}^{GL_m(F_v)}(\bigotimes_{i=1}^r \tau_{i,v}|\det{}_i|_v^{s_i}$. (Cf. section 6.3.) Consider an element $\tilde{f}$ which corresponds to a pure tensor $\otimes_v \tilde{f}_v$ in this factorization. So $\tilde{f}_v(\underline{s})$ is a smooth function $GL_m(F_v) \to V_{\otimes\tau_{i,v}}$ for each $\underline{s}$.) Then (6.5.2) equals

$$\prod_v \int_{U_{w_1}(F_v)} \lambda_v(\tilde{f}(\underline{s})(u_vg_v))\psi_W(u_v)\,du_v, \tag{6.5.3}$$

whenever each of the local integrals is convergent, and the infinite product is convergent (cf [Tate2] Theorem 3.3.1). By Propositions 3.1 and 3.2 of [Sha4], all of the local integrals are always convergent. (See also Lemma 2.3 and the remark at the end of section 2 of [Sha3].)

It is an application of Theorem 5.4 of [C-S] that the term corresponding to an unramified nonarchimedean place v in (6.5.2) is equal to $W_v^\circ(g_v) \cdot \prod_{i<j} L_v(s_i - s_j + 1, \tau_{i,v} \otimes \tilde{\tau}_{j,v})^{-1}$. The convergence of the infinite product is then an elementary exercise, as is the main equation in the statement of our present theorem.

The fact that f may be chosen so that the local Whittaker functions at the places in S do not vanish follows again from Propositions 3.1 and 3.2 of [Sha4] (see also the remark at the end of section 2 of [Sha3]).

7. Appendix II: Local results on Jacquet Functors

In this appendix, F is a non-archimedean local field of characteristic zero We denote the ring of integers and its unique maximal ideal by $\mathfrak{o}$, and $\mathfrak{p}$, respectively, and let $q_F := \#\mathfrak{o}/\mathfrak{p}$. The absolute value on F is normalized so that its image is $\{q_F^j : \ j \in \mathbb{Z}\}$. Also, ω is an unramified character of $F^\times$, τ is an irreducible unramified principal series representation of $GL_{2n}(F)$ such that $\tau \cong \tilde{\tau} \otimes \omega$, and ψ_0 is a nontrivial additive character of F.

For simplicity, we assume that the characteristic of the residue field $\mathfrak{o}/\mathfrak{p}$ is not equal to two. Hence there are four square classes in F, of which two contain units. If ϑ is a character of $N_\ell(F)$ for $1 \leq \ell \leq 2n$, then we may define the square class $\mathrm{Invt}(\vartheta)$ as in Definition 5.1.4 and it is an invariant which separates orbits of characters in general position. Where convenient, we may restrict attention to those ϑ such that $\mathrm{Invt}(\vartheta)$ contains units, as this condition is satisfied at almost all places by any global character. We also define abstract F-groups

$$G_{2n}^{\mathbf{a}} \qquad \mathbf{a} \in F^\times/(F^\times)^2,$$

and concrete subgroups

$$G_{2n}^{a} \subset G_{4n+1} \qquad a \in F^\times,$$

such that $G_{2n}^a \cong G_{2n}^{\mathbf{a}}$ $\forall a \in \mathbf{a}$, as in Definitions 5.1.5, and 5.1.9. The latter is defined using a character ψ_n^a given by the same formula as in Definition 5.1.8.

We require the additional technical hypothesis

$$(B(G_{4n+1}) \cap G_{2n}^a)(F) G_{2n}^a(\mathfrak{o}) = G_{2n}^a(F), \tag{7.0.4}$$

which is known (see [Tits], 3.9, and 3.3.2) to hold at all but finitely many non-Archimedean completions of a number field.

Throughout this section we shall express certain characters of reductive F-groups as complex linear combinations of rational characters. The identification is such that

$$\left(\sum_{i=1}^r s_i \chi_i\right)(h) := \prod_{i=1}^r |\chi_i(h)|^{s_i}.$$

Clearly, the coefficients $s_1, \dots s_r$ appearing in this expression are determined by the character at most up to $(2\pi i)/\log q_F$. If M is a Levi, then restriction gives an injective map $X(M) \to X(T)$. We shall frequently abuse notation and denote an element of $X(M)$ by the same symbol as its restriction to T. Finally, we let Ω denote a complex number such that $\omega(x) = |x|^\Omega$.

Lemma 3.6.1 may be reformulated as stating that $\tau \cong \mathrm{Ind}_{B(GL_{2n})(F)}^{GL_{2n}(F)} \mu$ for an unramified character μ, which is of one of the the following two forms:

$$\mu_1 e_1 + \dots + \mu_n e_n + (\Omega - \mu_n) e_{n+1} + \dots + (\Omega - \mu_1) e_{2n} \tag{7.0.5}$$

$$\mu_1 e_1 + \dots + \mu_{n-1} e_{n-1} + \frac{\Omega}{2} e_n + \left(\frac{\Omega}{2} + \frac{\pi i}{\log q_F}\right) e_{n+1} + (\Omega - \mu_{n-1}) e_{n+2} + \dots + (\Omega - \mu_1) e_{2n}. \tag{7.0.6}$$

In either case, by induction in stages,

$${}^{un}\,\mathrm{Ind}_{P(F)}^{G_{4n+1}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega \cong {}^{un}\,\mathrm{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)} \mu + \frac{1}{2}(e_1 + \dots + e_{2n}) + \Omega e_0.$$

(Here ^{un} indicates the unramified constituent, and P the Siegel parabolic of G_{4n+1}.)

Remark 7.0.7. *Because every unramified character is the square of another unramified character, it is possible to express τ as a twist of a self-dual representation, and deduce essentially all the results of this section from the "classical," self-dual case.*

Lemma 7.0.8. *If μ is of the form (7.0.5), then*

$$^{un}\operatorname{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)} \mu + \frac{1}{2}(e_1 + \cdots + e_{2n}) + \Omega e_0 \cong {}^{un}\operatorname{Ind}_{P_1(F)}^{G_{4n+1}(F)} \mu'$$

where P_1 is the standard parabolic with Levi isomorphic to $GL_2^n \times GL_1$, such that the roots of the Levi are $e_{2i-1} - e_{2i}$, $i = 1$ to n, and μ' is the rational character of this Levi given by

$$\mu' := \mu_1 \det{}_1 + \cdots + \mu_n \det{}_n + \Omega e_0.$$

Here $\det_i$ denotes the determinant of the GL_2-factor with unique root $e_{2i-1} - e_{2i}$.

Proof. Let

$$\tilde{\mu} = \mu + \frac{1}{2}(e_1 + \cdots + e_{2n}) + \Omega e_0 = \sum_{i=1}^{n} \left(\mu_i + \frac{1}{2}\right) e_i + \sum_{i=1}^{n} \left(\Omega - \mu_i + \frac{1}{2}\right) e_{n+i} + \Omega e_0.$$

Using the description of the Weyl action in Lemma 3.5.3 it is easily verified that this is in the same orbit as

$$\tilde{\mu}' := \sum_{i=1}^{n} \left[\left(\mu_i + \frac{1}{2}\right) e_{2i-1} + \left(\mu_i - \frac{1}{2}\right) e_{2i} \right] + \Omega e_0.$$

By the definition of the unramified constituent, then,

$$^{un}\operatorname{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)} \tilde{\mu} = {}^{un}\operatorname{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)} \tilde{\mu}'.$$

The lemma now follows from the well known (and easily verified) fact that

$$(7.0.9) \quad {}^{un}Ind_{B(GL_2)(F)}^{GL_2(F)} (\mu + \frac{1}{2})e_1' + (\mu - \frac{1}{2})e_2' = {}^{un} Ind_{B(GL_2)(F)}^{GL_2(F)} (\mu - \frac{1}{2})e_1' + (\mu + \frac{1}{2})e_2' = \mu \det,$$

where e_1' and e_2' are the usual basis for the lattice of rational characters of the torus of diagonal elements of GL_2. □

The next lemma is similar, but slightly more complicated. It makes use of alternative $\mathbb{Z}$-bases of the lattices of characters and cocharacters. Specifically, $\{e_1, \dots e_{2n-2}, f_1, f_2, f_0\}$, $\{e_1^*, \dots e_{2n-2}^*, f_1^*, f_2^*, f_0^*\}$, where

$$\begin{aligned} e_0 &= -f_1 & e_0^* &= -2f_0^* - f_1^* - f_2^* \\ e_{2n-1} &= -f_0 + f_1 + f_2 & e_{2n-1}^* &= -f_0^* \\ e_{2n} &= f_1 - f_2 & e_{2n}^* &= -f_0^* - f_2^*. \end{aligned}$$

The key feature of these $\mathbb{Z}$-bases is as follows. Recall that the group G_{4n+1} has a unique standard Levi isomorphic to $GL_2^{n-1} \times G_5$, with the based root datum of the G_5 component lying in the sublattices spanned by $\{e_{2n-1}, e_{2n}, e_0\}$, $\{e_{2n-1}^*, e_{2n}^*, e_0^*\}$. Now, G_5 and GSp_4 are the same F-group. When we write the based root datum of this Levi with respect to the new basis, the expression for the G_5 component matches the "standard form" for the based root datum of GSp_4 as in section 3.4. In particular, the character f_0 is the restriction to the torus of GSp_4 of the similitude factor (which is a generator for the rank-one lattice of rational characters of GSp_4), and there is a standard Levi, isomorphic to GL_2 such that its unique root is $f_1 - f_2$.

Remarks 7.0.10. *To avoid confusion, let us draw attention the following tricky point: we have defined a notion of "Siegel parabolic" and "Siegel Levi" for G_{2n+1}, any n. There is also a well known notion of "Siegel parabolic" and "Siegel Levi" for GSp_{2n}, any n, which is very widespread in the literature. The two groups G_5 and GSp_4 happen to coincide,* and the two notions of "Siegel parabolic" and "Siegel Levi" do **not.**

Lemma 7.0.11. *If μ is of the form (7.0.6), and $\tilde{\mu}$ is defined in terms of μ as in the proof of Lemma 7.0.8, then*

$$^{un}\,\mathrm{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)}\,\tilde{\mu} \cong {}^{un}\,\mathrm{Ind}_{P_2(F)}^{G_{4n+1}(F)}\,\mu''$$

where

$$\begin{aligned}\mu'' &= \sum_{i=1}^{n-1}\mu_i \det{}_i - \frac{\Omega-1}{2}f_0 + \left(-\frac{1}{2}+\frac{\pi i}{\log q_F}\right)\det{}_0\\ &= \sum_{i=1}^{n-1}\mu_i \det{}_i + \frac{\Omega-1}{2}e_{2n+1} - \left(\frac{\Omega}{2}+\frac{\pi i}{\log q_F}\right)\det{}_0\end{aligned}$$

where the notation is as follows: P_2 is the standard parabolic with Levi isomorphic to $GL_2^n \times GL_1$, such that the roots of the Levi are $e_{2i-1}-e_{2i}$, $i=1$ to $n-1$, and e_{2n}. (One might also describe this Levi as $GL_2^{n-1}\times GL_1 \times GSpin_3$.) As in Lemma 7.0.8 $\det{}_i$ *denotes the determinant of the GL_2-factor with unique root $e_{2i-1}-e_{2i}$, for $i=1$ to $n-1$, while* $\det{}_0$ *denotes the determinant of the GL_2 with unique root $e_{2n}=f_1-f_2$.*

Proof. This time $\tilde{\mu}$ is in the same Weyl orbit as

$$\begin{aligned}\tilde{\mu}'' &:= \sum_{i=1}^{n-1}\left[\left(\mu_i+\frac{1}{2}\right)e_{2i-1}+\left(\mu_i-\frac{1}{2}\right)e_{2i}\right]+\left(\frac{\Omega-1}{2}\right)e_{2n-1}+\left(\frac{\Omega-1}{2}+\frac{\pi i}{\log q_F}\right)e_{2n}+\Omega e_0\\ &= \sum_{i=1}^{n-1}\left[\left(\mu_i+\frac{1}{2}\right)e_{2i-1}+\left(\mu_i-\frac{1}{2}\right)e_{2i}\right]-\left(\frac{\Omega-1}{2}\right)f_0+\left(-1+\frac{\pi i}{\log q_F}\right)f_1-\frac{\pi i}{\log q_F}f_2.\end{aligned}$$

Using (7.0.9) again, in conjunction with the fact that $-\frac{\pi i}{\log q_F}f_2$ and $\frac{\pi i}{\log q_F}f_2$ are the same character, we obtain the lemma. □

Next, we need a slight extension of this. Let P_3 be the standard parabolic of G_{4n+1} with Levi isomorphic to $GL_2^{n-1}\times GSp_4$. Identify GSp_4 with the component of this Levi, and let $R = GSp_4 \cap P_2$. This is the subgroup known in the literature as the "Siegel" parabolic of GSp_4. When regarded as a parabolic of $GSpin_5$, it is the one for which we have introduced the notation $Q_1 = L_1N_1$. Its lattice of rational characters is spanned by f_0 and $\det{}_0$, defined as in Lemma 7.0.11. Let $\pi_0 = {}^{un}\,\mathrm{Ind}_{R(F)}^{GSp_4(F)}\left(\frac{1}{2}+\frac{\pi i}{\log q_F}\right)\det{}_0$. Extend π_0 trivially to a representation of the Levi of P_3.

Corollary 7.0.12.

$$^{un}\,\mathrm{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)}\,\tilde{\mu}'' \cong {}^{un}\,\mathrm{Ind}_{P_3(F)}^{G_{4n+1}(F)}\,\mu'''\otimes\pi_0,$$

where

$$\mu''' := \left(\sum_{i=1}^{n-1}\mu_i\det{}_i - \frac{\Omega-1}{2}f_0\right).$$

Proof. Induction in stages and the definition of the unramified constituent. □

An important fact about π_0 is the following:

Lemma 7.0.13. *The representation π_0 may be realized as a subrepresentation of*

$$\mathrm{Ind}_{R(F)}^{GSp_4(F)}\left(-\frac{1}{2}+\frac{\pi i}{\log q_F}\right)\det{}_0.$$

Proof. In fact, it is one of the spaces $R_2(V)$ introduced on p. 223 of [K-R]. This can be checked by direct computation. It also follows from Proposition 5.5 of [K-R], in that the intertwining operator is easily seen not to vanish on the spherical vector. □

Corollary 7.0.14. *The representation* $\mathrm{Ind}_{P_3(F)}^{G_{4n+1}(F)} \mu''' \otimes \pi_0$ *may be realized as a subrepresentation of*

$$\mathrm{Ind}_{P_2(F)}^{G_{4n+1}(F)} \mu''' + \left(-\frac{1}{2} + \frac{\pi i}{\log q_F}\right) \det{}_0.$$

A second important fact about the representation π_0 is the following:

Lemma 7.0.15. *Let ϑ be a character of the unipotent radical of R in general position. Regarding R as the parabolic Q_1 of G_5, the square class* $\mathrm{Invt}(\vartheta)$ *is defined. A sufficient condition for the vanishing of the twisted Jacquet module $\mathcal{J}_{N_1,\vartheta}(\pi_0)$ is that the Hilbert symbol* $(\cdot, \mathrm{Invt}(\vartheta))$ *not equal the unique nontrivial unramified quadratic character.*

Proof. This follows from [K-R], Lemma 3.5 (b), p. 226. (Here, we again use the fact that the unramified constituent of $\mathrm{Ind}_{R(F)}^{GSp_4(F)} \left(\frac{1}{2} + \frac{\pi i}{\log q_F}\right) \det{}_0$ is one of the spaces $R_2(V)$ introduced on p. 223 of [K-R].) $\square$

Proposition 7.0.16. *Let $\tau = Ind_{B(GL_{2n})(F)}^{GL_{2n}(F)} \mu$, with μ of the form (7.0.5), and let P denote the Siegel parabolic subgroup. Then for $\ell > n$ and ϑ in general postion, the Jacquet module $\mathcal{J}_{N_\ell,\vartheta}({}^{un}Ind_{P(F)}^{G_{4n+1}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega)$ is trivial. The same is true if $\ell = n$ and* $\mathrm{Invt}(\vartheta) \neq \square$.

Proof. By Lemma 7.0.8, it suffices to prove that the corresponding Jacquet module of $\mathrm{Ind}_{P_1(F)}^{G_{4n+1}(F)} \mu'$ vanishes. The space $Ind_{P_1(F)}^{G_{4n+1}(F)} \mu'$ has a filtration as a $Q_\ell(F)$-module, in terms of $Q_\ell(F)$-modules indexed by the elements of $(W \cap P_1)\backslash W/(W \cap Q_\ell)$. For any element x of $P_1(F)wQ_\ell F)$ the module corresponding to w is isomorphic to $c-ind_{x^{-1}P_1(F)x \cap Q_\ell(F)}^{Q_\ell(F)} (\mu' + \rho_{P_1}) \circ Ad(x)$. Here $Ad(x)$ denotes the map given by conjugation by x. It sends $x^{-1}P_1(F)x \cap Q_\ell(F)$ into $P_1(F)$. Also, here and throughout $c - ind$ denotes non-normalized compact induction. (See [Cass], section 6.3.)

Recall from 3.5 that the elements of the Weyl group of G_{4n+1} are (after the choice of pr) in natural one-to-one correspondence with the set of permutations $w \in \mathfrak{S}_{4n+1}$ satisfying,

(1) $w(4n + 1 - i) = 4n + 1 - w(i)$

As representatives for the double cosets $(W \cap P_1)\backslash W/(W \cap Q_\ell)$ we choose the element of minimal length in each. The permutations corresponding to these elements satisfy

(2) $w^{-1}(2i) > w^{-1}(2i - 1)$ for $i = 1$ to $2n$, and

(3) $\ell < i < j < 4n + 2 - \ell \implies w(i) < w(j)$.

Let I_w be the $Q_\ell(F)$-module obtained as

$$c - ind_{\dot{w}^{-1}P_1(F)\dot{w} \cap Q_\ell(F)}^{Q_\ell(F)} \left(\mu' + \rho_{P_1}\right) \circ Ad(\dot{w})$$

using any element $\dot{w}$ of $\mathrm{pr}^{-1}(\det w \cdot w)$. (Cf. section 3.5.)

A function f in I_w will map to zero under the natural projection to $\mathcal{J}_{N_\ell,\vartheta}(I_w)$ iff there exists a compact subgroup N_ℓ^0 of $N_\ell(F)$ such that

$$\int_{N_\ell^0} f(hn)\overline{\vartheta(n)}dn = 0 \qquad \forall h \in Q_\ell(F).$$

(See [Cass], section 3.2.) Let $h \cdot \vartheta(n) = \vartheta(h^{-1}nh)$. It is easy to see that the integral above vanishes for suitable N_ℓ^0 whenever

(7.0.17) $$h \cdot \vartheta|_{N_\ell(F) \cap w^{-1}P_1(F)w} \text{ is nontrivial.}$$

Furthermore, the function $h \mapsto h \cdot \vartheta$ is continuous in h, (the topology on the space of characters of $N_\ell(F)$ being defined by identifying it with a finite dimensional F-vector space, cf. section 3.8) so if this condition holds for all h in a compact set, then N_ℓ^0 can be made uniform in h.

Now, ϑ is in general position. Hence, so is $h \cdot \vartheta$ for every h. So, if we write

$$h \cdot \vartheta(u) = \psi_0(c_1 u_{1,2} + \cdots + c_{\ell-1} u_{\ell-1,\ell} + d_1 u_{\ell,\ell+1} + \cdots + d_{4n-2\ell+1} u_{\ell,4n-\ell+1}),$$

we have that $c_i \neq 0$ for all i and ${}^t\underline{d}w\underline{d} \neq 0$.

Clearly, the condition (7.0.17) holds for all h unless

(4) $w(1) > w(2) > \cdots > w(\ell)$.

Furthermore, because ${}^t\underline{d}w\underline{d} \neq 0$, there exists some i_0 with $\ell + 1 \leq i_0 \leq 2n$ such that $d_{i_0-\ell} \neq 0$ and $d_{4n+2+\ell-i_0} \neq 0$. From this we deduce that the condition (7.0.17) holds for all h unless w has the additional property

(5) There exists i_0 such that $w(\ell) > w(i_0)$ and $w(\ell) > w(4n + 2 - i_0)$.

However, if $\ell > n$ it is easy to check that no permutations with properties (1),(2), (4) and (5) exist.

Thus $\mathcal{J}_{N_\ell,\vartheta}(I_w) = \{0\}$ for all w and hence $\mathcal{J}_{N_\ell,\vartheta}({}^{un}Ind_{P(F)}^{G_{4n}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega) = \{0\}$ by exactness of the Jacquet functor.

If $\ell = n$, there is exactly one permutation w which satisfies (1)-(4). For this permutation, condition (4) is satisfied only with $i_0 = 4n + 2 - i_0 = 2n + 1$. The orbit of ϑ contains characters such that $d_i = 0$ for all $i \neq 2n + 1$ iff $\mathrm{Invt}(\vartheta) = \square$. $\square$

Proposition 7.0.18. *Let* $\tau = Ind_{B(GL_{2n})(F)}^{GL_{2n}(F)} \mu$*, with* μ *of the form* (7.0.6)*, and let* P *denote the Siegel parabolic subgroup. Then for* $\ell > n$ *and* ϑ *in general postion, the Jacquet module* $\mathcal{J}_{N_\ell,\vartheta}({}^{un}Ind_{P(F)}^{G_{4n+1}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega)$ *is trivial. The same is true if* $\ell = n$ *and* $\mathrm{Invt}(\vartheta) = \square$.

Proof. For $\ell > n$, the proof is similar to that of Proposition 7.0.16. Using Lemma 7.0.11 in place of Lemma 7.0.8, we consider a representation induced from a character of P_2 rather than P_1. The effect is that in place of condition (2) from the proof of Proposition 7.0.16, we have the condition

(2′) $w^{-1}(2i - 1) < w^{-1}(2i)$, $1 \leq i < n$, $\quad w^{-1}(2n) < w^{-1}(2n + 1)$.

The set of permutations satisfying (1),(2′),(3),(4) is again empty.

The proof of vanishing when $\ell = n$ and $\mathrm{Invt}(\vartheta) = \square$ is more nuanced. In this case we use both Lemma 7.0.11 and Corollary 7.0.12, obtaining *two* filtrations of

$$\mathrm{Ind}_{P_3(F)}^{G_{4n+1}(F)} \mu''' \otimes \pi_0 \subset \mathrm{Ind}_{P_2(F)}^{G_{4n+1}(F)} \mu'',$$

indexed by $(W \cap P_3)\backslash W/(W \cap Q_\ell)$ and $(W \cap P_2)\backslash W/(W \cap Q_\ell)$. The latter is a refinement of the former, in a manner which is compatible with the natural projection

$$(W \cap P_2)\backslash W/(W \cap Q_\ell) \to (W \cap P_3)\backslash W/(W \cap Q_\ell).$$

Let us denote the elements of the first filtration by I_w, $w \in (W \cap P_3)\backslash W/(W \cap Q_\ell)$, and the elements of the second by I'_w, $w \in (W \cap P_2)\backslash W/(W \cap Q_\ell)$.

Now, when $\ell = n$ there is a unique permuation w_0 satisfying (1)(2′),(3), (4),(5). It is the shortest element of the double coset containing the longest element of W. It follows that $\mathcal{J}_{N_n,\vartheta}(I'_w)$ vanishes for every $w \neq w_0$, and hence that $\mathcal{J}_{N_n,\vartheta}(I_w)$ vanishes for every w other than the shortest element of $(W \cap P_3) \cdot w_0 \cdot (W \cap Q_n)$, which we denote w'_0.

The permutation w'_0 can be described explicitly as follows:

$$w'_0(i) = \begin{cases} 4n + 2 - 2i & 1 \leq i \leq n - 1, \\ 2n - 1 & i = n, \\ 2i - 2n - 1 & n + 1 \leq i \leq 2n - 1,\ 2n + 3 \leq i \leq 3n + 1, \\ i & 2n \leq i \leq 2n + 2, \\ 2n + 3 & i = 3n + 2, \\ 8n + 4 - 2i & 3n + 3 \leq i \leq 4n + 1. \end{cases}$$

Furthermore, the space $I_{w_0'}$ is equal to the subspace of ${}^{un}\operatorname{Ind}_{P_3(F)}^{G_{4n+1}(F)}\mu'''\otimes\pi_0$ consisting of smooth functions having support in the open double coset $P_3(F)\cdot w_0'\cdot Q_n(F)$. Take such a function f and take $N_n^0\subset N_n(F)$, compact. Consider the integral

$$\int_{N_n^0} f(gn)\overline{\vartheta(n)}\,dn.$$

We may assume $g=w_0'q$ for some $q\in Q_n(F)$. Then we get

$$\int_{qN_n^0q^{-1}} f(w_0nq)\overline{q\cdot\vartheta(n)}\,dn,$$

where $q\cdot\vartheta(n)=\vartheta(q^{-1}nq)$. Hence, we consider

$$\int_{N_n^{0'}} f'(w_0n)\overline{\vartheta'(n)}\,dn, \tag{7.0.19}$$

for ϑ' a character of N_n such that $\operatorname{Invt}(\vartheta')=\square$, $f'\in I_{w_0'}$, and $N_n^{0'}\subset N_n(F)$ compact. Observe that $w_0N_nw_0^{-1}$ contains the unipotent radical U_R of the parabolic R of GSp_4 used to define π_0. Indeed, if $\hat{N}_n=\{u\in N_n: u_{n,2n}=u_{n,2n+1}=0\}$, then $\hat{N}_n$ is a normal subgroup of N_n and $N_n=w_0^{-1}U_Rw_0\cdot\hat{N}_n$. If $U\subset U_{\max}$, write $U(\mathfrak{p}^N)$ for $\{u\in U: u_{ij}\in\mathfrak{p}^N\forall i,j\}$.

For each $h\in G_{4n+1}(F)$, the function $g\mapsto f'(gh),\ g\in GSp_4(F)$ is an element of π_0. By Lemma 7.0.15, for each h there exists N such that

$$\int_{w_0^{-1}U_R(\mathfrak{p}^N)w_0} f'(w_0uh)\vartheta'(u)\,du=0.$$

Clearly, N depends on f' and ϑ', and hence, if $f'(g)=f(g\cdot q)$ and $\vartheta'=q\cdot\vartheta$, on q. However, f is smooth and has support which is compact modulo $P_3(F)$, so f' takes only finitely many values. Furthermore, the $q\cdot\vartheta$ is a continuous function of q in the sense discussed above. Thus, N may be made uniform in q. $\square$

Define a character ψ_n of $N_n(F)$ by the same formula as in Definition 5.1.8. In the proof of Lemma 5.1.10, we fixed a specific isomorphism $\operatorname{inc}: G_{2n}\to(L_n^{\psi_n})^0$. For the next proposition only, we let B denote the image under inc of the Borel $B(G_{2n})$ corresponding to our choices of maximal torus and simple roots for G_{2n}. It is equal to $(L_n^{\psi_n})^0\cap B(G_{4n+1})$. The corresponding maximal torus T is the subtorus $\langle e_i^*: i=0,\ \text{or}\ n+1\le i\le 2n\rangle$. Because of this $\sum_{i=0}^{2n}c_ie_i$ makes sense as a character of $T(F)$. (But depends only on $c_i,\ i=0,\ \text{or}\ n+1\le i\le 2n$.)

Proposition 7.0.20. *Let P_1, and μ' be defined as in Lemma 7.0.8. Then we have isomorphisms*

$$\mathcal{J}_{N_n,\psi_n}(\operatorname{Ind}_{P_1(F)}^{G_{4n+1}(F)}\mu')\cong\operatorname{Ind}_{B(F)}^{(L_n^{\psi_n})(F)}\mu^*\cong\operatorname{Ind}_{B(F)}^{(L_n^{\psi_n})(F)}\mu^{**}\qquad(\textit{ of } L_n^{\psi_n}-\textit{modules}),$$

$$\mathcal{J}_{N_n,\psi_n}(\operatorname{Ind}_{P_1(F)}^{G_{4n+1}(F)}\mu')\cong\operatorname{Ind}_{B(F)}^{(L_n^{\psi_n})^0(F)}\mu^*\oplus\operatorname{Ind}_{B(F)}^{(L_n^{\psi_n})^0(F)}\mu^{**}\qquad(\textit{ of } (L_n^{\psi_n})^0-\textit{modules}),$$

where

$$\mu^*=\sum_{i=1}^{n}\mu_ie_{n+i}+\Omega e_0,\qquad \mu^{**}=\sum_{i=1}^{n-1}\mu_ie_{n+i}+(\Omega-\mu_n)e_{2n}+\Omega e_0.$$

Proof. As before, we filter $\operatorname{Ind}_{P_1(F)}^{G_{4n+1}(F)}\mu'$ in terms of $Q_n(F)$-modules I_w. This time, $\mathcal{J}_{N_n,\psi_n}(I_w)=\{0\}$ for all w except possibly for one. This one Weyl element, which we denote w_0, corresponds to the unique permutation satisfying (1) and (2) of Proposition 7.0.16, together with $w_0(i)=4n-2i+2$ for $i=1$ to n. Exactness yields

$$\mathcal{J}_{N_n,\psi_n}\left({}^{un}Ind_{P(F)}^{G_{4n+1}(F)}\tau\otimes|\det|^{\frac{1}{2}}\boxtimes\omega\right)\cong\mathcal{J}_{N_n,\psi_n}(I_{w_0}).$$

(This is an isomorphism of $Q_n^{\psi_n}(F)$-modules, where $Q_n^{\psi_n} = N_n \cdot L_n^{\psi_n} \subset Q_n$, is the stabilizer of ψ_n in Q_n (cf. L^ϑ above).)

Now, recall that for each $h \in Q_n(F)$ the character $h \cdot \psi_n(u) = \psi_n(h^{-1}uh)$ is a character of N_n in general position, and as such determines coefficients ${}^hc_1, \dots, {}^hc_{n-1}$ and ${}^hd_1, \dots, {}^hd_{2n+1}$ as in (5.1.3). Clearly,

$$Q_n^o := \left\{ h \in Q_n(F) | \, d_i^h \neq 0 \text{ for some } i \neq n+1, \right\}$$

is open. Moreover, one may see from the description of w_0 that for h in this set the condition (7.0.17), which assures vanishing, is satisfied.

We have an exact sequence of $Q_n^{\psi_n}(F)$-modules

$$0 \to I_{w_0}^* \to I_{w_0} \to \bar{I}_{w_0} \to 0,$$

where I_w^* consists of those functions in I_w whose compact support happens to be contained in Q_n^o, and the third arrow is restriction to the complement of Q_n^o. This complement is slightly larger than $Q_n^{\psi_n}(F)$ in that it contains the full torus of $Q_n(F)$, but restriction of functions is an isomorphism of $Q_n^{\psi_n}(F)$-modules,

$$\bar{I}_{w_0} \to c - ind_{Q_n^{\psi_n}(F) \cap w_0^{-1} P_1(F) w_0}^{Q_n^{\psi_n}(F)} \left(\mu' + \rho_{P_1} \right) \circ Ad(w_0).$$

Clearly $\mathcal{J}_{N_n, \psi_n} \left(I_{w_0}^* \right) = \{0\}$, and hence we have the isomorphism

$$\mathcal{J}_{N_n, \psi_n} \left(Ind_{P_1(F)}^{G_{4n+1}(F)} \mu' \right) \cong \mathcal{J}_{N_n, \psi_n} \left(c - ind_{Q_n^{\psi_n}(F) \cap w_0^{-1} P_1(F) w_0}^{Q_n^{\psi_n}(F)} \left(\mu' + \rho_{P_1} \right) \circ Ad(w_0) \right)$$

of $Q_n^{\psi_n}$-modules.

Let us denote

$$c - ind_{Q_n^{\psi_n(F)} \cap w_0^{-1} P_1(F) w_0}^{Q_n^{\psi_n}(F)} \left(\mu' + \rho_{P_1} \right) \circ Ad(w_0)$$

by V. A straightforward computation shows that the functions in V satisfy

$$f(bq) = b^{\mu^* + \rho_B - J} f(q) \qquad \forall b \in B(F), \; q \in Q_n^{\psi_n}(F),$$

where

$$J = \sum_{i=1}^{n} (i - n - 1) e_{n+i}.$$

For $f \in V$, let

$$W(f)(q) = \int_{N_n(F) \cap w_0^{-1} \overline{U_{\max}}(F) w_0} f(uq) \bar{\psi}_n(u) du.$$

Then the character J computed above matches exactly the Jacobian of $Ad(b), \; b \in B(F)$, acting on $N_n(F) \cap w_0^{-1} \overline{U_{\max}}(F) w_0$. It follows that

$$W(f)(bq) = b^{\mu^* + \rho_B} f(g) \qquad \forall \; b \in B(F), \; q \in Q(F).$$

Now let $\mathcal{W}$ denote

$$\left\{ f : Q_n^{\psi_n}(F) \to \mathbb{C} \;\middle|\; \begin{array}{l} f(uq) = \psi_n(u) f(q) \; \forall \; u \in N_n(F), \; q \in Q_n^{\psi_n}(F), \\ f(bm) = b^{\mu^* + \rho_B} f(m) \; \forall \; b \in B(F), \; m \in L_n^{\psi_n}(F) \end{array} \right\}.$$

Then W maps V into $\mathcal{W}$.

Denote by $V(N_n, \psi_n)$ the kernel of the linear map $V \to \mathcal{J}_{N_n, \psi_n}(V)$. It is easy to show that $V(N_n, \psi_n)$ is contained in the kernel of W. In the Lemma 7.0.22 below, we show that in fact, they are equal. Restriction from $Q_n^{\psi_n}(F)$ to $L_n^{\psi_n}(F)$ is clearly an isomorphism $\mathcal{W} \to \mathrm{Ind}_{B(F)}^{L_n^{\psi_n}(F)} \mu^*$.

The proof that this is isomorphic to $\mathrm{Ind}_{B(F)}^{L_n^{\psi_n}(F)} \mu^{**}$ and decomposes into $(L_n^{\psi_n})^0$-modules in the manner described is straightforward. $\square$

The next proposition is similar. However, there is an interesting difference between the two. In the previous proposition, we let B denote the Borel subgroup $B(G_{4n+1}) \cap (L_n^{\psi_n})^0$ of $(L_n^{\psi_n})^0 \cong G_{2n}^{\mathbf{1}}$. For the next, we use it to denote $Q_{2n-1} \cap G_{2n}^a$, which is a Borel subgroup of G_{2n}^a. The corresponding maximal torus, $G_{2n}^a \cap L_{2n-1}$, is given by

$$\left\{ h_a \prod_{i=1}^{n-1} e_{n+i}^*(t_i) \cdot e_{2n}^* \left((x+y\sqrt{a}) \cdot (x-y\sqrt{a})^{-1} \right) e_0(x-y\sqrt{a}) h_a^{-1} : t_i \in F, \ x,y \in F, x^2-ay^2 \neq 0 \right\},$$

as in Lemma 5.1.10(3). Here $\sqrt{a}$ may be taken to be either of the solutions to $\zeta^2 = a$ in the algebraic closure of F. We assume $\sqrt{a} \notin F$. The lattice of F-rational characters of this torus is $\langle e_{n+i} : 1 \le i \le n-1, \ e_{2n} + 2e_0 \rangle$. The character $e_{2n} + 2e_0$ is the restriction of a rational character of the $L_{2n-1} \cong GL_1^{2n-1} \times GSpin_3$. To be precise, it is the *inverse* of the character $\det_0$ introduced earlier. (Cf. Lemma 7.0.11.) Thus, a general rational character of this torus may be expressed as

$$\sum_{i=1}^{n-1} c_i e_{n+i} + c_0 \det{}_0,$$

with $c_i \in \mathbb{Z}$. In particular map, the restriction map from $X(L_{2n-1})$ is surjective. A general unramified character of this torus may be expressed in the same form with $c_i \in \mathbb{C}$. Then c_i

Observe that for any t in this torus $\det{}_0(t)$ is a norm from $F(\sqrt{a})$. When a is in the square class which contains the non-square units (i.e., when $F(\sqrt{a})$ is the unique unramified quadratic extension of F,) the absolute value of a norm is always an even power of q_F, and so c_0 is defined only up to $\frac{\pi i}{\log q_F}$. (whereas the others are defined up to $\frac{2\pi i}{\log q_F}$ for $1 \le i \le n-1$.)

We also let $\tilde{B}$ denote $Q_{2n-1} \cap L_n^{\psi_n^a}$. (Recall that $G_{2n}^a := (L_n^{\psi_n^a})^0$.) It is not difficult to see that $L_{2n-1} \cap L_n^{\psi_n^a}$ is properly larger that $L_{2n-1} \cap (L_n^{\psi_n^a})^0$, i.e., contains elements of the non-identity component of $L_n^{\psi_n^a}$. A character of B may be extended trivially to $\tilde{B}$. And any character of $\tilde{B}$ which is obtained as the restriction of a character of Q_{2n-1} is such a trivial extension.

Proposition 7.0.21. *Let P_2, and μ'' be defined as in Lemma 7.0.11. Then we have isomorphisms*

$$\mathcal{J}_{N_n,\psi_n}(\mathrm{Ind}_{P_2(F)}^{G_{4n+1}(F)} \mu'') \cong \mathrm{Ind}_{\tilde{B}(F)}^{(L_n^{\psi_n^a})(F)} \mu^* \qquad (\text{ of } L_n^{\psi_n^a} - \text{modules}),$$

$$\mathcal{J}_{N_n,\psi_n}(\mathrm{Ind}_{P_2(F)}^{G_{4n+1}(F)} \mu'') \cong \mathrm{Ind}_{B(F)}^{G_{2n}^a(F)} \mu^* \qquad (\text{ of } (L_n^{\psi_n})^0 - \text{modules}),$$

where

$$\mu^* = \sum_{i=1}^{n-1} \mu_i e_{n+i} - \left(\frac{\Omega}{2} + \frac{\pi i}{\log q_F} \right) \det{}_0.$$

Proof. We use Lemma 7.0.11, and filter by Q_n-modules. As in Proposition 7.0.20, there is a unique permutation w_1 such that the corresponding Q_n-module I_{w_1} does not vanish. This permutation is given by

$$w_1(i) = \begin{cases} 4n+2-2i & 1 \le i \le n-1, \\ 2n+3 & i = n, \\ 2i-2n-1 & n+1 \le i \le 2n-1, \\ i & 2n \le i \le 2n+2, \\ 2i-2n-1 & 2n+3 \le i \le 3n+1, \\ 2n-1 & i = 3n+2, \\ 2(4n+2-i) & 3n+3 \le i \le 4n+1. \end{cases}$$

The group $Q_n \cap w_1^{-1} P_2 w_1$ contains L_{2n-1}. Since $L_{2n-1} \cdot Q_n^{\psi_n^a} = Q_n$, restriction of functions is an isomorphism of $Q_n^{\psi_n^a}$-modules,

$$I_{w_1} \to c - ind_{Q_n^{\psi_n^a} \cap w_1^{-1} P_2 w_1}^{Q_n^{\psi_n^a}} (\mu'' + \rho_{P_2}) \circ \mathrm{Ad}(w_1).$$

This time, let V denote

$$c - ind_{Q_n^{\psi_n^a} \cap w_1^{-1} P_2 w_1}^{Q_n^{\psi_n^a}} (\mu'' + \rho_{P_2}) \circ \mathrm{Ad}(w_1).$$

Once again the functions in V satisfy

$$f(bq) = b^{\mu^* + \rho_B - J} f(q) \qquad \forall b \in B(F),\ q \in Q_n^{\psi_n}(F),$$

with J as before. We define

$$W(f)(q) = \int_{N_n(F) \cap w_1^{-1} \overline{U_{\max}}(F) w_1} f(uq) \bar{\psi}_n(u) du,$$

and find that W maps V to

$$\mathcal{W} := \left\{ f : Q_n^{\psi_n}(F) \to \mathbb{C} \;\middle|\; \begin{array}{l} f(uq) = \psi_n(u) f(q)\ \forall\ u \in N_n(F),\ q \in Q_n^{\psi_n}(F), \\ f(bm) = b^{\mu^* + \rho_B} f(m)\ \forall\ b \in B(F),\ m \in L_n^{\psi_n}(F) \end{array} \right\},$$

which is easily seen to be isomorphic to each of the induced representations specified. As before, the kernel of the linear map $V \to \mathcal{J}_{N_n,\psi_n}(V)$ is contained in the kernel of W. In Lemma 7.0.22, we show that in fact, they are equal to complete the proof. $\square$

Lemma 7.0.22. *Let ϑ be a character of N_n in general position, H its stabilizer in L_n, U_1 and U_2 two subgroups of N_n such that $U_1 \cap U_2 = 1$ and $U_1 U_2 = U_2 U_1 = N_n$. Let B denote a Borel subgroup of the identity component of H and χ a character of B. Assume*

$$B(F) H(\mathfrak{o}) = H(F). \tag{7.0.23}$$

Let V denote a space of functions on $N_n(F) \cdot H(F)$ which are compactly supported modulo $U_1(F)$ on the left and satisfy

$$f(u_1 bq) = \chi(b) f(q) \qquad \forall u_1 \in U_1(F),\ b \in B(F),\ q \in H(F) N_n(F).$$

Let $V(N_n, \vartheta)$ denote the kernel of the usual projection from V to its twisted Jacquet module.

Let

$$W(f)(q) = \int_{U_2(F)} f(u_2 q) \bar{\vartheta}(u_2) du_2.$$

Then $Ker(W) \subset V(N_n, \vartheta)$.

Proof. We assume that

$$\int_{U_2(F)} f(uq)\bar{\vartheta}(u)du = 0,$$

for all $q \in H(F)N_n(F)$. What must be shown is that there is a compact subset C of $N_n(F)$ such that

$$\int_C f(gu)\bar{\vartheta}(u)du = 0,$$

for all $q \in H(F)N_n(F)$.

Consider first $m \in H(\mathfrak{o})$. Let $\mathfrak{p}$ denote the unique maximal ideal in $\mathfrak{o}$. If U is a unipotent subgroup and M an integer, we define

$$U(\mathfrak{p}^M) = \{u \in U(F) : u_{ij} \in \mathfrak{p}^M \ \forall i \neq j\}.$$

Observe that for each $M \in \mathbb{N}$, $N_n(\mathfrak{p}^M)$ is a subgroup of $N_n(F)$ which is preserved by conjugation by elements of $H(\mathfrak{o})$. We may choose M sufficiently large that $supp(f) \subset U_1(F)U_2(\mathfrak{p}^{-M})H(F)$. Then we prove the desired assertion with $C = N_n(\mathfrak{p}^{-M})$. Indeed, for $m \in H(\mathfrak{o})$, we have

$$\int_{N_n(\mathfrak{p}^{-M})} f(mu)\bar{\vartheta}(u)du = \int_{N(\mathfrak{p}^{-M})} f(um)\bar{\vartheta}(u)du,$$

because $Ad(m)$ preserves the subgroup $N_n(\mathfrak{p}^{-M})$, and has Jacobian 1. Let $c = \mathrm{Vol}(U_1(\mathfrak{p}^{-M}))$, which is finite. Then by U_1-invariance of f, the above equals

$$= c\int_{U_2(\mathfrak{p}^{-M})} f(um)\bar{\vartheta}(u)du.$$

This, in turn, is equal to

$$= c\int_{U_2(F)} f(um)\bar{\vartheta}(u)du,$$

since none of the points we have added to the domain of integration are in the support of f, and this last integral is equal to zero by hypothesis.

Next, suppose $q = u_2 m$ with $u_2 \in U_2(F)$ and $m \in H(\mathfrak{o})$. If $u_2 \in U_2(F) - U_2(\mathfrak{p}^{-M})$ then qu is not in the support of f for any $u \in U_2(\mathfrak{p}^{-M})$. On the other hand, if $u_2 \in U_2(\mathfrak{p}^{-M})$, then

$$\int_{N_n(\mathfrak{p}^{-M})} f(u_2 mu)\bar{\vartheta}(u)du = \int_{N_n(\mathfrak{p}^{-M})} f(u_2 um)\bar{\vartheta}(u)du$$

$$= \vartheta(u_2)\int_{N_n(\mathfrak{p}^{-M})} f(um)\bar{\vartheta}(u)du,$$

and now we continue as in the case $u_1 = 1$.

The result for general q now follows from the left-equivariance properties of f and (7.0.23). □

8. Appendix III: Identities of Unipotent Periods

8.1. **A Lemma Regarding Unipotent Periods.** We begin with a few remarks which are valid in the setting of section 3.8. There is a natural action of $G(F)$ on the space of unipotent periods $\mathcal{U}$ given by $\gamma \cdot (U, \psi) = (\gamma U \gamma^{-1}, \gamma \cdot \psi)$ where $\gamma \cdot \psi(u) = \psi(\gamma^{-1} u \gamma)$. We shall refer to this action as "conjugation." Obviously, unipotent periods which are conjugate are equivalent.

Lemma 8.1.1. *Suppose $U_1 \supset U_2 \supset (U_1, U_1)$ are unipotent subgroups of a reductive algebraic group G. Suppose H is a subgroup of G and let f be a smooth left $H(F)$-invariant function on $G(\mathbb{A})$. Suppose ψ_2 is a character of U_2 such that $\psi_2|_{(U_1,U_1)} \equiv 0$. Then the set $\mathrm{res}^{-1}(\psi_2)$ of characters of U_1 such that the restriction to U_2 is ψ_2 is nontrivial. (Here "res" is for "restriction" not "residue".) The elements of $\mathrm{res}^{-1}(\psi_2)$ are permuted by the action of $N_H(U_1)(F)$. The following are equivalent.*

(1) $f^{(U_2,\psi_2)} \equiv 0$
(2) $f^{(U_1,\psi_1)} \equiv 0 \; \forall \psi_1 \in \mathrm{res}^{-1}(\psi_2)$
(3) *For each $N_H(U_1)(F)$-orbit $\mathcal{O}$ in $\mathrm{res}^{-1}(\psi_2)$ $\exists \psi_1 \in \mathcal{O}$ with $f^{(U_1,\psi_1)} \equiv 0$*

Proof. It is obvious that 1 implies 2 and 3, and that 2 and 3 are equivalent. Consider

$$f^{(U_2,\psi_2)}(u_1 g) = \int_{U_2(F\backslash\mathbb{A})} f(u_2 u_1 g)\psi_2(u_2)du_2,$$

regarded as a function of u_1. It is left u_2 invariant and hence gives rise to a function of the compact abelian group $U_2(\mathbb{A})U_1(F)\backslash U_1(\mathbb{A})$. Denote this function by $\phi(u_1)$. Then

$$\phi(0) = \sum_{\chi} \int_{U_2(\mathbb{A})U_1(F)\backslash U_1(\mathbb{A})} \phi(u_1)\chi(u_1)du_1,$$

where "0" denotes the identity in $U_2(\mathbb{A})U_1(F)\backslash U_1(\mathbb{A})$, and the sum is over characters of $U_2(\mathbb{A})U_1(F)\backslash U_1(\mathbb{A})$. This, in turn, is equal to

$$\sum_{\chi} \int_D \int_{U_2(F\backslash\mathbb{A})} f(u_2 u_1 g)\psi_2(u_2)du_2 \chi(u_1)du_1,$$

for D a fundamental domain for the above quotient in $U_1(\mathbb{A})$. The group $U_1/(U_1,U_1)(F)$ is an F-vector space (cf. section 3.8) which can be decomposed into $U_2/(U_1,U_1)(F)$ and a complement. The F-dual of this vector space is identified, via the choice of ψ_0, with the space of characters of $U_1(\mathbb{A})$ which are trivial on $U_1(F)$. It follows that the sum above is equal to

$$= \sum_{\psi_1 \in \mathrm{res}^{-1}(\psi_2)} \int_{U_1(F\backslash\mathbb{A})} f(u_1 g)\psi_1(u_1)du_1.$$

The matter of replacing the sum over χ by one over $\psi_1 \in \mathrm{res}^{-1}(\psi_2)$ is clear from regarding $U_1/(U_1,U_1)(F)$ as a vector space which can be decomposed into $U_2/(U_1,U_1)$ and a complement. Now $2 \Rightarrow 1$ is immediate. $\square$

Corollary 8.1.2. *If $N_G(H)$ permutes the elements of $\mathrm{res}^{-1}(\psi_2)$ transitively, then $(U_2,\psi_2) \sim (U_2,\psi_1)$ for every $\psi_1 \in \mathrm{res}^{-1}(\psi_2)$.*

Definition 8.1.3. *Many of the applications of the above corollary are of a special type, and it will be convenient to introduce a term for them. The special situation is the following: one has three unipotent periods (U_i,ψ_i) for $i = 1,2,3$, such that $U_2 = U_1 \cap U_3$ and $\psi_1|_{U_2} = \psi_3|_{U_2} = \psi_2$. Furthermore, U_1 normalizes U_3 and permutes transitively, the set of characters ψ_3' such that $\psi_3'|_{U_2}$, and the same is true with the roles of 1 and 3 reversed. In this situation, the identity*

$$(U_1,\psi_1) \sim (U_2,\psi_2) \sim (U_3,\psi_3),$$

(which follows from Corollary 8.1.2) will be called a **swap,** *and we say that (U_1,ψ_1) "may be swapped for" (U_3,ψ_3), and vice versa.*

8.2. **A lemma regarding the projection, and a remark.**

Lemma 8.2.1. *The action of G_m on itself by conjugation factors through* pr.

Proof. One has only to check that the kernel of pr is in the center of G_m. When we regard G_m as a quotient of $Spin_m \times GL_1$, the quotient of pr is precisely the image of the GL_1 factor in the quotient. $\square$

Corollary 8.2.2. *Let u be a unipotent element of $G_m(\mathbb{A})$ and g any element of $G_m(\mathbb{A})$. Then* $\mathrm{pr}(gug^{-1})$ *is a unipotent element of $SO_m(\mathbb{A})$ and gug^{-1} is the unique unipotent element of its preimage in $G_m(\mathbb{A})$.*

Remark 8.2.3. *This fact, combined with the fact that* pr *is an isomorphism of varieties when restricted to the subvariety of unipotent elements of G_m, means that many statements may be proved for GSpin groups simply by taking the proof of the corresponding statement for special orthogonal groups and inserting the words "any preimage of" here and there.*

8.3. **Relations among Unipotent Periods used in Theorem 5.1.15.**

Lemma 8.3.1. *Let (U_1^a, ψ_1^a) and (U_2, ψ_2^a) be defined as in Theorem 5.1.15. Then $(U_1^a, \psi_1^a) \sim (U_2, \psi_2^a)$, for all $a \in F$.*

Proof. We regard a as fixed and omit it from the notation. We define some additional unipotent periods which appear at intermediate stages in the argument. Let U_4 be the subgroup defined by $u_{n,j} = 0$ for $j = n$ to $2n-1$ and $u_{2n,2n+1} = 0$. We define a character ψ_4 of U_4 by the same formula as ψ_1. Then (U_1, ψ_1) may be swapped for (U_4, ψ_4). (See definition 8.1.3.)

Now, for each k from 1 to n, define $(U_5^{(k)}, \psi_5^{(k)})$ as follows. First, for each k, the group $U_5^{(k)}$ is contained in the subgroup of $U_{\max}$ defined by, $u_{2n,2n+1} = 0$. In addition, $u_{n+k-1,j} = 0$ for $j < 2n$, and $u_{i,i+1} = 0$ if $n-k+1 \leq i < n+k-1$ and $i \equiv n-k+1 \mod 2$, and $\psi_5^{(k)}(u)$ equals

$$\psi_0 \left(\sum_{i=1}^{n-k} u_{i,i+1} + \sum_{i=n-k+1}^{n+k-2} u_{i,i+2} + u_{n+k-1,2n} + \frac{a}{2} u_{n+k-1,2n+1} + \sum_{i=n+k}^{2n-2} u_{i,i+1} + u_{2n-1,2n+2} \right).$$

(Note that one or more of the sums here may be empty.)

Next, let $U_6^{(k)}$ be the subgroup of $U_{\max}$ defined by the conditions $u_{2n,2n+1} = 0$, $u_{n+k-1,j} = 0$ for $j < 2n$, and $u_{i,i+1} = 0$ if $n-k+1 \leq i < n+k-1$ and $i \equiv n-k \mod 2$. The same formula which defines $\psi_5^{(k)}$ also defines a character of $U_6^{(k)}$. We denote this character by $\psi_6^{(k)}$.

We make the following observations:

- $(U_5^{(1)}, \psi_5^{(1)})$ is precisely (U_4, ψ_4).
- For each k, $(U_5^{(k)}, \psi_5^{(k)})$ is conjugate to $(U_6^{(k+1)}, \psi_6^{(k+1)})$. The conjugation is accomplished by any preimage of the permutation matrix which transposes i and $i+1$ for $n-k \leq i < n+k$ and $i \equiv n-k \mod 2$.
- $(U_6^{(k)}, \psi_6^{(k)})$ may be swapped for $(U_5^{(k)}, \psi_5^{(k)})$.

Thus $(U_4, \psi_4) \sim (U_5^{(n-1)}, \psi_5^{(n-1)})$.

Now, let $U_2' = U_5^{(n-1)}$, and let

$$\psi_2'(u) = \psi(u_{1,3} + \cdots + u_{2n-2,2n} + u_{2n-2,2n+1} + \frac{a}{2} u_{2n-1,2n} + u_{2n-1,2n+2}).$$

Then $(U_5^{(n-1)}, \psi_5^{(n-1)}$ is conjugate to (U_2', ψ_2'), which may be swapped for (U_2, ψ_2). ☐

Lemma 8.3.2. *Let (U_3, ψ_3) and (U_2, ψ_2^0) be defined as in Theorem 5.1.15. Then*

$$(U_3, \psi_3) \in \langle (U_2, \psi_2^0), \{(N_\ell, \vartheta) : n \leq \ell < 2n \text{ and } \vartheta \text{ in general position.}\} \rangle.$$

Proof. To prove this assertion we introduce some additional unipotent periods. For $k = n$ to $2n-1$ let $U_7^{(k)}$ denote the subgroup of $U_{\max}$ defined by $u_{2n,2n+1} = 0$, and $u_{j,2n} = 0$ for $k+1 \leq i \leq 2n-1$, and let

$$\psi_7^{(k)}(u) = \psi_0 \left(\sum_{i=1}^{k-1} u_{i,i+1} + u_{k,2n} + \sum_{i=k+1}^{2n-2} u_{i,i+1} + u_{2n-1,2n+2} \right).$$

Let $U_8^{(k)}$ denote the subgroup defined by by $u_{2n-1,2n+1} = 0$, $u_{k,j} = 0$ for $k+1 \leq j < 2n$, and let $U_9^{(k)}$ denote the subgroup defined by the additional condition $u_{k,2n} = 0$. The same formula which defines $\psi_7^{(k)}$ may be used to specify a character of $U_8^{(k)}$, which we denote $\psi_8^{(k)}$. In addition, let

$$\psi_0\left(\sum_{i=1}^{k-1} u_{i,i+1} + u_{k,2n+2} + \sum_{i=k+1}^{2n-1} u_{i,i+1}\right),$$

be denoted by $\tilde{\psi}_8^{(k)}$ for $u \in U_8^{(k)}$ or $\psi_9^{(k)}$ for $u \in U_9^{(k)}$.

Now, we need the following observations:

- $(U_7^{(n)}, \psi_7^{(n)})$ is just the period (U_1^0, ψ_1^0) from theorem 5.1.15, and so is equivalent to (U_2, ψ_2^0) by the previous result.
- For each k, $(U_7^{(n)}, \psi_7^{(n)})$ is conjugate to $(U_9^{(k+1)}, \psi_9^{(k+1)})$. (One conjugates by a preimage of a permutation matrix and then by a toral element to fix a minus sign which is introduced.)
- $(U_8^{(k+1)}, \tilde{\psi}_8^{(k+1)})$ is spanned by $(U_9^{(k+1)}, \psi_9^{(k+1)})$ and $\{(N_k, \vartheta) : \vartheta \text{ in general position}\}$. More precisely, if ϑ is any extension of $\psi_9^{(k+1)}$ which is *not* in general position, then the restriction of ϑ to U_8 is $\tilde{\psi}_8^{(k+1)}$). (Cf. Corollary 8.1.2.)
- $(U_8^{(k)}, \tilde{\psi}_8^{(k)})$ is conjugate to $(U_8^{(k)}, \psi_8^{(k)})$.
- $(U_8^{(k)}, \psi_8^{(k)})$ may be swapped for $(U_7^{(k)}, \psi_7^{(k)})$.

We deduce that (U_2, ψ_2^0) divides $(U_8^{(2n-1)}, \psi_8^{(2n-1)})$, a period which differs from (U_3, ψ_3) only in that integration over $u_{2n,2n+1}$ is omitted. Thus (U_3, ψ_3) is the constant term in the Fourier expansion of $(U_8^{(2n-1)}, \psi_8^{(2n-1)})$, in the variable $u_{2n,2n+1}$, while all of the nonconstant terms are Whittaker integrals with respect to various generic characters of $U_{\max}$. As $\mathcal{E}_{-1}(\tau, \omega)$ is non-generic, they all vanish. The result follows. □

Lemma 8.3.3. *Take $a \in F^\times$. We regard a as fixed throughout and, for the most part we suppress it from the notation. As in Theorem 5.1.15, let V_i denote the unipotent radical of the standard parabolic of G_{4n+1} having Levi isomorphic to $GL_i \times G_{4n-2i+1}$ (for $1 \le i \le 2n$). For $1 \le j < 2n$, let V_i^{4n-2j} denote the unipotent radical of the standard maximal parabolic of G^a_{4n-2j} having Levi isomorphic to $GL_i \times G^a_{4n-2j-2i}$ (for $1 \le i \le 2n-j-2$ in the split case and $1 \le i \le 2n-j-2$ in the nonsplit cases). Let (N_ℓ, ψ^a_ℓ) be the period used to define the descent, as usual, and let $(N_\ell, \psi^a_\ell)^{(4n-2k+1)}$ denote the analogue for $G_{4n-2k+1}$, embedded into G_{4n+1} inside the Levi of a maximal parabolic.*

Then, $(V_k^{2n}, \mathbf{1}) \circ (N_n, \psi_n)$ is an element of

$$\langle (N_{n+k}, \psi_{n+k}), \{(N_{n+j}, \psi_{n+j})^{(4n-2k+2j)} \circ (V_{k-j}, \mathbf{1}) : \ 1 \le j < k\}\rangle.$$

Proof. In this proof, we shall not need to refer to any of the unipotent periods defined previously. On the other hand we will need to consider several new unipotent periods.

Let $m = (m_1, m_2, m_3)$ be a triple of integers satisfying: $0 \le m_1 < m_2 \le m_3 + 1 \le 2n$. We associate to this data a unipotent group U_m and two characters ψ_m, ψ'_m as follows:

- U_m is defined by the condition that $u_{i,j} = 0$ whenever $m_1 < i < m_2 - 1$ and $j < m_2$, or $m_3 < i$,
- $\psi_m(u) = \psi_0\left(\sum_{i=1}^{m_1} u_{i,i+1} + u_{m_1+1,m_2} + \sum_{i=m_2}^{m_3-1} u_{i,i+1} + u_{m_3,2n} + \frac{a}{2} u_{m_3,2n+2}\right)$,
- $\psi'_m(u) = \psi_0\left(\sum_{i=1}^{m_1-1} u_{i,i+1} + u_{m_1,m_2-1} + \sum_{i=m_2=1}^{m_3-1} u_{i,i+1} + u_{m_3,2n} + \frac{a}{2} u_{m_3,2n+2}\right)$.

Then (U_m, ψ'_m) is conjugate to (U_m, ψ_m) and may be swapped for $(U_{m'}, \psi_{m'})$, where $(m_1, m_2, m_3)' = (m_1 - 1, m_2 - 1, m_3)$. Furthermore, for any $k < n$, $(V_k^{2n}, \mathbf{1}) \circ (N_n, \psi_n)$ is an integral over the subgroup of $U_{n,n+k+1,n+k}$ defined by the conditions, $u_{i,2n} = -\frac{a}{2} u_{i,2n+2}$, for $n < i \le n+k$. It may be swapped for the period (U_m, ψ'') corresponding to $m = (n-1, n+k+1, n+k)$, and

$$\psi''(u) = \psi_0\left(\sum_{i=1}^{n-1} u_{i,i+1} + u_{n,2n} + \frac{a}{2}u_{n,2n+2}\right),$$

and this period is conjugate to (U_m, ψ'_m) for this value of m. It follows that $(V_k^{2n}, \mathbf{1}) \circ (N_n, \psi_n)$ is equivalent to (U_m, ψ'_m) for the triple $m = (0, k+2, n+k)$.

Now, it's easy to see that $(U_{(0,1,m_3)}, \psi'_{(0,1,m_3)}) = (N_{m_3}, \psi^a_{m_3})$, and that for $m_2 > 2$ there are two orbits of extensions of $\psi_{(0,m_2,m_3)}$ to $U_{(0,m_2-1,m_3)}$, namely, the one containing $\psi'_{(0,m_2-1,m_3)}$, and the trivial extension, which yields the period $(N_{m_3-m_2+2}, \psi^a_{m_3-m_2+2})^{(4n-2m_2+5)} \circ (V_{m_2-2}, \mathbf{1})$. This proves the assertions regarding all cases except for the two parabolics with Levi isomorphic to $GL_1 \times GL_n$ in the split case.

As noted previously, it is enough to consider one of them, because they are conjugate in G_{4n+1}. Furthermore, we may conjugate by h_a, and use the more convenient embedding of $G_{2n}^{\square}$ into G_{4n+1} as $(L_n^{\psi_n})^0$.

For this case we take $m \in \mathbb{Z}$ with $0 \le m \le n$, and define U_m to be the subgroup of $U_{\max}$ defined by $u_{i,j} = 0$ whenever $m < i < j \le m+n+1$. Take

$$\psi'_m(u) = \psi_0\left(\sum i = 1^{m-1} u_{i,i+1} + u_{m,m+n+1} + \sum_{i=m+n+2}^{2n} u_{i,i+1}\right),$$

$$\psi''_m(u) = \psi_0\left(\sum i = 1^{m} u_{i,i+1} + u_{m+1,m+n+2} + \sum_{i=m+n+3}^{2n} u_{i,i+1}\right).$$

Then $(V_n^{2n}, \mathbf{1}) \circ (N_n, \psi_n) = (U_n, \psi'_n)$. Furthermore (U_m, ψ'_m) is conjugate to (U_m, ψ''_m) and may be swapped for (U_{m-1}, ψ''_{m-1}). Furthermore, (U_0, ψ'_0) is easily seen to be in the span of the periods

$$(U_{\max}^{4n-2k+1}, \vartheta) \circ (V_k, \mathbf{1})$$

for $0 \le k < n$ and ϑ a generic character of the maximal unipotent subgroup of $G_{4n-2k+1}$ (embedded into G_{4n+1}) as a component of a standard Levi as usual. This completes the proof. $\square$

8.4. **Relation of periods on U_2 via theta functions.** The next relation of unipotent periods differs from all the others, both in the nature of the statement and in the nature of the proof. We described in section 3.8 how a character of $U(F\backslash\mathbb{A})$, where U is a unipotent subgroup of a reductive group G, may be thought of as an element of an F-vector space equipped with an algebraic action of $N_G(U)$. For purposes of this discussion it is more useful to identify this character with an element of a space having an action of all of G, which is compatible with the action of $G(F)$ on unipotent periods by conjugation (as in 8.1), and this may be done using the coadjoint representation of G on the F-dual, $\mathfrak{g}_F^*$, of its Lie algebra.

Observe that if an equivalence between (U_1, ψ_1) and (U_2, ψ_2) can be proved using conjugation and swapping, then ψ_1 and ψ_2 correspond to points in the same orbit of $G(F)$ acting on $\mathfrak{g}_F^*$. The manner in which U_1 and U_2 will be related is not as easy to describe, but one may note for example that they will have the same dimension.

So far, we have proved relations of two forms

- Equivalencies, in which the unipotent subgroup U is replaced by another of the same dimension, and the character ψ by another in the same orbit.
- Relations where one replaces U by a group of properly larger dimension, and considers all orbits of extensions of ψ.

The statement that (U_2,ψ_2^0) is spanned by $\{(U_2,\psi_2^a):\ a\in F^\times\}$ is of a different nature, and it is proved by a different method, which was shown to us by David Ginzburg.

Let V denote the subgroup of $U_{\max}$ defined by $u_{2i-1,2i}=0$ for $1\le i\le n$. We begin by defining a certain representation of $V(\mathbb{A})$. It will be convenient to introduce explicitly the isomorphisms of various root subgroups of $U_{\max}$ with $\mathbb{G}_a$ corresponding to our coordinates u_{ij}. Thus, let $x_{ij}:\mathbb{G}_a\to U_{\max}$ be defined by the condition that

$$(x_{ij}(r))_{k,\ell}=\delta_{i,k}\delta_{k,\ell}r,\qquad \text{for } 1\le i<j\le 4n+1-i,\quad 1\le k<\ell\le 4n+1-k.$$

Let U_{ij} denote its image.

The main thing is to define the action of the subgroup of U_H of V consisting of those elements such that $u_{ij}=0$ whenever $i<2n-1$. This subgroup is the product of $U_{2n-1,2n+1},U_{2n-1,2n+2}$ and $U_{2n,2n+1}$. It is a Heisenberg group in three variables, with center $U_{2n-2,2n+2}$.

As is well known, $U_H(\mathbb{A})$ has a unique isomorphism of class of representations π satisfying

$$\pi(x_{2n-1,2n+2}(r))v=\psi_0(r)v,$$

and there is a representation ω_{ψ_0} in this class given by action on the space $\mathcal{S}(\mathbb{A})$ of Schwartz functions on $\mathbb{A}$ such that

$$\omega_{\psi_0}(x_{2n-1,2n+1}(r))\phi(x)=\psi_0(rx)\phi(x),\qquad \text{and}$$
$$\omega_{\psi_0}(x_{2n,2n+1}(r))\phi(x)=\phi(x+r).$$

This may then be extended to an action of all of $V(\mathbb{A})$ by decreeing u acts by the character

$$\psi_{V_2}(u):=\psi_0\left(\sum_{i=1}^{2n-2}u_{i,i+2}+u_{2n-1,2n+2}\right)$$

whenever u is in the subgroup $V_2(\mathbb{A})$ of $V(\mathbb{A})$ defined by $u_{2n-1,2n+1}=u_{2n,2n+1}=0$. Observe that this character is the common restriction of all the characters ψ_2^a.

The group V is the unipotent radical of a certain parabolic R. Let L denote its Levi factor. It acts on the space of characters of V_2. The stabilizer is isomorphic to SL_2. Its image under pr consists of matrices of the form

$$\operatorname{diag}(g,\dots,g,1,{}_tg,\dots,{}_tg),\qquad g\in SL_2.$$

Denote this stabilizer S_ψ. Then ω_{ψ_0} extends to a projective representation of $V(\mathbb{A})\rtimes S_\psi(\mathbb{A})$ or a genuine representation of $V(\mathbb{A})\rtimes\widetilde{S}_\psi(\mathbb{A})$, where $\widetilde{\ }$ denotes the metaplectic double cover. It is known that $S_\psi(F)$ lifts to a subgroup of $\widetilde{S}_\psi(\mathbb{A})$. The representation ω_{ψ_0} has an automorphic realization given by theta functions

$$\theta_\phi^{\psi_0}(g)=\sum_{\xi\in F}\omega_{\psi_0}(g)\phi(\xi).$$

For $\phi\in\mathcal{S}(\mathbb{A})$ and $\varphi\in C^\infty(G_{4n+1}(F\backslash\mathbb{A}))$ we may now define

$$\Theta_\phi(\varphi):S_\psi(F)\backslash\widetilde{S}_\psi(\mathbb{A})\to\mathbb{C}$$

by

$$\Theta_\phi(\varphi)(\tilde h)=\int_{V(F\backslash\mathbb{A})}\varphi(v\tilde{\mathrm{pr}}(\tilde h))\theta_\phi^{\psi_0}(v\tilde h)\,dv,$$

where $\tilde{\mathrm{pr}}$ denotes the projection $\widetilde{S}_\psi(\mathbb{A})\to S_\psi(\mathbb{A})$. Observe that the subgroup U_2 is the product of a codimension one subgroup of V and $U_{\max}^{S_\psi}:=S_\psi\cap U_{\max}$, which is a maximal unipotent of S_ψ. The group $U_{\max}^{S_\psi}(\mathbb{A})$ lifts to a subgroup of $\widetilde{S}_\psi(\mathbb{A})$. For $a\in F$ let

$$\Theta_\phi^a(\varphi)=\int_{U_{\max}^{S_\psi}(F\backslash\mathbb{A})}\Theta_\phi(\varphi)(u)\psi_0(au_{12})\,du.$$

Lemma 8.4.1. *For $a \in F$, and $\varphi \in C^\infty(G_{4n+1}(F\backslash\mathbb{A}))$,*

$$\varphi^{(U_2,\psi_2^a)} \equiv 0 \qquad \Longleftrightarrow \qquad \Theta_\phi^a(\rho(g)\varphi) = 0 \quad \forall\, \phi \in \mathcal{S}(\mathbb{A}).$$

Proof.

$$\Theta_\phi^a(\varphi)(h) = \int_{U_{\max}^{S_\psi}(F\backslash\mathbb{A})} \int_{V(F\backslash\mathbb{A})} \sum_\xi \varphi(vuh)\omega_{\psi_0}(vuh)\phi(\xi) dv\psi_0(au_{1,2})\, du$$

$$\int_{U_{\max}^{S_\psi}(F\backslash\mathbb{A})} \int_{V(F\backslash\mathbb{A})} \sum_\xi \varphi(x_{2n,2n+1}(\xi)vuh)\omega_{\psi_0}(x_{2n,2n+1}(\xi)vuh)\phi(0) dv\psi_0(au_{12})\, du.$$

We may rewrite $U_{\max}^{S_\psi} \cdot V$ as $U_2 \cdot U_{2n,2n+1}$, obtaining

$$\sum_\xi \int_{(F\backslash\mathbb{A})} \int_{U_2(F\backslash\mathbb{A})} \varphi(x_{2n,2n+1}(\xi)u_2x_{2n,2n+1}(r)h)\omega_{\psi_0}(x_{2n,2n+1}(\xi)u_2x_{2n,2n+1}(r)h)\phi(0)du_2\, dr.$$

$$= \sum_\xi \int_{(F\backslash\mathbb{A})} \int_{U_2(F\backslash\mathbb{A})} \varphi(u_2x_{2n,2n+1}(\xi+r)h)\omega_{\psi_0}(u_2x_{2n,2n+1}(\xi r)h)\phi(0)du_2\, dr.$$

$$= \int_{\mathbb{A}} \int_{U_2(F\backslash\mathbb{A})} \varphi(u_2x_{2n,2n+1}(r)h)\omega_{\psi_0}(u_2x_{2n,2n+1}(r)h)\phi(0)du_2\, dr.$$

But from the description of the action ω_{ψ_0} given above we see at once that

$$\omega_{\psi_0}(u_2x_{2n,2n+1}(r)h)\phi(0) = \psi_2^a(u_2)\omega_{\psi_0}(x_{2n,2n+1}(r)h)\phi(0),$$

so we have

$$\int_{\mathbb{A}} \varphi^{(U_2,\psi_2^a)}(x_{2n,2n+1}(r)h)\omega_{\psi_0}(x_{2n,2n+1}(r)h)\phi(0)\, dr.$$

Our assertion now follows, for a smooth function whose integral against every Schwartz function is the zero function (and vice versa). □

Corollary 8.4.2. *Let the group U_2, and the character ψ_2^a for each $a \in F$ be defined as in the main theorem. Then $(U_2, \psi_2^0) \in \langle\{(U_2, \psi_2^a) : a \in F^\times\}\rangle$.*

Proof. In light of lemma 8.4.1, this now follows from the fact that a genuine function on $\widetilde{SL}_2(\mathbb{A})$ can not be equal to its constant term. □

References

[Asg] M. Asgari, Local L-functions for Split Spinor Groups, Canad. J. Math. Vol 54 (4), 2002 pp. 673-693.

[Asg-Sha1] M. Asgari and F. Shahidi, *Generic Transfer For general Spin Groups*, Duke Math. Journal. Vol. 132, No. 1, 2006, pp. 137- 190.

[Asg-Sha2] M. Asgari and F. Shahidi, *Generic Transfer From $GSp(4)$ to $GL(4)$* , Compos. Math. 142 (2006), no. 3, 541–550.

[Banks1] W. Banks, Twisted symmetric-square L-functions and the nonexistence of Siegel zeros on GL(3), Duke Math. J. 87 (1997), no. 2, 343-353.

[Banks2] W. Banks, Exceptional Representations on the Metaplectic Group. Dissertation, Stanford University 1994

[Banks3] W. Banks, private communication.

[BG] D. Bump, D. Ginzburg Symmetric square L-functions on GL(r). Ann. of Math. (2) 136 (1992), no. 1, 137–205.

[BZ1] I. N. Bernšteĭn and A. V. Zelevinskiĭ. Representations of the group $GL(n, F)$, where F is a local non-Archimedean field. *Uspehi Mat. Nauk*, 31(3(189)):5–70, 1976.

[BZ2] I. N. Bernšteĭn and A. V. Zelevinskiĭ, Induced representations of reductive $\mathfrak{p}$-adic groups. I. Ann. Sci. École Norm. Sup. (4) 10 (1977), no. 4, 441–472.

[Car] P. Cartier, Representations of p-adic groups: a survey. Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, pp. 111–155, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979. Available online at `http://www.ams.org/online_bks/pspum331/`

[Cass] W. Casselman, Introduction to the theory of admissible representations of p-adic reductive groups, Available at `http://www.math.ubc.ca/ cass/research.html`

[C-S] W. Casselman, J. Shalika, The unramified principal series of p-adic groups. II. The Whittaker function. Compositio Math. 41 (1980), no. 2, 207–231.

[C-K-PS-S1] J. Cogdell, H. Kim, I. Piatetski-Shapiro, F. Shahidi, *On lifting from classical groups to* GL_N, Publ. Math. Inst. Hautes Études Sci., 93 (2001), pp.5-30.

[C-K-PS-S2] J. Cogdell, H. Kim, I.I. Piatetski-Shapiro, F. Shahidi, Functoriality for the classical groups. Publ. Math. Inst. Hautes Études Sci. No. 99 (2004), 163–233.

[Gan-Tak] W.T. Gan, S. Takeda, The Local Langlands Conjecture for GSp(4), preprint, available at `http://arxiv.org/abs/0706.0952`

[Gel-Sha] S. Gelbart, F. Shahidi, *Analytic properties of automorphic L-functions.* Perspectives in Mathematics, 6. Academic Press, Inc., Boston, MA, 1988.

[Gel-So] On Whittaker models and the vanishing of Fourier coefficients of cusp forms, Proc. Indian Acad. Sci. Math. Sci. 97 (1987), no. 1-3, 67–74 (1988).

[GK75] I. M. Gelfand and D. A. Kajdan. Representations of the group $\mathrm{GL}(n, K)$ where K is a local field. In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 95–118. Halsted, New York, 1975.

[Gi-PS-R] D. Ginzburg, I.I. Piatetski-Shapiro, and S. Rallis, L functions for the orthogonal group. Mem. Amer. Math. Soc. 128 (1997), no. 611, MR1357823 (98m:11041)

[GRS1] D. Ginzburg, S. Rallis, and D. Soudry, *On a correspondence between cuspidal representations of* GL_{2n} *and* $\widetilde{\mathrm{Sp}}_{2n}$,, J. Amer. Math. Soc.,12(1999), n.3, pp. 849-907

[GRS2] D. Ginzburg, S. Rallis, and D. Soudry, *Lifting cusp forms on* GL_{2n} *to* $\widetilde{\mathrm{Sp}}_{2n}$*: the unramified correspondence*, Duke Math. J., 100 (1999),n. 2, pp. 243-266

[GRS3] D. Ginzburg, S. Rallis and D. Soudry, *On explicit lifts from cusp forms from GL_m to classical groups* , Annals of Math., 150(1999) 807-866. Available online at `http://arxiv.org/pdf/math.NT/9911264`

[GRS4] D. Ginzburg, S. Rallis, D. Soudry , *Generic automorphic forms on* $\mathrm{SO}(2n+1)$*: functorial lift to* $\mathrm{GL}(2n)$*, endoscopy, and base change*, Internat. Math. Res. Notices, 14 (2001), pp.729-764.

[GRS5] D. Ginzburg, S. Rallis, and D. Soudry, *Endoscopic representations of* $\widetilde{\mathrm{Sp}}_{2n}$, J. Inst. Math. Jussieu, vol. 1 2002, no. 1, pp. 77-123.

[GRS6] D. Ginzburg, S. Rallis, D. Soudry, On the automorphic theta representation for simply laced groups. Israel Journal of Mathematics 100 (1997), 61-116.

[H-T] M. Harris, and R. Taylor, *The geometry and cohomology of some simple Shimura varieties.* With an appendix by Vladimir G. Berkovich. Annals of Mathematics Studies, 151. Princeton University Press, Princeton, NJ, 2001.

[Henn1] G. Henniart, Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique. Invent. Math. 139 (2000), no. 2, 439–455.

[Henn2] G. Henniart, Correspondance de Langlands et fonctions L des carrés extérieur et symétrique, IHES preprint M03-20, available at `http://www.ihes.fr/PREPRINTS/M03/M03-20.pdf`

[I] J. Igusa, A Classification of Spinors Up to Dimension Twelve, Amer. J. Math., Vol. 92, No. 4 (1970), 997-1028.

[Ja] H. Jacquet, Generic representations, in *Non-commutative harmonic analysis (Actes Colloq., Marseille-Luminy, 1976)*, pp. 91–101. Lecture Notes in Math., Vol. 587, Springer, Berlin, 1977.

[Ja-Sh1] H. Jacquet, J. Shalika, Exterior square L functions, Automorphic Forms, Shimura Varieties and L-Functions, Vol.2 Academic Press (1990), 143-226.

[Ja-Sh2] H. Jacquet, J. Shalika, On Euler products and the classification of automorphic representations. I. Amer. J. Math. 103 (1981), no. 3, 499–558.

[Ja-Sh3] H. Jacquet, J. Shalika, On Euler products and the classification of automorphic representations. II. Amer. J. Math. 103 (1981), no. 4, 777–815.

[Ji-So] D. Jiang, D. Soudry, The local converse theorem for $\mathrm{SO}(2n+1)$ and applications. Ann. of Math. (2) 157 (2003), no. 3, 743–806.

[Kaz] D. Kazhdan, On lifting, in *Lie group representations, II (College Park, Md., 1982/1983)*, 209–249, Lecture Notes in Math., 1041, Springer, Berlin, 1984.

[Kim1] H. Kim, Langlands-Shahidi Method and Poles of Automorphic L-functions: Application to Exterior Square L-functions, Canad. J. Math. Vol. **51** (4), 1999, pp. 835-849.

[Kim2] Kim, Henry H. Automorphic L-functions, in *Lectures on automorphic L-functions*, 97-201, Fields Inst. Monogr., 20, Amer. Math. Soc., Providence, RI, 2004.

[Kim-Sh] H. Kim, F. Shahidi, On simplicity of poles of automorphic L-functions. J. Ramanujan Math. Soc. 19 (2004), no. 4, 267–280.

[Kn] A. Knapp, *Representation theory of semisimple groups. An overview based on examples.* Princeton Mathematical Series, 36. Princeton University Press, Princeton, NJ, 1986.

[K-R] S. Kudla, S. Rallis, Ramified degenerate principal series representations for $\mathrm{Sp}(n)$. Israel J. Math. 78 (1992), no. 2-3, 209–256.

[L-L] Labesse, J.-P.; Langlands, R. P. L-indistinguishability for SL(2). Canad. J. Math. 31 (1979), no. 4, 726–785. Available online at `http://www.sunsite.ubc.ca/DigitalMathArchive/Langlands/`

[L1] R. Langlands, Euler Products, Yale University Press, (1971). Available online at `http://www.sunsite.ubc.ca/DigitalMathArchive/Langlands/`

[L2] R. Langlands, "On the classification of irreducible representations of real algebraic groups," *Representation theory and harmonic analysis on semisimple Lie groups*, pp. 101–170, Math. Surveys Monogr., 31, Amer. Math. Soc., Providence, RI, 1989. Available online at `http://www.sunsite.ubc.ca/DigitalMathArchive/Langlands/`

[L3] R. P. Langlands, On the notion of an automorphic representation, in *Proc. Sympos. Pure Math., XXXIII, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1,* pp. 189–207, Amer. Math. Soc., Providence, R.I., 1979. Available online at `http://www.ams.org/online_bks/pspum331/` and `http://www.sunsite.ubc.ca/DigitalMathArchive/Langlands/`

[L4] R. P. Langlands, Automorphic representations, Shimura varieties, and motives. Ein Märchen, in *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, Part 2, pp. 205–246, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979. Available online at `http://www.ams.org/online_bks/pspum332/` and `http://www.sunsite.ubc.ca/DigitalMathArchive/Langlands/`

[MW1] C. Moeglin and J.-L. Waldspurger *Spectral Decomposition and Eisenstein Series,* Cambridge University Press, 1995.

[MW2] C. Moeglin, J.-L. Waldspurger, Le spectre résiduel de $\mathrm{GL}(n)$, *Ann. Sci. École Norm. Sup.* 22 (1989), 605–674. MR 91b:22028

[PS] I.I. Piatetski-Shapiro, Multiplicity one theorems, in *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1,* pp. 209–212, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979. Available online at `http://www.ams.org/online_bks/pspum331/`

[PraRam] D. Prasad and D. Ramakrishnan, preprint, 2006. Available online at `http://www.math.caltech.edu/people/dinakar.html`.

[Sha1] F. Shahidi, Langlands-Shahidi method. *Automorphic forms and applications,* P. Sarnak, F. Shahidi ed. IAS/Park City Math. Ser., 12, Amer. Math. Soc., Providence, RI, 2007, pp.299–330

[Sha2] F. Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups. Ann. of Math. (2) 132 (1990), no. 2, 273-330.

[Sha3] F. Shahidi, Local coefficients as Artin factors for real groups. Duke Math. J. 52 (1985), no. 4, 973–1007.

[Sha4] F. Shahidi, On certain L-functions. Amer. J. Math. 103 (1981), no. 2, 297–355.

[Sha5] F. Shahidi, Whittaker models for real groups, Duke Math. J. 47 (1980), no. 1, 99–125.

[Sil] A. Silberger, *Introduction to harmonic analysis on reductive p-adic groups. Based on lectures by Harish-Chandra at the Institute for Advanced Study, 1971–1973.* Mathematical Notes, 23. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1979

[Spr] T.A. Springer, *Linear Algebraic Groups , 2nd Ed* Birkhäuser, 1998.

[So1] D. Soudry, On Langlands functoriality from classical groups to GL_n. Automorphic forms. I. Astérisque No. 298 (2005), 335–390.

[So2] D. Soudry, Rankin-Selberg convolutions for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$: local theory. Mem. Amer. Math. Soc. 105 (1993), no. 500

[Tad1] M. Tadić, Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case). Ann. Sci. cole Norm. Sup. (4) 19 (1986), no. 3, 335–382.

[Tad2] M. Tadić, An external approach to unitary representations. Bull. Amer. Math. Soc. (N.S.) 28 (1993), no. 2, 215–252.

[Tate1] J. Tate, Number theoretic background. Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 3–26, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979. Available online at `http://www.ams.org/online_bks/pspum332/`

[Tate2] J. Tate, Fourier analysis in number fields, and Hecke's zeta-functions. 1967 *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)* Cassels, Frohlich ed., Thompson, Washington, D.C., pp. 305–347.

[Tits] J. Tits, Reductive groups over local fields. Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, pp. 29–69, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979. Available online at `http://www.ams.org/online_bks/pspum332/`

[Vog] D. Vogan, Gelfand-Kirillov dimension for Harish-Chandra modules. Invent. Math. 48 (1978), no. 1, 75–98.

[Z] A. V. Zelevinskiĭ, Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$. Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 2, 165–210.

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