Multi-variable Rankin-Selberg Integrals for Orthogonal Groups

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1 Introduction

In this paper we begin the study of a family of multi-variable Rankin-Selberg integrals in similitude Orthogonal groups. Unfortunately, as it seems right now, this family of integrals produce L functions only for low rank groups. To describe the construction, let π be a cuspidal generic irreducible representation of the group $GSO_{2n}(\mathbf{A})$. Let P denote the standard Siegel parabolic subgroup of $GSO_{2n}(\mathbf{A})$. Thus P has the Levi decomposition $P = (GL_1 \times GL_n) \cdot U(P)$. Let $E_P(g, s)$ denote the Eisenstein series defined on the group $GSO_{2n}(\mathbf{A})$, which is associated to the induced representation $Ind_{P(\mathbf{A})}^{GSO_{2n}(\mathbf{A})} \delta_P^s$. The family of integrals we consider is given by

$$\int\limits_{Z(\mathbf{A})GSO_{2n}(F)\backslash GSO_{2n}(\mathbf{A})} \varphi(g)E_Q(g, w)E_P(g, s)dg \qquad (1)$$

Here, the function $\varphi(g)$ is a vector in the space of π and $E_Q(g, w)$ is a certain Eisenstein series which depends on the value of n. In other words, in each case we will need to choose a different representation for $E_Q(g, w)$. Also, s and w are complex variables and Z is the center of the group GSO_{2n}. For simplicity we shall assume that π has a trivial central character.

One of our main results is to show that for a suitable choice of the representation $E_Q(g, w)$, integral (1) is Eulerian. At this point we can show that only when $n \leq 6$. The L functions obtained by these integrals are Spin L functions. The cases $n = 2, 3$ are trivial. In fact, when $n = 3$ we get the usual Rankin product integral where we view GSO_6 as GL_4. In this case one can actually replace $E_Q(g, w)$ with any generic automorphic representation of the group $GSO_6(\mathbf{A})$. In [G1] a construction for the Spin L functions is given for the groups GSO_{10} and GSO_{12}.

In this paper we shall work out integral (1) for the group GSO_8. As it turns out, the Eisenstein series $E_Q(g, w)$ actually depends on two complex variables. Hence integral (1) represents a product of three L functions. The standard L function appears once, and the Spin L function which corresponds to the fourth fundamental representation of $GSpin_8(\mathbf{C})$ appears twice.

In section two we introduce the global integral and show it to be Eulerian with Whittaker model. In the third section we carry out the unramified computation. These two sections are quite standard. In the last section we give an application of our construction. We relate the functorial lift from the exceptional group G_2 to GSO_8 with a certain period integral and show that this is all related to existence of poles of certain L functions. More precisely, we prove

Main Theorem: *(Theorem 4.3) Let π be an irreducible generic cuspidal representation of the group $GSO_8(\mathbf{A})$ which has a trivial central character. Then the following are equivalent:*
1) Both partial L functions, $L^S(\pi, Spin, s)$ and $L^S(\pi, St, s)$ have a simple pole at $s = 1$.
2) The period integral $\mathcal{P}_{\varphi,\phi}$ (see section 4 for definition) is nonzero for some choice of data.
3) The representation π is the functorial lift from a cuspidal generic representation of the exceptional group $G_2(\mathbf{A})$.

As was mentioned above one can produce Eulerian integrals of the type (1) also for the groups GSO_{10} and GSO_{12}. The second named author intends to study these cases in the near future.

2 The Global Integral

Let $G = GSO_8$. Let π denote a cuspidal irreducible generic representation of $G(\mathbf{A})$. For simplicity, we shall assume that π has a trivial central character. To define the Eisenstein series we first consider the following parabolic subgroups of G. Let P denote the maximal standard parabolic subgroup of G with Levi factorization $P = (GL_1 \times GL_4)U(P)$. We shall denote by Q the maximal standard parabolic of G with Levi decomposition $Q = (GL_2 \times GSO_4)U(Q)$. Let $E_Q(g, s_1, s_2)$ denote the Eisenstein series defined on the group

$G(\mathbf{A})$ corresponding to the induced representation $I(s_1,s_2) = Ind_{Q(\mathbf{A})}^{G(\mathbf{A})}(Ind_{B_2(\mathbf{A})}^{GL_2(\mathbf{A})}\delta_2^{s_1})\delta_Q^{s_2}$. Here B_2 is the standard Borel subgroup of GL_2 and s_i are complex variables. Also δ_2 and δ_Q are the modulus functions of B_2 and Q respectively. Next we define the Siegel Eisenstein series $E_P(g,s_3)$ which corresponds to the induced representation $I(s_3) = Ind_{P(\mathbf{A})}^{G(\mathbf{A})}\delta_P^{s_3}$.

The global integral we consider is

$$\int\limits_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi(g)E_Q(g,s_1,s_2)E_P(g,s_3)dg \tag{2}$$

where Z is the center of the group G.

To factorize this integral we first fix some notation. In terms of matrices we consider the group G relative to the form defined by the matrix J which has ones along the other diagonal and zeros elsewhere. For $1\le i\le 4$, let α_i denote the four simple roots of the group G. Let $x_{\alpha_i}(r)$ denote the one dimensional unipotent subgroup corresponding to the root α_i. We label the roots such that

$$x_{\alpha_1}(r) = I + re'_{1,2} \quad x_{\alpha_2}(r) = I + re'_{2,3} \quad x_{\alpha_3}(r) = I + re'_{3,4} \quad x_{\alpha_4}(r) = I + re'_{3,5}$$

Here I is the 8×8 identity matrix and $e_{i,j} = e_{i,j} - e_{9-j,9-i}$. For $1\le i\le 4$ let $w[i]$ denote the simple reflection corresponding to the simple root α_i. We shall write $w[i_1i_2\ldots i_r]$ for $w[i_1]w[i_2]\ldots w[i_r]$.

Let ψ denote a non-trivial additive character of $F\backslash\mathbf{A}$. For $g\in G(\mathbf{A})$, $f_{s_1,s_2}\in I(s_1,s_2)$ and $f_{s_3}\in I(s_3)$ we define $f^R_{s_1,s_2}(g)$ to equal

$$\int\limits_{(F\backslash A)^4} f_{s_1,s_2}(w[2134]x_{\alpha_1}(r_1)x_{\alpha_3}(r_2)x_{\alpha_4}(r_3)x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(r_4)g)\psi^{-1}(r_1+r_2+r_3)dr_i \tag{3}$$

and

$$f^L_{s_3}(g) = \int\limits_{(F\backslash A)^2} f_{s_3}(w[42]x_{\alpha_2}(l_1)x_{\alpha_2+\alpha_4}(l_2)g)\psi^{-1}(l_1)dl_i \tag{4}$$

We have the following

Theorem 2.1: *With the above notations and for $Re(s_i)$ large we have*

$$\int\limits_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi(g)E_Q(g,s_1,s_2)E_P(g,s_3)dg = \int\limits_{Z(\mathbf{A})U(\mathbf{A})\backslash G(\mathbf{A})} W_\varphi(g)f^R_{s_1,s_2}(g)f^L_{s_3}(g)dg \tag{5}$$

Here W_φ is the Whittaker function corresponding to φ and U is the maximal unipotent subgroup of G which consists of upper triangular matrices.

Proof: We start by unfolding the two Eisenstein series. The first thing is to analyze the contribution to the integral from each of the representatives of the space $Q\backslash G/P$. This space contains three representatives which we can choose to be $e, w[24]$ and $w[21324]$. It is not hard to check that the first two contribute zero to the integral. As for the third one, the stabilizer is the group N' defined as the subgroup of G consisting of all matrices of the form

$$N' = \{ \begin{pmatrix} \lambda A_1 & C_1 & & C_2 \\ & \lambda A_2 & & \\ & & A_2^* & C_1^* \\ & & & A_1^* \end{pmatrix} \ : \ A_1, A_2 \in GL_2, \quad \lambda \in GL_1, \quad (C_1, C_2) \in Mat_2\}$$

Changing variables $g \mapsto w[4]g$ and further unfolding the Eisenstein series on G corresponding to $I(s_1, s_2)$ we obtain that the left hand side of (5) equals

$$\int\limits_{Z(\mathbf{A})B_2(F)GL_2(F)N(F)\backslash G(\mathbf{A})} \varphi(g) f_{s_1,s_2}(w[2132]g) f_{s_3}(w[4]g)dg \tag{6}$$

Here $B_2 \times GL_2$ and N are embedded in G as

$$(b,h) = \begin{pmatrix} b & & & \\ & h & & \\ & & h^* & \\ & & & b^* \end{pmatrix} \qquad n = \begin{pmatrix} I & & C_1 & C_2 \\ & I & & C_1^* \\ & & I & \\ & & & I \end{pmatrix}$$

where $C_1, C_2 \in Mat_2$ such that the above matrix is in G and I is the two by two identity matrix. Factoring the integration over $N(F)\backslash N(\mathbf{A})$ we denote by $\varphi^N(g)$ the constant term of φ along N. Next, we expand $\varphi^N(g)$ along $x_{\alpha_4}(r)$ and then along $(I + r_1 e'_{1,3})(I + r_2 e'_{1,4})$ with points in $F\backslash\mathbf{A}$. It is easy to see that the constant terms will contribute zero by the cuspidality of φ. Thus, (6) equals

$$\int\limits_{Z(\mathbf{A})GL_1^2(F)N_1(\mathbf{A})\backslash G(\mathbf{A})} \varphi^{U_1,\psi}(g) f_{s_1,s_2}(w[2132]g) f_{s_3}(w[4]g)dg \tag{7}$$

Here GL_1^2 is embedded in G as the group of all diagonal matrices $diag(a, b, a, 1, a, 1, ab^{-1}, 1)$. The group $N_1 =< N, x_{\alpha_1}(r), x_{\alpha_3}(r) >$. Also, we denote

$$U_1 =< x_{\alpha_1}(r), x_{\alpha_1+\alpha_2}(r), x_{\alpha_1+\alpha_2+\alpha_3}(r), x_{\alpha_3}(r), U(P) >$$

where $U(P)$ is the unipotent radical of the parabolic subgroup P. Finally,

$$\varphi^{U_1,\psi}(g) = \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} \varphi(u_1 g)\psi_{U_1}(u_1)du_1$$

where $\psi_{U_1}(u_1) = \psi_{U_1}(x_{\alpha_4}(r_1)x_{\alpha_1+\alpha_2}(r_2)u_1') = \psi(r_1 + r_2)$ and u_1' is any product of all other one dimensional unipotent subgroups in U_1.

Next we consider the Fourier expansion of $\varphi^{U_1,\psi}(g)$ along the group $I + re_{3,2}'$ with points in $F\backslash\mathbf{A}$. Using the left invariance property of φ under $G(F)$ we obtain after a suitable conjugation

$$\varphi^{U_1,\psi}(g) = \int\limits_{\mathbf{A}} \varphi^{U_2,\psi_{U_2}}(x_{\alpha_2+\alpha_4}(r)g)dr$$

Here U_2 is the unipotent group defined as follows. Let U_1' denote the subgroup of U_1 obtained by omitting the one dimension unipotent subgroup corresponding to the root $\alpha_2+\alpha_4$. Then define $U_2 =< U_1', x_{-\alpha_2}(r) >$. Also, the character ψ_{U_2} is defined as the restriction of ψ_{U_1} to the group U_1'. Plugging this identity into (7) and collapsing the adelic integration we obtain

$$\int\limits_{Z(\mathbf{A})GL_1^2(F)N_2(\mathbf{A})\backslash G(\mathbf{A})} \varphi^{U_2,\psi_{U_2}}(g)f_{s_1,s_2}(w[2132]g)f_{s_3}(w[4]g)dg \tag{8}$$

Here N_2 is the subgroup of N_1 generated by all one dimensional unipotent subgroups in N_1 omitting the root $\alpha_2 + \alpha_4$. Next, using the left invariance property of φ we obtain by conjugation $\varphi^{U_2,\psi_{U_2}}(g) = \varphi^{U_3,\psi_{U_3}}(w[42]g)$. Here U_3 is the unipotent radical of the standard parabolic subgroup of G whose Levi part is $GL_1^2 \times GSO_4$. We also define $\psi_{U_3}(u_3) = \psi(r_1+r_2)$ where we write $u_3 = x_{\alpha_1}(r_1)x_{\alpha_2}(r_2)u_3'$ and u_3' is any product of all other one unipotent subgroups in U_3. We plug this into (8). Then we expand along the unipotent subgroup $x_{\alpha_3}(r_1)x_{\alpha_4}(r_2)$ with points in $F\backslash\mathbf{A}$ to obtain by collapsing the summation over $GL_1^2(F)$ and using cuspidality

$$\int\limits_{Z(\mathbf{A})N_2(\mathbf{A})\backslash G(\mathbf{A})} W_\varphi(w[42]g)f_{s_1,s_2}(w[2132]g)f_{s_3}(w[4]g)dg \tag{9}$$

We change variables $g \mapsto w[42]g$. This changes the domain of integration to the domain $Z(\mathbf{A})N_3(\mathbf{A})\backslash G(\mathbf{A})$ where $N_3 = w[42]N_2w[24]$. Since N_3 is a subgroup of the maximal unipotent subgroup U of G, we can factor the integration domain along $N_3\backslash U$. Using the left invariance properties of the functions W_φ, f_{s_1,s_2} and f_3 we obtain identity (5). ■

3 The Unramified Computation

In this section we consider the local unramified integral which results from identity (5). Let F be a local finite field where all data are unramified. To be more precise, let π denote an unramified irreducible representation of the local group G. We assume that π has a trivial central character. Let $I(s_1, s_2)$ denote the induced representation $Ind_Q^G(Ind_{B_2}^{GL_2}\delta_2^{s_1})\delta_Q^{s_2}$ and let $I(s_3)$ denote the induced representation $Ind_P^G\delta_P^{s_3}$. All subgroups in the above representations were defined in section 2.1. From Theorem 2.1 we are led to consider the integral

$$I = \int\limits_{ZU\backslash G} W_\pi(g) f^R_{s_1,s_2}(g) f^L_{s_3}(g) dg \tag{10}$$

Here $f^R_{s_1,s_2}(g)$ and $f^L_{s_3}(g)$ are the local functionals of the global ones as defined in (3) and (4).

We shall denote by $L(\pi, Spin, s)$ the local Spin L function corresponding to the fourth fundamental representation of the group $GSpin_8(\mathbf{C})$, which is the L group of the group G. This representation is defined exactly as in [G1] page 773. By $L(\pi, St, s)$ we shall denote the local standard L function of $GSpin_8(\mathbf{C})$. Both representations, the Spin representation and the Standard representation, are an eight dimensional irreducible representations of the group $GSpin_8(\mathbf{C})$. Also, by $\zeta(s)$ we shall denote the local zeta function.

The main result of this section is

Proposition 3.1: *For all unramified data, and for $Re(s_i)$ large, integral I equals*

$$\frac{L(\pi, Spin, s_1)L(\pi, Spin, 5s_2-2)L(\pi, St, 3s_3-1)}{\zeta(2s_1)\zeta(s_1+5s_2-1)\zeta(s_1+5s_2-2)\zeta(-s_1+5s_2-1)\zeta(-s_1+5s_2)\zeta(10s_2-4)\zeta(6s_3)\zeta(6s_3-2)}$$

Proof: Let T denote the maximal torus of G. Using the Iwasawa decomposition, integral I equals

$$I = \int\limits_{Z\backslash T} W_\pi(t) f^R_{s_1,s_2}(t) f^L_{s_3}(t)\delta_B^{-1}(t) dt \tag{11}$$

where B is the Borel subgroup of G which consists of upper triangular matrices. We parameterize the an element t in $Z\backslash T$ as $t = diag(ab_1, ab_2, ab_3, a, 1, b_3^{-1}, b_2^{-1}, b_1^{-1})$. In this case $\delta_B^{-1}(t) = |ab_1|^{-6}|b_2|^{-4}|b_3|^{-2}$. We start by computing

$$f^L_{s_3}(t) = \int\limits_{F^2} f_{s_3}(w[42]x_{\alpha_2}(l_1)x_{\alpha_2+\alpha_4}(l_2)t)\psi^{-1}(l_1)dl_1dl_2$$

Conjugating the matrix $x_{\alpha_2+\alpha_4}(l_2)$ to the left we obtain, as inner integration, the following intertwining operator

$$\int\limits_F f_{s_3}(w[4]x_{\alpha_4}(l_2)g)dl_2$$

A simple computation shows that this intertwining operator maps the space $Ind_B^G\delta_P^{s_3}$ to the space $Ind_B^G\chi_{s_3}$ where $\chi_{s_3}(t) = |b_1b_2b_3^{-1}|^{3s_3}|ab_3|$. If K is the maximal compact subgroup of G

then this intertwining operators maps the K fixed vector in one space to the K fixed vector in the other space. Using the usual factorizations, we thus obtain

$$f^L_{s_3}(t)=\frac{\zeta(6s_3-1)}{\zeta(6s_3)}\int\limits_F f^0_{s_3}(w[2]x_{\alpha_2}(l_1)t)\psi^{-1}(l_1)dl_1$$

where $f^0_{s_3}$ is the K fixed vector in $Ind_B^G\chi_{s_3}$. To compute this integral, we break the integration domain into $|l_1|\le 1$ and into $|l_1|>1$ and proceeding as in [G1] pages 775-776, this last integral equals

$$\frac{\zeta(6s_3-2)}{\zeta(6s_3-1)}|a||b_1|^{3s_3}|b_2|^{-3s_3+2}|b_3|^{3s_3-1}(1-|b_2b_3^{-1}|^{6s_3-2}q^{-6s_3+2})$$

Here $q=|p|^{-1}$ where p is a generator of the maximal ideal inside the ring of integers of F. Combining all this we obtain

$$f^L_{s_3}(t)=\frac{\zeta(6s_3-2)}{\zeta(6s_3)}|a||b_1|^{3s_3}|b_2|^{-3s_3+2}|b_3|^{3s_3-1}(1-|b_2b_3^{-1}|^{6s_3-2}q^{-6s_3+2})$$

Next we repeat the same calculation, this time with $f^R_{s_1,s_2}(t)$. This computation is more involved but is done exactly the same way. By conjugating the root $x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(r_4)$ to the left (see (3)) we obtain an intertwining operator which we compute as we did above. The integration along the other three roots, which involves the character ψ^{-1}, is done as in [G1] pages 775-776. We thus obtain,

$$f^R_{s_1,s_2}(t)=\frac{\zeta(s_1+5s_2-2)\zeta(-s_1+5s_2-2)^2}{\zeta(s_1+5s_2-1)\zeta(-s_1+5s_2)\zeta(-s_1+5s_2-1)}|a|^{s_1+2}|b_1|^2|b_2|^{s_1+5s_2-1}|b_3|^{s_1-5s_2+3}\times$$

$$(1-|b_1b_2^{-1}|^{s_1+5s_2-2}q^{-s_1-5s_2+2})(1-|b_3|^{-s_1+5s_2-2}q^{s_1-5s_2+2})(1-|ab_3|^{-s_1+5s_2-2}q^{s_1-5s_2+2})$$

Denote $K_\pi(t)=W_\pi(t)\delta_B^{-1/2}(t)$. Plugging all this into (11), integral I equals

$$\frac{\zeta(6s_3-2)\zeta(s_1+5s_2-2)\zeta(-s_1+5s_2-2)^2}{\zeta(6s_3)\zeta(s_1+5s_2-1)\zeta(-s_1+5s_2)\zeta(-s_1+5s_2-1)}\int\limits_{Z\backslash T}K_\pi(t)z(a,b_1,b_2,b_3)dt$$

where

$$z(a,b_1,b_2,b_3)=|a|^{s_1}|b_1|^{3s_3-1}|b_2|^{s_1+5s_2-3s_3-1}|b_3|^{s_1-5s_2+3s_3+1}(1-|b_2b_3^{-1}|^{6s_3-2}q^{-6s_3+2})\times$$

$$(1-|b_1b_2^{-1}|^{s_1+5s_2-2}q^{-s_1-5s_2+2})(1-|b_3|^{-s_1+5s_2-2}q^{s_1-5s_2+2})(1-|ab_3|^{-s_1+5s_2-2}q^{s_1-5s_2+2})$$

Consider the following change of variables in T. We set $ab_3 \mapsto t_1$, $b_1b_2^{-1} \mapsto t_2$, $b_2b_3^{-1} \mapsto t_3$ and $b_3 \mapsto t_4$. Under this change of variables the torus T is parameterized as $t = diag(t_1t_2t_3t_4, t_1t_3t_4, t_1t_4, t_1, t_4, 1, t_3^{-1}, t_2^{-1}t_3^{-1})$. Thus, the above integral equals

$$\int\limits_{Z\backslash T} K_\pi(t)z(t_1,t_2,t_3,t_4)dt$$

where now

$$z(t_1,t_2,t_3,t_4) = |t_1|^{s_1}|t_2|^{3s_3-1}|t_3|^{s_1+5s_2-2}|t_4|^{s_1+3s_3-1}(1-|t_3|^{6s_3-2}q^{-6s_3+2})\times$$

$$(1-|t_2|^{s_1+5s_2-2}q^{-s_1-5s_2+2})(1-|t_1|^{-s_1+5s_2-2}q^{s_1-5s_2+2})(1-|t_4|^{-s_1+5s_2-2}q^{s_1-5s_2+2})$$

For $1 \le i \le 4$ write $t_i = p^{n_i}$. We shall also denote $x = q^{-s_1}, y = q^{-5s_2+2}$ and $z = q^{-3s_3+1}$. It follows from the Casselman-Shalika formula [C-S], that $K_\pi(t) = (n_2,n_3,n_4,n_1)$ where (n_2,n_3,n_4,n_1) equals the trace of the irreducible representation $n_2\varpi_1 + n_3\varpi_2 + n_4\varpi_3 + n_1\varpi_4$ evaluated in the semi-simple conjugacy class of $GSpin_8(\mathbf{C})$ associated with the representation π. Here ϖ_i is the $i-th$ fundamental representation of $GSpin_8(\mathbf{C})$.

Hence the above integral equals

$$\sum_{n_i=0}^{\infty}(n_2,n_3,n_4,n_1)x^{n_1+n_3+n_4}y^{n_3}z^{n_2+n_4}(1-(x^{-1}y)^{n_1+1})(1-(xy)^{n_2+1})(1-z^{2n_3+2})(1-(x^{-1}y)^{n_4+1})$$

Cancelling the zeta factors on both sides, to prove the identity stated it is enough to prove the identity

$$\sum_{n_i=0}^{\infty}(n_2,n_3,n_4,n_1)x^{n_1+n_3+n_4}y^{n_3}z^{n_2+n_4}\left(\frac{1-(x^{-1}y)^{n_1+1}}{1-x^{-1}y}\right)\times$$

$$\left(\frac{1-(xy)^{n_2+1}}{1-xy}\right)\left(\frac{1-z^{2(n_3+1)}}{1-z^2}\right)\left(\frac{1-(x^{-1}y)^{n_4+1}}{1-x^{-1}y}\right)=$$

$$(1-xy)\frac{L(\pi,Spin,s_1)}{\zeta(2s_1)}\frac{L(\pi,Spin,5s_2-2)}{\zeta(10s_2-4)}\frac{L(\pi,St,3s_3-1)}{\zeta(6s_3-2)}$$

Using the decomposition of the symmetric algebras as given in [B], we have

$$\frac{L(\pi,Spin,s_1)}{\zeta(2s_1)} = \sum_{m_1=0}^{\infty}(0,0,0,m_1)x^{m_1} \qquad \frac{L(\pi,Spin,5s_2-2)}{\zeta(10s_2-4)} = \sum_{m_2=0}^{\infty}(0,0,0,m_2)y^{m_2}$$

$$\frac{L(\pi,St,3s_3-1)}{\zeta(6s_3-2)} = \sum_{m_3=0}^{\infty}(m_3,0,0,0)z^{m_3}$$

Thus we need to prove the identity

$$\sum_{n_i=0}^{\infty}(n_2,n_3,n_4,n_1)x^{n_1+n_3+n_4}y^{n_3}z^{n_2+n_4}\left(\frac{1-(x^{-1}y)^{n_1+1}}{1-x^{-1}y}\right)\times$$

$$\left(\frac{1-(xy)^{n_2+1}}{1-xy}\right)\left(\frac{1-z^{2(n_3+1)}}{1-z^2}\right)\left(\frac{1-(x^{-1}y)^{n_4+1}}{1-x^{-1}y}\right)=$$

$$(1-xy)\sum_{m_1=0}^{\infty}(0,0,0,m_1)x^{m_1}\sum_{m_2=0}^{\infty}(0,0,0,m_2)y^{m_2}\sum_{m_3=0}^{\infty}(m_3,0,0,0)z^{m_3}$$

Finally, this identity can be written as

$$\sum_{n_i=0}^{\infty}(n_2,n_3,n_4,n_1)x^{n_1+n_3+n_4}y^{n_3}z^{n_2+n_4}\left(\frac{1-(x^{-1}y)^{n_1+1}}{1-x^{-1}y}\right)\times$$

$$\left(\frac{1-(xy)^{n_2+1}}{1-xy}\right)\left(\frac{1-z^{2(n_3+1)}}{1-z^2}\right)\left(\frac{1-(x^{-1}y)^{n_4+1}}{1-x^{-1}y}\right)=$$

$$(1-xy)\sum_{m_i=0}^{\infty}(0,0,0,m_1)\otimes(0,0,0,m_2)\otimes(m_3,0,0,0)x^{m_1}y^{m_2}z^{m_3} \tag{12}$$

Here and henceforth, by abuse of notations, we denote by (n_2,n_3,n_4,n_1) the representation itself.

To prove this identity, we need two lemmas.

Lemma 3.2

$$(0,0,0,m_1)\otimes(0,0,0,m_2)=\bigoplus_{r=0}^{\min(m_1,m_2)}\bigoplus_{i=0}^{\min(m_1,m_2)-r}(0,r,0,|m_2-m_1|+2i),$$

and **Lemma 3.3**

$$(0,r,0,s)\otimes(m,0,0,0)=\bigoplus_{a,b,c,k}(m-k-a+b,r+k-a-2b-c,-k+a+b+2c,s-k+a+b).$$

where the sum is over quadruples (a,b,c,k) satisfying the bounds

$$0\le\quad k\quad\le m \tag{13}$$

$$0\le\quad a\quad\le\min(r,m-k) \tag{14}$$

$$\max(0,k-a-s)\le\quad b\quad\le\min(k,r-a) \tag{15}$$

$$\max(0,k-b-a)\le\quad c\quad\le\min(k-b,s). \tag{16}$$

Both lemmas are proved using the formulae in [B-K-W]. Black, King, and Wybourne have obtained their formulae via branching rules for the restriction from SO_8 to U_4. Using their formula, one proceeds as follows. First, one obtains a sum of products of representations of U_4. These are parametrized by "composite Young diagrams," $\{\bar{\mu}; \lambda\}$, that is, by pairs of partitions μ and λ such that the total number of parts does not exceed 4. A representation of $U_4(\mathbf{C})$ may also be specified by a pair, consisting of an irreductible representation of $SU_4(\mathbf{C})$ and an integer (the power of the determinant to twist by). The relationship between the two parametrizations is

$$\{\bar{\mu}; \lambda\} \leftrightarrow (\{\lambda_1 + \mu_1, \dots, \lambda_p + \mu_1, \mu_1, \dots, \mu_1, \mu_1 - \mu_q, \dots, \mu_{q-1} - \mu_q\}, \det{}^{-\mu_q}),$$

and the parametrization of irreducible representations of SU_k by partitions with at most $k-1$ parts is as usual. One then takes the product of the representations of U_4 by the Littlewood-Richardson rule. One then must interpret the results as "representations" of SO_8, using "modification rules." Some composite Young diagrams correspond to actual representations. Others give a representation with the coefficient -1 to cancel one of the other terms in the sum, and still others correspond to zero.

We include only the proof of the lemma 3.3. By a similar but easier argument, one may deduce a formula for $(m_1, 0, 0, 0) \otimes (m_2, 0, 0, 0)$ which is equivalent, by triality, to lemma 3.2.

Proof of Lemma 3.3: Black, King and Wybourne define the notation λ/ξ as follows. Let $m^{\nu}_{\lambda\mu}$ be the constants that appear in the Littlewood-Richardson rule. Thus

$$\{\lambda\} \cdot \{\mu\} = \sum_{\nu} m^{\nu}_{\lambda\mu} \{\nu\}.$$

Then

$$\{\lambda/\xi\} = \sum_{\nu} m^{\lambda}_{\nu\xi} \{\nu\}.$$

Black, King and Wybourne denote $(m, 0, 0, 0)$ by $[m]$, while $(0, r, 0, s)$ is denoted by $[\mu]_-$ when s is even, and $[\Delta; \mu]_-$ if s is odd, where $\mu = \mu_1^2 \mu_3^2$, with $\mu_1 = r + \lfloor \frac{s}{2} \rfloor, \mu_3 = \lfloor \frac{s}{2} \rfloor$. The relevant formulae are

$$[\lambda] \times [\mu]_- = \sum_{\xi} [\{\bar{\xi}; (\lambda/\xi B)\} \cdot \{\mu\}]_-,$$

in the "tensor" case (s even) and

$$[\lambda] \times [\Delta; \mu]_- = \sum_{\xi} [\Delta; \{\bar{\xi}; (\lambda/\xi B)\} \cdot \{\mu\}]_-,$$

in the “spinor” case (s odd). Here B is the sum of all partitions such that each part appears an even number of times (e.g. $\{4,4,1,1\}$). Since $\lambda = (m)$ has only one part, we may drop the B. Further, λ/ξ is trivial unless $\xi = \{k,0,0,0\}$ with $k \leq m$. Thus, we obtain,

$$\sum_{k=0}^{m} [\{\bar{k}; m-k\} \cdot \{\mu_1^2 \mu_3^2\}]_-.$$

To compute $\{\bar{k}; m-k\} \cdot \{\mu_1^2\mu_3^2\}$ we compute $\{m,k,k\} \cdot \{(\mu_1-\mu_3)^2\}$, and then twist by $\det^{-k+\mu_3}$. The Littlewood-Richardson rule gives

$$\{m,k,k\} \cdot \{(\mu_1-\mu_3)^2\} = \sum_{\substack{a+b\leq\mu_1-\mu_3 \\ b+c\leq k\leq a+b+c \\ a+k\leq m}} \{\mu_1-\mu_3+m-a, \mu_1-\mu_3-b+k, a+b+c, k-c\} \quad (17)$$

Twisting by $\det^{\mu_3-k}$ amounts to adding $\mu_3 - k$ to each term, which yields

$$\{\mu_1+m-a-k, \mu_1-b, \mu_3+a+b+c-k, \mu_3-c\}.$$

The next step is to interpret this using modification rules, which are different in the “tensor” and “spinor” cases. We will restrict our attention to the “tensor” case. The “spinor” case is similar, but things are shifted by 1, the end result being that where $2\mu_3$ appears in the “tensor” case, $2\mu_3+1$ appears in the “spinor” case. These are the respective values of s.

If $c \leq \mu_3$, we have a partition, and the corresponding representation of $Spin_8$, in our highest weight notation, is

$$(m-k-a+b, \mu_1-\mu_3+k-a-2b-c, 2c+a+b-k, a+b+2\mu_3-k).$$

If $c > \mu_3$, we must apply a modification rule to the “composite partition”

$$[\xi;\lambda]_- = [\overline{\mu_3-c}; \mu_1+m-a-k, \mu_1-b, \mu_3+a+b+c-k]_-.$$

Since the partition ξ has only one part the modification rule for $[\bar{\xi};\lambda]_-$ is easy to describe. If $\xi = \lambda_3+1, \lambda_2+2$ or λ_3+3, it is zero. If $\xi \leq \lambda_3$ it is $[\lambda_1,\lambda_2,\lambda_3,\xi]_+$. If $\lambda_3+1 < \xi < \lambda_2+2$ we get $-[\lambda_1,\lambda_2,\xi-1,\lambda_3+1]_+$, in the other two cases we get $[\lambda_1,\xi-2,\lambda_2+1,\lambda_3+1]_+$ and $-[\xi-3,\lambda_1+1,\lambda_2+1,\lambda_3+1]_+$. Let us say that $[\bar{\xi};\lambda]$ or (a,b,c,k) is Type 1,2,3 or 4, respectively, based on which of these cases applies. In particular, for $\mu_3 \leq c \leq 2\mu_3$, we get

$$[\mu_1+m-a-k, \mu_1-b, \mu_3+a+b+c-k, c-\mu_3]_+,$$

which is again

$$(m-k-a+b, \mu_1-\mu_3+k-a-2b-c, 2c+a+b-k, a+b+2\mu_3-k).$$

Our task is now to show that the terms with $c > 2\mu_3$ all cancel with one another. (Note that if $c \leq 2\mu_3$ then necessarily $b \geq k - 2\mu_3 - a$ and $k \leq \mu_1 + \mu_3$.) This may be done by constructing explicit bijections between the set of Type 1 quadruples appearing that satisfy $c > 2\mu_3$ and a subset of the set of Type 2 quadruples, between the remaining Type 2 quadruples and a subset of the set of Type 3 quadruples, and between the remaining Type 3 quadruples and the set of Type 4 quadruples. Given a Type i pair $[\bar{\xi}; \lambda]$, it is easy to construct pairs of each of the other types which are mapped to the same partition under the modification rules. It is also easy to see that the map $(a, b, c, k) \mapsto [\bar{\xi}; \lambda]$ is injective, and one gets bijections on the space of quadruples. What remains to check is that these correspondences match quadruples that appear in the sum (17) with one another.

For example, the map $(a, b, c, k) \mapsto (a', b', c', k')$ defined by

$$\begin{aligned} a' &= k - b - 2\mu_3 - 1 \\ b' &= b \\ c' &= a + b + c - k + 2\mu_3 + 1 \\ k' &= a + b + 2\mu_3 + 1. \end{aligned}$$

matches Type 1 quadruples appearing in (17) such that $c > 2\mu_3$ with a subset of the set of Type 2 quadruples appearing in (17), namely those satisfying, $k \leq \mu_1 + \mu_3 + 1$. The remaining bijections are similar. ■

We may now proceed to the proof of (12). We first apply Lemma 3.2 to

$$\sum_{m_1, m_2 = 0}^{\infty} (0, 0, 0, m_1) \otimes (0, 0, 0, m_2) \otimes (m_3, 0, 0, 0) x^{m_1} y^{m_2},$$

obtaining

$$\sum_{r,i=0}^{\infty} \sum_{m_1, m_2 = r+i}^{\infty} (0, r, 0, |m_1 - m_2| + 2i) \otimes (m_3, 0, 0, 0) x^{m_1} y^{m_2}.$$

Now, we consider the set of triples (m_1, m_2, i) such that $|m_1 - m_2| + 2i$ is equal to a fixed number s. It's clear that $(m_1 + m_2)$ must have the same parity as s. Furthermore,

$$(m_1 + m_2) = |m_1 - m_2| + 2i + 2(\min(m_1, m_2) - i) \geq s + 2r,$$

because both m_1 and m_2 are at least $(r + i)$. Put $k = (m_1 + m_2 - s - 2r)/2$. Then k is a nonnegative integer, and every nonnegative integer occurs as a value of k. Triples (m_1, m_2, i) satisfying $m_1 + m_2 = 2k + 2r + s$ and $|m_1 - m_2| + 2i = s$ are in bijection with pairs (m_1, m_2) such that $m_1 + m_2 = 2k + 2r + s$ and $|m_1 - m_2| \leq s$. The bound on $|m_1 - m_2|$ is equivalent to $m_1, m_2 \geq r+k$. Put $j = m_1 - r - k$. Then j runs from 0 to s, and $m_2 - r - k = s - j$. We have shown:

$$\sum_{m_1,m_2=0}^{\infty} (0,0,0,m_1) \otimes (0,0,0,m_2) \otimes (m_3,0,0,0) x^{m_1} y^{m_2},$$

$$= \sum_{r,s,k=0}^{\infty} (0,r,0,s) \otimes (m_3,0,0,0)(xy)^{r+k} \sum_{j=0}^{s} x^j y^{s-j}. \tag{18}$$

The sum over k gives $(1-xy)^{-1}$. It follows that

$$(1-xy) \sum_{m_i=0}^{\infty} (0,0,0,m_1) \otimes (0,0,0,m_2) \otimes (m_3,0,0,0) x^{m_1} y^{m_2} z^{m_3} =$$

$$\sum_{m_3,r,s,k=0}^{\infty} (0,r,0,s) \otimes (m_3,0,0,0)(xy)^r z^{m_3} \left(\sum_{j=0}^{s} x^j y^{s-j} \right).$$

Next we apply lemma 3.3. Since we're summing over all r, s, m_3, the inequalities (13)-(16) may be simplified somewhat, yielding the summation

$$\sum_{a,b,c=0}^{\infty} \sum_{r=a+b}^{\infty} \sum_{s=c}^{\infty} \sum_{k=b+c}^{a+b+c} \sum_{m_3=k+a}^{\infty}.$$

Let us introduce new variables

$$\begin{aligned} \mu_3 &= m_3 - a - k \\ \rho &= r - a - b \\ \kappa &= k - b - c \\ \alpha &= a - \kappa = a + b + c - k \\ \sigma &= s - c. \end{aligned}$$

Then we obtain

$$\sum_{m_3,r,s,k=0}^{\infty} (0,r,0,s) \otimes (m_3,0,0,0)(xy)^r z^{m_3} \left(\sum_{j=0}^{s} x^j y^{s-j} \right)$$

$$= \sum (\mu_3 + b, \rho + \kappa, \alpha + c, \alpha + \sigma)(xy)^{\rho+\alpha+\kappa+b} z^{\mu_3+\alpha+b+c+2\kappa} \left(\sum_{j=0}^{c+\sigma} x^j y^{c+\sigma-j} \right),$$

where the summation is from 0 to ∞ in all variables (except, of course, j.) Now,

$$\sum_{\mu_3,b=0}^{\infty} = \sum_{n_2=0}^{\infty}\sum_{b=0}^{n_2},$$

$$\sum_{\rho,\kappa=0}^{\infty} = \sum_{n_3=0}^{\infty}\sum_{\kappa=0}^{n_3}.$$

The sums on b and κ yield

$$\left(\frac{1-(xy)^{n_2+1}}{1-xy}\right)\left(\frac{1-z^{2(n_3+1)}}{1-z^2}\right).$$

Finally, one may show that

$$\sum_{\substack{\alpha,c,\sigma\geq 0\\ \alpha+c=n_4,\alpha+\sigma=n_1}}\ \sum_{j=0}^{c+\sigma} x^{\alpha+j}y^{\alpha+c+\sigma-j} = x^{n_1+n_4}\left(\frac{1-(x^{-1}y)^{n_1+1}}{1-x^{-1}y}\right)\left(\frac{1-(x^{-1}y)^{n_1+1}}{1-x^{-1}y}\right).$$

by checking that both sides are equal to

$$\sum_{I+J=n_1+n_4} \min(I,J,n_1,n_4)x^I y^J.$$

The identity (12) follows.

4 Poles of L functions

In this section we will characterize all generic irreducible cuspidal representations π of the group $G = GSO_8$ such that both, the Standard and the Spin L functions has a pole.

We start with a certain local result.

Lemma 4.1: *Let F be a local field. For any choice of complex numbers s_1, s_2 and s_3 there is a choice of data such that integral* (10) *is nonzero.*

Proof: This is quite standard. We refer the reader to [G-S] for details for a similar case. ■

Let S denote a set of places such that outside of S all data is unramified. We denote by $L^S(\pi, Spin, s_1)$ the partial Spin L function and by $L^S(\pi, St, 3s_3-1)$ the partial Standard L function of π. It follows from [G2], that these two L functions can have at most a simple pole at the points $s_1 = 1$ and $s_3 = 2/3$ respectively. Before stating our results concerning certain periods, we recall some basic facts about residues of Eisenstein series.

We start with the Eisenstein series $E_P(g, s_3)$. It follows from the results of [K-R], that this Eisenstein series has a simple pole at $s_3 = 1$ and $s_3 = 2/3$. The residue at the first point is the constant function and if follows from [G-R-S3] that the residue at $s_3 = 2/3$ is the minimal representation of GSO_8. Thus if we denote this representation by $\theta(g)$, we have $\theta(g) = Res_{s_3=2/3}(g, s_3)$.

Next we consider the Eisenstein series $E_Q(g, s_1, s_2)$. Taking the residue at the point $s_1 = 1$ and using the fact that the residue of the GL_2 Eisenstein series is the constant function, we thus obtain $E_Q(g, s_2) = Res_{s_1=1}E_Q(g, s_1, s_2)$. Here $E_Q(g, s_2)$ is the Eisenstein series defined on the group $GSO_8(\mathbf{A})$ which corresponds to the induced representation $Ind_{Q(\mathbf{A})}^{GSO_8(\mathbf{A})}\delta_Q^{s_2}$. From this discussion and from sections one and two we obtain

Proposition 4.2: *Suppose that the partial L functions $L^S(\pi, Spin, s_1)$ and $L^S(\pi, St, 3s_3-1)$ have simple poles at the points $s_1 = 1$ and $s_3 = 2/3$ respectively. Then there is a choice of data such that the integral*

$$\int\limits_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi(g)\theta(g)E_Q(g, s_2)dg \tag{19}$$

is not zero. ■

We now unfold integral (19). For $Re(s_2)$ large we unfold the Eisenstein series and we obtain

$$\int\limits_{Z(\mathbf{A})Q(F)\backslash G(\mathbf{A})} \varphi(g)\theta(g)f_Q(g, s_2)dg \tag{20}$$

Consider the unipotent subgroup of G generated by all matrices of the form $x(r) = I_8 + re'_{1,7}$. We expand the theta representation along this group and we obtain

$$\theta(g) = \int\limits_{F\backslash\mathbf{A}} \theta(x(r)g)dr + \sum_{\alpha\in F^*}\int\limits_{F\backslash\mathbf{A}} \theta(x(r)g)\psi(\alpha r)dr$$

Plugging this expansion into (20) it follows from the smallness properties of $\theta(g)$ that the contribution of the constant term is zero. On the remaining terms the rational points of the group $GL_2 \times GSO_4$, the Levi part of Q, acts with one orbit and the stabilizer are the rational points of the group $H = (GL_2 \times GSO_4)^0$. Here the zero indicates that the similitude factor of both groups is the same. Thus (20) equals

$$\int\limits_{Z(\mathbf{A})H(F)V(F)\backslash G(\mathbf{A})} \varphi(g) \int\limits_{F\backslash\mathbf{A}} \theta(x(r)g)\psi(r)dr f_Q(g, s_2)dg \tag{21}$$

Here V is the unipotent radical of the parabolic subgroup Q. Arguing in a similar way as in [G-R-S2] page 610 formula (4.3) we have the identity

$$\int\limits_{F\backslash\mathbf{A}} \theta(x(r)m)\psi(r)dr = \theta_\phi^\psi(m)$$

and this identity holds for all $m \in HV$. Here the function $\theta_\phi^\psi(m)$ is the theta representation defined on the group $\widetilde{Sp}_8(\mathbf{A})$. The function ϕ is a Schwartz function of $\mathbf{A}^4$. Plugging this into (21) we obtain the integral

$$\mathcal{P}_{\varphi,\phi} = \int\limits_{Z(\mathbf{A})H(F)\backslash H(\mathbf{A})} \int\limits_{V(F)\backslash V(\mathbf{A})} \varphi(vh)\theta_\phi^\psi(vh)dvdh$$

as an inner integration. We have proved a part of the following

Theorem 4.3: *Let π be an irreducible generic cuspidal representation of the group $G(\mathbf{A})$ which has a trivial central character. Then the following are equivalent:*

1) Both partial L functions, $L^S(\pi, Spin, s_1)$ and $L^S(\pi, St, 3s_3-1)$ have simple poles at $s_1 = 1$ and $s_3 = 2/3$ respectively.

2) The period integral $\mathcal{P}_{\varphi,\phi}$ is nonzero for some choice of data.

3) The representation π is the functorial lift from a cuspidal generic representation of the exceptional group $G_2(\mathbf{A})$.

Proof: We proved that 1) implies 2). Suppose that $\mathcal{P}_{\varphi,\phi}$ is nonzero for some choice of data. Let us first prove that $L^S(\pi, St, s)$ has a simple pole at $s = 1$. To do that we consider the global integral

$$\int\limits_{Z(\mathbf{A})H(F)\backslash H(\mathbf{A})} \int\limits_{V(F)\backslash V(\mathbf{A})} \varphi(v(h_1,h_2))\theta_\phi^\psi(v(h_1,h_2))E(h_1,s)dvdh_1dh_2 \qquad (22)$$

where $(h_1, h_2) \in H$ with $h_1 \in GL_2$ and $h_2 \in GSO_4$ and $E(h_1, s)$ is the Eisenstein series on GL_2 which corresponds to the induced representation $Ind_{B_2(\mathbf{A})}^{GL_2(\mathbf{A})}\delta_{B_2}^s$. Since $\mathcal{P}_{\varphi,\phi}$ is the residue of integral (22) it follows from the assumption that (22) is not zero for $Re(s)$ large. Unfolding the Eisenstein series and then the theta series we obtain

$$\int\limits_{Z(\mathbf{A})GSO_4(F)N(\mathbf{A})\backslash H(\mathbf{A})} \int\limits_{Y(\mathbf{A})\backslash V(\mathbf{A})} \int\limits_{L(F)\backslash L(\mathbf{A})} \varphi(lv(h_1,h_2))\psi_1(l)\omega_\psi(v(h_1,h_2))\phi(0)f_s(h_1)dldvdh_1dh_2 \qquad (23)$$

Here N is the maximal unipotent subgroup of GL_2 and L consists of all unipotent matrices in G of the form $t(l_1, \ldots, l_6) = I_8 + \sum_{i=1}^6 l_i e'_{1,i+1}$. The group Y is the subgroup of L which consists of all matrices of the form $t(0, l_2, \ldots, l_6)$ and $\psi_1(l) = \psi(l_6)$. Also ω_ψ is the Weil representation defined on the group $Sp_6(\mathbf{A})$. Conjugating by a suitable Weyl element w, we obtain as inner integration

$$\int\limits_{L(F)\backslash L(\mathbf{A})} \varphi(lwv(h_1,h_2))\psi_L(l)dl \qquad (24)$$

where now $\psi_L(l) = \psi(l_1)$. Let R be the unipotent subgroup of G which consists of all matrices of the form $k(r_1, r_2, r_3, r_4) = I_8 + \sum_{i=1}^4 r_i e'_{2,i+2}$. We expand the above integral along the group R. The group $GSO_4(F)$ acts on this expansion with three orbits. First the constant term. By cuspidality we get zero contribution to integral (23). Similarly, the orbit which corresponds to the nonzero vectors of length zero. It will also contribute zero to integral (23). Thus we are left with the orbit which corresponds to vectors with nonzero length. The stabilizer inside $GSO_4(F)$ contains the group SO_3 and we thus obtain as an inner integration to (23), the integral

$$\int\limits_{SO_3(F)\backslash SO_3(\mathbf{A})} \int\limits_{L(F)\backslash L(\mathbf{A})} \int\limits_{R(F)\backslash R(\mathbf{A})} \varphi(lr(1,m))\psi_L(l)\psi_R(r)drdldm \tag{25}$$

where $\psi_R(r) = \psi_R(k(r_1, r_2, r_3, r_4) = \psi(r_2 + r_3)$. From the assumptions it thus follows that integral (25) is nonzero for some choice of zero. Arguing as in [G-R-S1] theorem 3.4 we obtain that $L^S(\pi, St, s)$ has a simple pole at $s = 1$.

What is more important to us is that if we view π as a cuspidal representation of the group $SO_8(\mathbf{A})$ then its theta lift to $Sp_6(\mathbf{A})$ is a nonzero generic cuspidal representation. Indeed, this follows from [G-R-S1] proposition 3.2. In other words the integral

$$f(m) = \int\limits_{SO_8(F)\backslash SO_8(\mathbf{A})} \varphi(g)\widetilde{\theta}_\phi^\psi((m,g))dg \tag{26}$$

is not zero for some choice of data. Here $m \in Sp_6$ and $\widetilde{\theta}_\phi^\psi$ is the theta function defined on the group $\widetilde{Sp}_{48}(\mathbf{A})$ and ϕ is a Schwartz function on $\mathbf{A}^{24}$.

Let U denote the unipotent subgroup of the standard maximal parabolic subgroup of Sp_6 whose Levi part is $GL_2 \times SL_2$. In matrices we have

$$U = \left\{ u = \begin{pmatrix} 1 & & x_1 & x_2 & y_1 & y_2 \\ & 1 & x_3 & x_4 & y_2 & * \\ & & 1 & & * & * \\ & & & 1 & * & * \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right\}$$

where the * indicates that the matrix is in Sp_6. Define a character ψ_U of U by $\psi_U(u) = \psi(x_1 + x_4)$. It is easy to see that the stabilizer of ψ_U inside $GL_2 \times SL_2$ is SL_2 embedded diagonally. We shall now compute the integral

$$f^{SL_2 U,\psi}(m) = \int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{U(F)\backslash U(\mathbf{A})} f(urm)\psi_U(u)dudr$$

Plugging (26) into this we obtain

$$f^{SL_2U,\psi}(m) = \int\limits_{SO_8(F)\backslash SO_8(\mathbf{A})} \int\limits_{SL_2(F)\backslash SL_2(\mathbf{A})} \int\limits_{U(F)\backslash U(\mathbf{A})} \varphi(g)\widetilde{\theta}_\phi^\psi((urm,g))\psi_U(u)dudrdg$$

Arguing as in [G-R-S2] pages 552-553 we deduce that the right hand side converges absolutely after a suitable normalization. We unfold the theta function and use the well known formulas for the Weil representation as can be found, for example, in [M-V-W]. We have

$$\widetilde{\theta}_\phi^\psi((urm,g)) = \sum_{\delta_1,\delta_2,\delta_3\in F^8} \omega_\psi((urm,g))\phi(\delta_1,\delta_2,\delta_3)$$

Consider the polarization where the group SO_8 acts linearly on each of the vectors δ_i. Performing the integral over the variables y_i in U we may restrict the summations to all $\delta_i \in F^8$ such that $(\delta_1,\delta_1) = (\delta_1,\delta_2) = (\delta_2,\delta_2) = 0$ The group $SL_2(F)\times SO_8(F)$ acts on this set of vectors with various orbits. One can check that all orbits contribute zero except the orbit which corresponds to $\delta_i = \delta_i^0$ where $\delta_1^0 = (0,0,0,0,0,0,0,1)$ and $\delta_2^0 = (0,0,0,0,0,0,1,0)$. In this case the stabilizer inside $SL_2\times SO_8$ is the group H_0V where $H_0 = SL_2\times SO_4$ and V is the subgroup of SO_8 as defined right after (21). The group SL_2 is embedded diagonally inside $SL_2\times SO_8$ where inside the SO_8 it is embedded inside the group H defined right before (21). The group SO_4 is embedded inside the group H in the obvious way. Thus $f^{SL_2U,\psi}(m)$ equals

$$\int\limits_{H_0V(F)\backslash(SL_2\times SO_8)(\mathbf{A})} \int\limits_{T(\mathbf{A})U(F)\backslash U(\mathbf{A})} \varphi(g)\sum_{\xi\in F^6}\omega_\psi((urm,g))\phi(\delta_1^0,\delta_2^0,(0,1,\xi)\psi_U(u)dudrdg$$

Here T is the subgroup of U which consists of the subgroup generated by all matrices where $x_1 = x_3 = 0$. We also performed the integration with respect to x_2 and x_4 which gives the conditions $(\delta_1^0,\delta_3) = 0$ and $(\delta_2^0,\delta_3) = 1$ to deduce that $\delta_3 = (0,1,\xi)$ where $\xi\in F^6$.

The last two unipotent variables in U, the variables x_1 and x_4 act linearly and we obtain that $f^{SL_2U,\psi}(m)$ equals

$$\int\limits_{H_0(\mathbf{A})\backslash(SL_2\times SO_8)(\mathbf{A})} \int\limits_{\mathbf{A}^2} \int\limits_{H_0(F)\backslash H_0(\mathbf{A})} \int\limits_{V(F)\backslash V(\mathbf{A})} \varphi(vhg)\times$$

$$\sum_{\xi\in F^4}\omega_\psi(vh(rm,g))\phi(\delta_1^0,\delta_2^0,(0,1,\xi,x_1,x_2)\psi(x_2)dx_idrdg$$

We claim that there is a choice of data such that $f^{SL_2U,\psi}(m)$ is nonzero. Suppose not. This means that the above integral vanishes for all choice of data. Choosing the Schwartz functions in an appropriate way, the vanishing assumption implies that the inner integration over H_0 and V is zero for all choice of data. Analyzing the action of the groups H_0 and V on the Schwartz function in the above integration we deduce that the inner integration over H_0 and V can be realized as acting on $\theta_{\phi_1}^{\psi}$. Here $\theta_{\phi_1}^{\psi}$ is the theta representation on the group $\widetilde{Sp}_8(\mathbf{A})$ and ϕ_1 is a Schwartz function on $\mathbf{A}^4$. In other words the embedding of H_0 in the above integral is compatible with the embedding of H_0 inside Sp_8 as the tensor product, and the action of V is compatible with the action of the Heisenberg group with nine variables. From all this we deduce that the vanishing assumption we made, implies that the integral

$$\mathcal{P}'_{\varphi,\phi} = \int\limits_{H_0(F)\backslash H_0(\mathbf{A})} \int\limits_{V(F)\backslash V(\mathbf{A})} \varphi(vh)\theta_{\phi}^{\psi}(vh)dvdh$$

is zero for all choice of data. However, if we factor out the similitude element inside $\mathcal{P}_{\varphi,\phi}$ we obtain $\mathcal{P}'_{\varphi,\phi}$ as inner integration. Thus $\mathcal{P}'_{\varphi,\phi}$ is not zero for some choice of zero. We derived a contradiction which implies that $f^{SL_2U,\psi}(m)$ is not zero for some choice of data.

Let τ denote the representation of $Sp_6(\mathbf{A})$ generated by all functions of the form $f(m)$ as defined in (26). As was mentioned above τ is a generic cuspidal representation which has a nonzero period integral with respect to the group SL_2U and the character ψ_U. By this we mean that the integral $f^{SL_2U,\psi}(m)$ is not zero for some choice of data.

We now consider the lifting of τ to the exceptional group G_2. We do this as in [G-R-S3] and [G-J]. Let θ_{E_7} denote the minimal representation of the exceptional group E_7 as was constructed in [G-R-S3]. We construct the integral

$$\mathcal{F}(x) = \int\limits_{Sp_6(F)\backslash Sp_6(\mathbf{A})} f(m)\theta_{E_7}((x,m))dm \tag{27}$$

Here $x \in G_2$. Let σ be the representation of $G_2(\mathbf{A})$ generated by all functions $\mathcal{F}(x)$. It follows from [G-J] theorem 3.1 that σ is a cuspidal representation of $G_2(\mathbf{A})$. Let us remark that even though all statements in [G-J] are made for the group GSp_6, all the following statements are obtained in a similar way for the group Sp_6. From [G-J] theorem 3.3 and from the fact that $f^{SL_2U,\psi}(m)$ is not zero for some choice of data, it follows that σ is a generic representation. In particular σ is not zero.

Finally, both lifting from SO_8 to Sp_6 and from Sp_6 to G_2 given by the above constructions are functorial. Hence we proved that 2) implies 3).

To complete the proof of the theorem, we need to show that 3) implies 1). Let π be a generic cuspidal representation of the group $G(\mathbf{A})$ which is the functorial lift of a cuspidal generic representation σ of $G_2(\mathbf{A})$. It follows that $L^S(\pi, Spin, s) = L^S(\pi, St, s) =$

$L^S(\sigma,s)\zeta^S(s)$ where $L^S(\sigma,s)$ is the standard seven degree L function of G_2. It follows from [G-R-S2] that σ lifts to a cuspidal generic representation τ of PGL_3 or Sp_6. In the latter case, it follows from [C-K-PS-S] that τ lifts to a cuspidal representation of GL_7.(In that paper the lifting was established for odd orthogonal groups. However, it is expected to be similar for symplectic groups and we shall assume their result in our case.) From all this we deduce that $L^S(\sigma,s)$ is not zero at $s=1$. Hence, both L functions $L^S(\pi,Spin,s)$ and $L^S(\pi,St,s)$ have a simple pole at $s=1$. This completes the proof of the theorem. ■

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