The Adjoint L function for GL_5

David Ginzburg
School of Mathematical Sciences
Sackler Faculty of Exact Sciences
Tel-Aviv University, Israel 69978
and
Joseph Hundley
Mathematics Department
Southern Illinois University Carbondale
1245 Lincoln Drive
Carbondale, IL 62901

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It is a fact of classical invariant theory that, if we consider the adjoint representation of $PGL_n(\mathbb{C})$, and the second fundamental representation of $PSp_{2n}(\mathbb{C})$, (often denoted $\wedge_0^2$) then we find that the algebras of invariants for these two representations are isomorphic. This can be seen, for example, in the table on p. 260 of [P-V]. More precisely, each is a free algebra, having $n-1$ generators of degrees $2, 3, 4, \ldots, n$. It was observed in [G-R] that when such an isomorphism of algebras of invariants exists, it is an indication that *if* Rankin-Selberg constructions happen to exist for the L-functions corresponding to both cases, then they are likely to make use of the same Eisenstein series.

In the family at hand, there are already two examples for this: when $n = 3$, the adjoint L function was constructed in [G1] and the L function for $\wedge_0^2$ was constructed in [G-R], using the same Eisenstein series, which is defined on the exceptional group G_2. Similarly, for $n = 4$, the adjoint L function was constructed in [B-G1] and the L function for $\wedge_0^2$ was constructed in [B-G2]. These constructions use the same Eisenstein series, which is defined on the exceptional group F_4.

In these notes we describe some recent progress towards the construction of Rankin-Selberg integrals corresponding to the case $n = 5$. As far as we know, these will be the first Rankin-Selberg integrals corresponding to L group representations such that the algebra of invariants has more than three generators. This time, the Eisenstein series is defined on the exceptional group E_8.

Our first integral is of the form

$$\int_{Spin_{11}(F)\backslash Spin_{11}(\mathbb{A})} \varphi(g) \int_{U(F)\backslash U(\mathbb{A})} E(ug, s)\theta_\phi^\psi(\ell(u)g)\psi_{U_P}(u)dudg, \tag{1}$$

where U is the unipotent radical of a certain parabolic subgroup of E_8, and ψ_{U_P} is a character of U. The group $Spin_{11}$ is embedded in the Levi part of this parabolic as the stabilizer of a certain character of a certain subgroup of U. Also, θ_ϕ^ψ is a theta function defined on the semidirect product of the metaplectic group $\widetilde{Sp}_{32}(\mathbb{A})$ and the Heisenberg group $H_{33}(\mathbb{A})$ in 33 variables, while ℓ is a

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projection of U onto H_{33} and j is an embedding of $Spin_{11}(\mathbb{A})$ into $\widetilde{Sp}_{32}(\mathbb{A})$. Finally, the function $\varphi(g)$ is a generic cusp form defined on the group $GSpin_{11}(\mathbb{A})$, while $E(g, s)$ is an Eisenstein series on $E_8(\mathbb{A})$. See section 2 for precise notation.

Our second integral is similar but simpler: using the same Eisenstein series, and the same group U, we choose a character ψ_U having stabilizer isomorphic to SL_5. The integral is

$$\int_{SL_5(F)\backslash SL_5(\mathbb{A})} \varphi(g) \int_{U(F)\backslash U(\mathbb{A})} E(ug, s)\psi_U(u)dudg. \tag{2}$$

This time φ is a cusp form defined on $GL_5(\mathbb{A})$.

The relationship between our two integrals is another example of a phenomenon observed in [G-H1]: if we replace φ in (1) by an Eisenstein series induced from a cuspidal representation of the parabolic of $Spin_{11}$, having Levi part isomorphic to GL_5, then our GL_5 construction may be regarded, in a suitable formal sense, as a convergent sub-integral of our $Spin_{11}$ construction. We give some details in section 3. Incidentally, it is possible to relate the construction of [G-R] section 4 with that of [G1], and the construction of [B-G2] with that of [B-G1] in a similar fashion.

In addition to the L function attached to the second fundamental representation of $Sp_{2n}(\mathbb{C})$, there is another family of L functions which seems to share a special affinity with the family of adjoint L function. These are the ratios $\zeta_K(s)/\zeta_F(s)$ of Dedekind zeta functions, which may also be expressed as products of Artin L functions. Here K is a commutative algebra of degree n over F.

This second observation goes back at least to [Ja-Z]. The paper [J-R] of Jiang and Rallis makes the connection with the theory of integral representations, and we refer the reader there for a more detailed discussion. What was noted in [J-R] is that, at least for values of n up to 5, the two families seem to be related on a nuts-and-bolts level in the theory of integral representations, in addition to the relationships suggested by [Ja-Z]. Indeed, the main Theorem of [J-R] may be formulated as stating that for $n = 3$ the ratio $\zeta_K(s)/\zeta_F(s)$ has an integral representation using the same Eisenstein series (in this case, defined on the group G_2) that was used in [G1] to obtain a Rankin-Selberg integral for the adjoint L function of GL_3. For the case $n = 2$, Jiang and Rallis point to [Si, Sh, Ge-Ja], before noting that work of Wright and Yukie [W-Y] constitutes evidence that a similar approach might work for $n = 4$ and 5. We expand on this observation slightly.

An integral representation for the ratio $\zeta_K(s)/\zeta_F(s)$ arises from a correspondence between algebras K and $M(F)$ orbits of characters of $N(F)\backslash N(\mathbb{A})$, where $P = MN$ is a parabolic subgroup in a split reductive algebraic group G defined over F. This G is Sp_4 for $n = 2$, and G_2 for $n = 3$. For $n = 4$ and 5, a similar correspondence is defined in [W-Y], with the role of G being played by F_4 and E_8 respectively. (Actually, what Wright and Yukie attach to an orbit is not an algebra of degree n but a field extension of degree at most n, but the overlap– field extensions of degree exactly n– is the main interest anyway.) Thus it may be regarded as further evidence of the Jiang-Rallis philosophy that Rankin-Selberg integrals representing the corresponding adjoint L functions have been found in precisely these two groups.
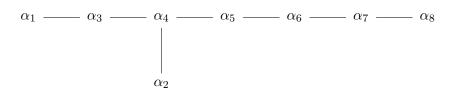
Part of the *heuristic* similarity between adjoint L functions and the products of Artin L functions $\zeta_K(s)/\zeta_F(s)$ rests on the fact that the adjoint L function may also be expressed as a quotient: $L(s, \pi \times \tilde\pi)/\zeta_F(s)$, where $\tilde\pi$ is the contragredient of π. The same is true of the L function for the representation $\wedge_2^0$, attached to an automorphic representation of $Spin_{11}(\mathbb{A})$: it is $L(s, \pi, \wedge^2)/\zeta_F(s)$.

We now describe the content of this paper. In sections 1 and 2 we define our two integrals, and we state the results concerning their unfoldings. In section 3 we give some details of the formal

manipulations that may be used to relate them. In section 4 we describe the part of the unramified computations for the GL_5 integral which is complete.

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1 The GL_5 global integral

The group E_8 has 8 simple roots and 120 positive roots. We number the simple roots

$$
\alpha_1 \;\text{------}\; \alpha_3 \;\text{------}\; \alpha_4 \;\text{------}\; \alpha_5 \;\text{------}\; \alpha_6 \;\text{------}\; \alpha_7 \;\text{------}\; \alpha_8
$$
$$
\alpha_2
$$

and refer to all other roots as octuples of integers in terms of this basis. For each root α there is a one dimensional unipotent subgroup U_α. We will wish to make use of a family of isomorphisms $x_\alpha : \mathbb{G}_a \to U_\alpha$. It will be important to know the coefficients $n_{\alpha,\beta}$ such that

$$
[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(n_{\alpha,\beta}rs).
$$

The isomorphisms may be chosen so that all the coefficients are $0, 1$, or -1 as described in [Gk-Se]

We now describe the elements of our Rankin-Selberg integral. First, let Q denote the maximal parabolic subgroup of E_8 whose unipotent radical contains the subgroup U_{α_5}. The derived group of its Levi part is isomorphic to $SL_5 \times SL_4$. We consider the Eisenstein series $E(g,s)$ associated to the induced representation $\mathrm{Ind}_{Q(\mathbb{A})}^{E_8(\mathbb{A})} \delta_Q^s$. Next, let P denote the parabolic subgroup whose unipotent radical, U_P, contains the groups U_{α_1} and U_{α_8}, and whose Levi part contains U_{α_i} for the remaining values of i. The derived group of this Levi part is isomorphic to $Spin_{12}$. We fix an additive character ψ of $F\backslash\mathbb{A}$, and define a character of $U_P(F)\backslash U_P(\mathbb{A})$ by

$$
\psi_U(x_\alpha(r)) = \begin{cases} \psi(r), & \alpha = \alpha_1, \alpha_8, 12343210, 01122221 \\ 1, & \text{otherwise.} \end{cases}
$$

The identity component of the stabilizer of this character in the Levi part of P is generated by the groups $U_{\pm\alpha_i}$ for $i = 2, 4, 5, 6$. In particular, it is isomorphic to SL_5. Using the maps x_α fixed above, we may pin down a specific isomorphism, which we use to identify elements of this subgroup of E_8 with 5×5 matrices in SL_5.

The Fourier coefficient

$$
\int_{U(F)\backslash U(\mathbb{A})} E(ug,s)\psi_U(u)du
$$

is left $SL_5(F)$-invariant. Restricting the g variable to $SL_5(\mathbb{A})$ we obtain a smooth automorphic function on SL_5 of moderate growth, which we now integrate against a cusp form, in the space of an irreducible automorphic cusipidal representation of $GL_5(\mathbb{A})$. Thus, our integral is

$$
\int_{SL_5(F)\backslash SL_5(\mathbb{A})} \varphi(g) \int_{U(F)\backslash U(\mathbb{A})} E(ug,s)\psi_U(u)dudg. \tag{3}
$$

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In regards to the integration over $U(F)\backslash U(\mathbb{A})$, we wish to make the following remark. In [G-R-S1] and [G2], a procedure is described which begins with a nilpotent orbit, and associates to it a set of Fourier coefficients. Here, the "nilpotent orbits" are for the coadjoint action of a group on its Lie algebra over the algebraic closure. The details in [G-R-S1, G2] are for classical groups and make use of the labeling of nilpotent orbits by partitions, but the same idea works for exceptional groups, where the most convenient labeling is by weighted Dynkin diagrams. The above integral defines a Fourier coefficient which is associated, by this procedure to the nilpotent orbit in $\mathfrak{e}_8$, associated to the weighted Dynkin diagram

$$2 \relbar\joinrel\relbar 0 \relbar\joinrel\relbar 0 \relbar\joinrel\relbar 0 \relbar\joinrel\relbar 0 \relbar\joinrel\relbar 0 \relbar\joinrel\relbar 2$$
$$|$$
$$0$$

As noted on p. 405 of [C], the stabilizer of a point in this orbit is a group of type A_4.

We unfold integral (3). The result is as follows. Let $\tilde{w}_0$ denote the shortest element of the double coset

$$W_Q w_\ell W_{SL_5},$$

where w_ℓ is the longest element of the Weyl group. Let V denote the maximal unipotent subgroup of SL_5 corresponding to our choice of positive roots. It is identified with the group of 5×5 upper triangular unipotent matrices. Let V' to be the subgroup consisting of elements of the form

$$\begin{pmatrix} 1 & v_1 & v_2 & v_3 & v_4 \\ & 1 & v_5 & v_6 & v_7 \\ & & 1 & -v_5 & v_2 \\ & & & 1 & -v_1 \\ & & & & 1 \end{pmatrix}.$$

Finally, let $U_0 = U \cap \tilde{w}_0^{-1}\overline{U}_P\tilde{w}_0 = \prod_{\alpha>0:\tilde{w}_0\alpha<0} U_\alpha$. Then we have

Theorem: *For $Re(s)$ large, integral (3) is equal to*

$$I(W_\varphi, f, s) := \int_{V'(\mathbb{A})\backslash SL_5(\mathbb{A})} W_\varphi(h) \int_{U_0(\mathbb{A})} f(\tilde{w}_0\delta_0 u_0 h, s)\psi_U(u_0)du_0 \, dh, \qquad (4)$$

where W_φ is the Whittaker function of φ and

$$\delta_0 = x_{00001110}(1)x_{01111000}(-1)x_{00111100}(1)x_{01121110}(-1).$$

From this it essentially follows that, for suitable data, the original integral is factorizable– i.e. that it may be expressed as an infinite product of corresponding local integrals.

2 The $GSpin_{11}$ global integral

Next we describe a similar construction, involving a cusp form in the space of an irreducible automorphic cuspidal generic representation of $GSpin_{11}(\mathbb{A})$.

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We make use of the same parabolic P as before. The derived group of the Levi part of P is isomorphic to $Spin_{12}$ and so the Levi part itself is isomorphic to a quotient of $Spin_{12} \times GL_1^2$. Let ψ_{U_P} be the character of U_P defined by
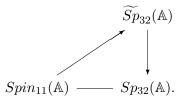
$$\psi_{U_P}(x_\alpha(r)) = \begin{cases} \psi(r) & \alpha = 01011111, 00111111 \\ 1 & \text{otherwise.} \end{cases}$$

The stabilizer of this character in the Levi part is isomorphic to $GSpin_{11}$.

The next element of our Rankin-Selberg integral is a theta function. To define it, we notice that a certain subgroup U_H of U_P has the structure of a Heisenberg group. Specifically, U_H is the product of the subgroups U_α associated to roots $\alpha = n_1\alpha_1 + \cdots + n_8\alpha_8$, such that $n_8 = 0$ and $n_1 \neq 0$. Because of the Heisenberg group structure, $U_H/U_{22343210}$ is a 32 dimensional symplectic vector space, with the skew-symmetric bilinear form being given by commutator. By Sp_{32} we shall mean the group of automorphisms of this symplectic vector space.

Above we noted a subgroup of the Levi part of P isomorphic to $GSpin_{11}$. It acts on U_H by conjugation. The subgroup of elements which fix the subgroup $U_{22343210}$, and hence correspond to elements of Sp_{32}, is precisely the derived group, isomorphic to $Spin_{11}$. This map of $Spin_{11}$ into Sp_{32} is essentially the spin representation, and in particular is an embedding. Thus $Spin_{11}$ is simultaneously identified with a certain subgroup of E_8, and with a certain subgroup of Sp_{32}.

Now, having fixed an additive character above, we obtain a Schrödinger-Weil representation ω_ψ of the semidirect product of $U_H(\mathbb{A})$ and $\widetilde{Sp}_{32}(\mathbb{A})$, where $\widetilde{Sp}_{32}(\mathbb{A})$ denotes the metaplectic double cover of $Sp_{32}(\mathbb{A})$. This representation has an automorphic realization by theta functions, θ_ϕ^ψ, where ϕ is a Schwartz function, and ψ is the additive character fixed above. Furthermore, there is a map $j : Spin_{11}(\mathbb{A}) \to \widetilde{Sp}_{32}(\mathbb{A})$ such that the following diagram commutes:

$$\widetilde{Sp}_{32}(\mathbb{A})$$

$$Spin_{11}(\mathbb{A}) \quad\text{------}\quad Sp_{32}(\mathbb{A}).$$

There is a natural projection from U_P to U_H which we denote ℓ. Thus, a theta function θ_ϕ^ψ may be evaluated at $\ell(u)j(h)$ for $u \in U(\mathbb{A})$ and $h \in Spin_{11}(\mathbb{A})$.

The integral we consider is

$$\int_{Spin_{11}(F)\backslash Spin_{11}(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \varphi(h)\theta_\phi^\psi(\ell(u)j(h))E(uh,s)\psi_U(u)\,du\,dh, \tag{5}$$

where φ is a cusp form in the space of a generic irreducible cuspidal automorphic representation π of $GSpin_{11}(\mathbb{A})$. This time, the integral over U defines a Fourier coefficient obtained from the nilpotent orbit of $\mathfrak{e}_8$ associated to the weighted Dynkin diagram

$$2 \text{ ---- } 0 \text{ ---- } 0 \text{ ---- } 0 \text{ ---- } 0 \text{ ---- } 0 \text{ ---- } 1$$
$$|$$
$$0$$

.

The result of the unfolding is similar. Let w be the shortest element of $Qw_\ell P$, which is of length 83, and $U_0 = U \cap w^{-1}\overline{U}_Q w$, the product of all the groups U_α such that $\alpha > 0$ and $w\alpha < 0$.

Then for a certain codimension 5 subgroup V'' of V, defined similarly to V' above, and a suitable element ξ_0 of F^{16}, we have

Theorem: *For $Re(s)$ large, the integral (5) is equal to*

$$\int_{V''(\mathbb{A})\backslash Spin_{11}(\mathbb{A})} \int_{U_0(\mathbb{A})} \omega_\psi(\ell(u)j(h))\phi(\xi_0)f(wuh,s)\psi_U(u)$$

$$W_\varphi(x_{-00000100}(x_3)x_{-00011100}(x_1)x_{-00010000}(x_2)w[645]h)dudh.$$

Once again, the factorization as a product of local integrals is now essentially immediate.

3 Formal manipulations

In this section, we describe how to obtain the GL_5 integral from the $GSpin_{11}$ integral via certain formal manipulations. We emphasize that every integral appearing at an intermediate stage is divergent, so this procedure is purely formal.

Let τ be an automorphic representation of $GL_5(\mathbb{A})$ and $E_\tau(g,s)$ an Eisenstein series defined on $Spin_{11}(\mathbb{A})$ using the section $f_\tau(g,s) \in Ind_{R(\mathbb{A})}^{G(\mathbb{A})}\tau \otimes \delta_R^s$. Here R is the standard maximal parabolic subgroup of $Spin_{11}$ having Levi part isomorphic to GL_5. We plug $E_\tau(g,s)$ into (5) for φ, and formally unfold the resulting divergent integral. We obtain

$$\int_{R(F)\backslash Spin_{11}(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} f_\tau(g,s)\theta_\phi^\psi(\ell(u)j(g))E(ug,s)\psi_U(u)dudg.$$

Next we replace integration over $R(F)\backslash Spin_{11}(\mathbb{A})$ by integration over $GL_5(F)\backslash GL_5(\mathbb{A}) \times U_R(F)\backslash U_R(\mathbb{A})$ where GL_5 is identified with the Levi part of R and U_R is its unipotent radical. At the same time, we replace $f_\tau(g,s)$ by a cusp from φ_τ in the space of τ. This step may be thought of as plugging in the Iwasawa decompostion of $Spin_{11}$ with respect to the parabolic R and passing to an "inner integral."

We allow ourselves to plug in simplified, formal versions of the sorts of identities which commonly arise in the unfolding of Rankin-Selberg integrals and other similar computations.

For example, the next step is to replace integration over the whole group U against a theta function by integration over a subgroup against a character. The group is obtained by deleting 16 of the subgroups U_α from the Heisenberg part, and the character is the one which is trivial on every U_α except 22343210 and $\psi(r)$ on $x_{22343210}(r)$. This may be regarded as a simplified formal version of the type of identities found on p. 5 of [G-R-S1].

We also use a simplified formal version of the technique of "exchange of roots." (Cf. [G-L], section 2.1, [G-R-S2] pp. 748-750.) The idea is to replace integration over one unipotent group U_1 against a character ψ_1 by integration over a more advantageous group U_2 against a corresponding character ψ_2 by proving an identity of the form

$$\int_{U_1(F)\backslash U_1(\mathbb{A})} f(u_1 g)\psi_1(u_1)du_1 = \int_{Z(\mathbb{A})}\int_{U_2(F)\backslash U_2(\mathbb{A})} f(u_2 z g)\psi_2(u_2)du_2 dz.$$

This is possible under suitable hypotheses on the groups U_1, U_2 and Z. For a more thorough discussion see the references cited above. In our simplified formal version, we omit the z integral.

This may be thought of as passing once again to an "inner integral," corresponding to a fixed value of z.

Conjugation by $w[34254365428765431]$ (here we specify an element of the Weyl group as a word in the simple reflections) maps $\{22343210, 01011111, 00111111\}$ to $\{\alpha_1, 01122221, 12343210\}$. It is then possible to exchange roots so as to obtain integration over the subgroup of U generated by $\{U_\alpha \subset U : \alpha \neq \alpha_8\}$. We then perform a Fourier expansion on U_{α_8}.

We ignore the term corresponding to the trivial character. The remaining terms are permuted transitively by our (conjugated) GL_5, which acts by det on this one dimensional subgroup. The stabilizer is SL_5, embedded exactly as in section 1. In this manner, formally, we have obtained the GL_5 integral.

4   Unramified computations in the GL_5 case.

We consider the local analogue of integral (4), at a place where the data is unramified. (So, W is now the normalized spherical vector in the Whittaker model of a local representaion, etc.) We wish to show that $I(W, f, s) = L(s, \pi, Ad)/N(s)$ where $N(s)$ is the normalizing factor of the Eisenstein series, given by

$$\zeta(11s)\zeta(11s-1)\zeta(11s-2)^2\zeta(11s-3)^2\zeta(11s-4)\zeta(22s-6)\zeta(22s-7)\zeta(22s-8)^2\zeta(22s-9)\zeta(22s-10)$$

$$\zeta(33s-12)\zeta(33s-13)\zeta(33s-14)\zeta(33s-15)\zeta(44s-18)\zeta(44s-20)\zeta(55s-25)$$

Performing a series of fairly standard steps, such as plugging in the Iwasawa decomposition and a suitable form of the Casselman-Shalika formula, we may express $I(W, f, s)$ as:

$$\sum_{\lambda \in \Lambda_R^{++}} \chi_\lambda(t_\pi) x^{\ell(\lambda)} I(W, f, s; \lambda), \tag{6}$$

where λ is summed over dominant weights of $PGL_5(\mathbb{C})$ (i.e., dominant weights of $GL_5(\mathbb{C})$ which are in the span of the roots), χ_λ denotes the trace of the irreducible finite dimensional representation of $GL_5(\mathbb{C})$ having highest weight λ, and ℓ is an integer-valued linear function on the root lattice. Also, $x = q^{-11s+5}$, where q is the absolute value of a uniformizer. The heart of the matter is the "inner integral" $I(\lambda)$. Let $\lambda = \sum_{i=1}^4 m_i \beta_i$, where $\beta_1, \ldots, \beta_4$ are the simple roots of $GL_5(\mathbb{C})$, and choose a uniformizer p. Then $I(\lambda, s)$ can be written as

$$\int_{F^3} \psi(p^{m_1-m_3+m_4} r_1 - p^{-m_1+m_2} r_2) \int_{U_1} \psi_{U_1}(u_1) f(\tilde{w}_1 x_{\alpha_7}(r_1) x_{\alpha_4}(r_2) x_{00011100}(r_3) \delta(m) u_0, s) du_0 dr_i \tag{7}$$

with U_1 being a conjugate of U_0 above, ψ_{U_1} a character, and

$$\delta(m) = x_{-\alpha_2}(p^{-m_2+m_3}) x_{-\alpha_5}(p^{m_1-m_2+m_3-m_4}) x_{-\alpha_7}(p^{-m_1+m_4}) x_{-\alpha_3}(p^{m_2-m_3}).$$

The delicate part of the dependence of $I(\lambda, s)$ on λ comes from $\delta(m)$. As m varies, the shape of the Iwasawa decomposition of this element varies, and results in several different cases for the shape of the integral. The easiest is when $m_1 = m_4$ and $m_2 = m_3$. This case corresponds to weights λ which are symmetric under the outer automorphism of $GL_5(\mathbb{C})$ that reverses the order of the simple roots,

i.e., weights which, in terms of the fundamental weights, are of the form $\lambda = (m, k, k, m)$. For such weights, we find that $N(s)I(W, f, s, \lambda)$ equals

$$
\frac{x^{k+2m}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} \Bigg[\delta_{k,0} x^6 (1-x^{m-1})(1-x^m)(1-x^{m+1})(1-x^{m+2})
$$

$$
+ \frac{(1-x^{k+1})(1-x^{k+2})(1-x^{k+3})}{(1-x)(1-x^2)(1-x^3)} (1-x^{m+1})(1-x^{m+2})(1-x^{m+3})(1-x^{m+4})
$$

$$
+ x^3 \frac{(1-x^{k+1})(1-x^k)(1-x^{k+3})}{(1-x)^2(1-x^2)} (1-x^m)(1-x^{m+1})(1-x^{m+2})(1-x^{m+3})
$$

$$
- x^7 \frac{(1-x^{k-1})(1-x^{k+1})(1-x^{k+2})}{(1-x)^2(1-x^2)} (1-x^{m-1})(1-x^m)(1-x^{m+1})(1-x^{m+2})
$$

$$
- x^{12} \frac{(1-x^{k-1})(1-x^k)(1-x^{k+1})}{(1-x)(1-x^2)(1-x^3)} (1-x^{m-2})(1-x^{m-1})(1-x^m)(1-x^{m+1}) \Bigg].
$$

(Here $\delta_{k,0}$ is the Kronecker delta.)

Turning our attention to the other side of the desired identity, we need an analog of the identity appearing near the top of p.155 in [B-G1]. The identity for our case is

$$
\frac{(1-x)}{(1-x^5)} \sum_{k_i=0}^{\infty} x^{k_1+2k_2+3k_3+4k_4} Trace(t_\pi | V_{k_1,k_2,k_3,k_4} \otimes V_{k_4,k_3,k_2,k_1}) = \sum_{n=0}^{\infty} Trace(t_\pi | sym^n Ad), \quad (8)
$$

where V_{k_1,k_2,k_3,k_4} is the irreducible finite-dimensional representation of $GL_5(\mathbb{C})$ having highest weight $\sum_i k_i \varpi_i$. (In fact, this is an identity of functions, which is then evaluated at t_π.)

Fix a dominant weight λ. We may consider the "coefficient" of $\chi_\lambda(t_\pi) = Trace(t_\pi | V_\lambda)$ –a power series in x – on either side of (8). Let $J(\lambda)$ denote this coefficient, and let $\tilde{I}(W, f, s, \lambda) = N(s)I(W, f, s, \lambda)$ (and extend by zero to dominant weights which are not in the root lattice). Then our main assertion may be repackaged as the assertion that $J(\lambda) = \tilde{I}(W, f, s, \lambda)$ for all λ.

Proposition: *This holds for $\lambda = (m, k, k, m)$.*

In fact, one may prove an equality of the "symmetric pieces":

$$
\sum_{k,m=0}^{\infty} \tilde{I}(W, f, s, m, k, k, m) x^{\ell(m,k,k,m)} \chi_{(m,k,k,m)}(t_\pi) = \sum_{k,m=0}^{\infty} J(m, k, k, m) \chi_{(m,k,k,m)}(t_\pi),
$$

by replacing $\chi_{(m,k,k,m)}(t_\pi)$ by $T_1^k T_2^m$, where T_1 and T_2 are indeterminates. The resulting power series in 3 variables may then be summed to obtain explicit rational functions, which turn out to be equal.

We remark that the desired equality can be verified by hand for small asymmetrical λ as well. Also, the quantity $\tilde{I}(W, f, s, \lambda)$ which was introduced on an ad hoc basis above also appears naturally as an "inner integration" if we interpret part of the unipotent integration in $I(W, f, s, \lambda)$ above as an intertwining operator, and part as the Jacquet integral for the Whittaker function on the $GL_4 \times GL_5$ Levi part.

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