# COUNTING STRUCTURES IN THE MÖBIUS LADDER

John P. McSorley

Department of Mathematical Sciences, Michigan Technological University, 1400 Townsend Dr., Houghton, MI 49931-1295.

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### Abstract

The Möbius ladder,  $M_n$ , is a simple cubic graph on 2n vertices. We present a technique which enables us to count exactly many different structures of  $M_n$ , and somewhat unifies counting in  $M_n$ . We also provide new combinatorial interpretations of some sequences, and ask some questions concerning extremal properties of cubic graphs.

### Introduction

The *Möbius ladder*,  $M_n$ , is a simple cubic graph on 2n vertices. It is shown in Fig 1a, and the representation of it used in this paper appears in Fig. 1b.

This class of graphs is interesting both for mathematicians, see Biggs [1], where  $M_n$  is often used as an example; and chemists, see Hosoya and Harary [7], and the references therein.

A spanning edge-subgraph of a simple graph is a subset of the edges of the graph, together with all of its vertices. All structures considered here, except those in §19 and §20, are spanning, labelled edge-subgraphs; henceforth referred to simply as edge-subgraphs.

We present a counting technique which enables us to count exactly many different structures of  $M_n$ .

### Fig. 1a and 1b here.

### Preliminaries

Consider Fig. 1b. The edges ii' for i = 1, ..., n are *diagonals*, all other edges are *outside edges*. The graph  $M_n$  contains n diagonals and 2n outside edges. Unless otherwise stated, we will always assume that  $n \ge 2$  and  $1 \le k \le n$ .

Now consider the 2n vertices labelled 1, 2, 3, ..., n, 1', 2', 3', ..., n' arranged clockwise around a circle, see Fig. 2a. A (n, k)-graph, G, is a subgraph of  $M_n$  with vertex set these 2n vertices, and edge set any k diagonals and any number of outside edges.

#### Fig. 2a and 2b here.

Call vertices which lie on a diagonal diagonal vertices. Moving clockwise, call the edge-subgraph of G between two consecutive diagonal vertices, u and v, a *join*. Denote it by [u, v), see Fig. 2b; clearly G has 2k joins. The join [u, v) does not contain vertex v, hence every vertex and every outside edge of G lie in exactly one join. Call the join [u, v) complete if all possible outside edges are present in it, *incomplete* otherwise. Thus, a complete join is a path with its last vertex removed.

Call the join [u, v) empty if it contains no edges, and G empty if all of its joins are empty. We will often use K to denote an empty (n, k)-graph. The joins [u, v) and [u', v')(see Fig. 2b) are opposite and comprise a join-pair. Such a join-pair is incomplete if at least one of [u, v) or [u', v') is incomplete. The graph G is join-pair incomplete if each of its k join-pairs is incomplete.

Let the non-diagonal vertices between u and v be labelled  $u_2, \ldots, u_r$ . We define the *size* of the join  $[u, v) = [u, u_2, \ldots, u_r, v)$  to be r, which is the maximum possible number of edges in the join. A join of size 1 is an edge or a non-edge; a join is even/odd if its size is even/odd. It is straightforward to prove the following theorem:

**Theorem A.** Let G be a (n, k)-graph:

- (i) G is acyclic if and only if G is join-pair incomplete and has  $\leq k 1$  complete joins;
- (ii) if G is connected then G has  $\geq k 1$  complete joins;
- (iii) if G is acyclic and connected then G is join-pair incomplete and has exactly k-1 complete joins.

The following is a key definition and theorem.

A k-composition of  $n, x_1 + \cdots + x_k$ , is an expression of n as an ordered sum of k positive integers, each  $x_j \ge 1$ . Throughout this paper we let  $\underline{x} = x_1 \cdots x_k = x_1 + \cdots + x_k$  be an arbitrary k-composition of n, there are  $\binom{n-1}{k-1}$  such compositions.

Let  $\theta(n,k)$  be the number of empty (n,k)-graphs.

Consider the vertices  $[n] = \{1, 2, ..., n\}$  of a (n, k)-graph. Now the diagonal vertices without the ' form a k-set of [n]. Conversely, a k-set of [n] determines k diagonals by choosing each vertex in the k-set to be a diagonal vertex. Thus  $\theta(n, k) = {n \choose k}$ . However, we need the following theorem, central to the main idea of this paper.

**Theorem B.** Let  $\theta(n,k)$  denote the number of empty (n,k)-graphs, then

$$\theta(n,k) = \frac{n}{k} \sum_{\underline{x}} 1 = \frac{n}{k} \sum_{\underline{x}} \prod_{j=1}^{k} 1^2.$$

**Proof.** Let K be an empty (n, k)-graph which contains the diagonal 11', see Fig. 3. Let us start at vertex 1 and move clockwise, as we move we record the size of the joins which we pass through. Let the *j*-th join be  $X_j$ , and let  $|X_j| = x_j$  for  $1 \le j \le k$ . After k joins we obtain  $x_1 \cdots x_k$ , a k-composition of n. Conversely, any k-composition of  $n, y_1 \cdots y_k$ , uniquely determines a (n, k)-graph which contains 11', by starting at 1 and (moving clockwise) letting the size of the *j*-th join be  $y_j$  for  $1 \le j \le k$ . Hence, the number of (n, k)-graphs containing 11' is  $\sum_x 1$ .

Similarly, for any fixed  $i, 2 \leq i \leq n$ , the number of (n, k)-graphs containing ii' is  $\sum_{\underline{x}} 1$ .

Hence  $\sum_{i=1}^{n} \left| \left\{ \begin{array}{c} (n,k) - \text{graphs} \\ \text{containing } ii' \end{array} \right\} \right| = n \sum_{\underline{x}} 1$ . But each (n,k)-graph is counted k times in the left-hand side of this equation. Hence  $k\theta(n,k) = n \sum_{x} 1$ , i.e.,  $\theta(n,k) = \frac{n}{k} \sum_{x} 1$ .

Now, an empty join can be made into an empty join in  $\overline{1}$  way, and an empty join-pair into an empty join-pair in  $1^2$  way(s). So the product  $\prod_{j=1}^{k} 1^2$  gives the number of ways the k join-pairs of K can be extended to an empty (n, k)-graph. Hence the result.

### Fig. 3 here.

Let  $[x^n]H(x)$  be the coefficient of  $x^n$  in H(x). We have the following result, where ' denotes differentiation.

# Theorem C.

(i) 
$$\sum_{\underline{x}} \prod_{j=1}^{n} \sigma(x_j) = [x^n] \{ \sigma(1)x + \sigma(2)x^2 + \cdots \}^k.$$
  
(ii) Let  $H(x) = xJ(x)$ , where  $J(0)$  and  $J'(0)$  are defined.  
Then  $[x^n] \sum_{k=1}^{n} \frac{n}{k} H(x)^k = [x^n] \frac{xH'(x)}{1 - H(x)}.$ 

Now let the elements of [n] be arranged in a circle. A non-consecutive, cyclic – ncc – set of [n] is a subset of elements of [n] which are non-consecutive when chosen from this circular arrangement. Let  $\psi(n, t)$  denote the number of ncc t-sets of [n], where  $0 \le t \le \lfloor \frac{n}{2} \rfloor$ .

So 
$$\psi(n,t) = \frac{n}{n-t} {\binom{n-t}{t}}$$
 and  $\mathcal{L}(n) = \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \psi(n,t) = \left((1+\sqrt{5})/2\right)^n + \left((1-\sqrt{5})/2\right)^n$ ,  
where  $\mathcal{L}(n)$  is the *n*-th Lucas number are non-24 and 46 of Comptet [2] and non-72.

where  $\mathcal{L}(n)$  is the *n*-th Lucas number; see pp. 24 and 46 of Comtet [3], and pp. 73 and 246 of Stanley [10].

In the following sections, when counting **Generic Structures** – gs – of  $M_n$ , we let gs(n,k) denote the number of generic structures of  $M_n$  with k diagonals, and gs(n) the total number of generic structures of  $M_n$ .

The idea of this paper is to count generic structures in  $M_n$  by extending an empty (n, k)-graph, K, to a generic structure by adding edges to its empty joins, whilst keeping k fixed. For example in §2, if K is extended to a forest of  $M_n$ , then a join of the extended K can be any arbitrary edge-subgraph of the complete join; for a tree (§8), a join must be either complete or complete less one edge. So, we first find gs(n,k) by summing over  $\underline{x}$ , and then sum over k to determine g(n).

Even though some of these structures have been counted before, and thus some counts appear in the literature, it seems that this technique is new, and somewhat unifies the approach to counting structures in  $M_n$ ; it also works for the cubic prism. For a unified approach to counting structures in  $K_n$ , the complete graph on n vertices, see Harary and Palmer [6].

We are now ready to start counting.

### 1. Edge-Subgraphs -e

Clearly  $M_n$  has 3n edges, so  $e(n) = 2^{3n} = 8^n$ .

We now illustrate our technique by redetermining e(n).

First consider k = 0. Let  $C_{2n}$  denote the cycle with 2n vertices and 2n edges; any edge-subgraph of  $C_{2n}$  is an edge-subgraph of  $M_n$  with no diagonals, hence  $e(n,0) = 4^n$ .

For  $k \ge 1$ , let K be the empty (n, k)-graph that contains diagonal 11' and determines the composition  $\underline{x} = x_1 \cdots x_k$ , see Fig. 3, and let  $[k] = \{1, 2, \ldots, k\}$ . Now, if K is to be extended to an edge-subgraph, then a join of size r of the extended K can be any of the  $2^r$ edge-subgraphs of the complete join. From Theorem B, the number of empty (n, k)-graphs is  $\frac{n}{k} \sum_{\underline{x}} \prod_{j=1}^{k} 1^2$ , so, using the multiplication principle, we have

$$e(n,k) = \frac{n}{k} \sum_{\underline{x}} \prod_{j=1}^{k} (2^{x_j})^2.$$
 (1)

So 
$$e(n) = e(n,0) + \sum_{k=1}^{n} e(n,k) = 4^n + \sum_{k=1}^{n} \left(\frac{n}{k} \sum_{\underline{x}} \prod_{j=1}^{k} 4^{x_j}\right) = 8^n.$$

### **2.** Forests -f

A forest is an acyclic edge-subgraph. If a (n, k)-graph, F, is a forest of  $M_n$  then an incomplete join of F is an arbitrary edge-subgraph of the complete join, (except the complete join).

Using Theorem A(i), we have:

### **Lemma 2.1.** Let F be a (n, k)-graph. Then F is a forest of $M_n$ if and only if

- (i) F is join-pair incomplete;
- (ii) F has  $\leq k 1$  complete joins; and
- (iii) each incomplete join of F is an arbitrary incomplete join.

If an incomplete join has size r then it can be any of  $2^r - 1$  edge-subgraphs.

Any edge-subgraph of  $C_{2n}$ , except  $C_{2n}$  itself, is a forest of  $M_n$ ; hence,  $f(n,0) = 4^n - 1$ . Now for  $k \ge 1$ . Let t be an arbitrary variable; for  $\underline{x} = x_1 \cdots x_k$ , with n and k fixed, define

$$\pi(\underline{x},t) = \pi(x_1\cdots x_k,t) = \prod_{j=1}^k (t^{x_j}-1),$$

and

$$\alpha(n,k,t) = \sum_{\underline{x}} \pi(\underline{x},t).$$

Then, using Theorem C(i)

$$\alpha(n,k,t) = [x^n]\{(t-1)x + (t^2 - 1)x^2 + \dots\}^k$$
$$= [x^n] \left\{ \frac{(t-1)x}{(1-x)(1-tx)} \right\}^k = [x^n] A_{[t]}(x)^k, \text{ say.}$$
(2)

Again, let K be as in Fig. 3. Consider the (n, k)-graph  $\overline{F}_s$  that has s complete joins on the right-hand side of 11' for some s, where  $0 \le s \le k - 1$ , and no complete joins on the left-hand side.

By the above lemma  $\overline{F}_s$  is a forest and the number of such forests is

$$\pi(\underline{x},2) \cdot \sum_{S \subset [k]} \left\{ \frac{\pi(\underline{x},2)}{\prod_{j \in S} (2^{x_j}-1)} \right\},\,$$

where S is a s-set of [k], and  $\prod_{i=1}^{n} (2^{x_j} - 1) = 1$ .

The first factor in the expression above gives the number of possible edge-subgraphs on the left-hand side of 11' in  $\overline{F}_s$ , and the second factor the number on the right-hand side.

Now  $\overline{F}_s$  is join-pair incomplete and so the s complete joins on the right-hand side of 11' can be arranged in  $2^s$  ways: for each complete join [u, v), switch it with its incomplete opposite join [u', v'), (if a(a + 1) is an edge in [u, v) then, after switching, a'(a + 1)' is an edge in [u', v'), and vice versa). Conversely, if a forest contains s complete joins then  $0 \le s \le k-1$  and it is join-pair incomplete, so we may form such a graph  $\overline{F}_s$  by switching all complete joins from the left-hand side of 11' to the right-hand side.

Hence, incorporating the factor of  $2^s$  into the above expression and simplifying, the total number of forests which can be formed from K is

$$\pi(\underline{x}, 2)^{2} \sum_{S \subset [k]} \prod_{j \in S} \left( \frac{2}{2^{x_{j}} - 1} \right)$$
(3)  
$$= \pi(\underline{x}, 2)^{2} \left\{ \prod_{j=1}^{k} \left( 1 + \frac{2}{2^{x_{j}} - 1} \right) - \prod_{j=1}^{k} \left( \frac{2}{2^{x_{j}} - 1} \right) \right\}$$
$$= \pi(\underline{x}, 2) \left\{ \prod_{j=1}^{k} (2^{x_{j}} + 1) - 2^{k} \right\}$$
(4)  
$$= \pi(\underline{x}, 4) - 2^{k} \pi(\underline{x}, 2).$$

Now, using Theorem B, the number of empty (n, k)-graphs is  $\frac{n}{k} \sum_{x} 1$ .

Hence

$$f(n,k) = \frac{n}{k} \sum_{\underline{x}} \{\pi(\underline{x},4) - 2^k \pi(\underline{x},2)\} = \frac{n}{k} \{\alpha(n,k,4) - 2^k \alpha(n,k,2)\}.$$
 (5)

Using (2) and (5), we have:

### Lemma 2.2.

$$f(n,k) = \begin{cases} 4^n - 1, & k = 0; \\ [x^n] \frac{n}{k} \{ A_{[4]}(x)^k - (2A_{[2]}(x))^k \}, & k \ge 1. \end{cases}$$

Then

$$f(n) = 4^{n} - 1 + [x^{n}] \left\{ \sum_{k=1}^{n} \frac{n}{k} \{A_{[4]}(x)^{k} - (2A_{[2]}(x))^{k}\} \right\},\$$

now use Theorem C(ii) twice,

$$= 4^{n} - 1 + [x^{n}] \left\{ \frac{x A'_{[4]}(x)}{1 - A_{[4]}(x)} - \frac{2x A'_{[2]}(x)}{1 - 2A_{[2]}(x)} \right\}$$
  
=  $4^{n} - 1 + [x^{n}] \frac{x(1 + 4x - 40x^{2} + 16x^{3} + 16x^{4})}{(1 - 2x)(1 - 4x)(1 - 5x + 2x^{2})(1 - 8x + 4x^{2})}$ .

Thus, the above rational function is the ordinary generating function for the total number of forests of  $M_n$  with  $k \ge 1$ .

Finally, partial fractions give:

### Theorem 2.3.

$$f(n) = (4 + \sqrt{12})^n + (4 - \sqrt{12})^n + 2^n - \left((5 + \sqrt{17})/2\right)^n - \left((5 - \sqrt{17})/2\right)^n - 1.$$

The numbers  $\{f(n) : n \ge 2\}$  yield the sequence  $\{38, 328, 2686, 21224, \ldots\}$  which does not appear in Sloane and Plouffe [9]. The numbers  $\{f(n,n)\} = \{3^n - 2^n\}$  appear as sequence M3887, so we have a new combinatorial interpretation of this sequence.

Note that from p. 103 of Biggs [1], and Biggs, Damerell, and Sands [2], the Möbius ladders form a recursive family of graphs, i.e., their Tutte polynomials satisfy a homogeneous linear recurrence relation, and so can be computed. Hence, see p. 104 of [1], we may count the number of forests to give the same as above.

So, for large n, f(n) is approximately  $(7.4641)^n$ . For an arbitrary cubic graph on 2n vertices the number of forests is less than  $8^n$ . A natural question is then: Does there exist a family of cubic graphs,  $G_n$ , on 2n vertices, such that, for some  $n_0$  and all  $n > n_0$ , the number of forests of  $G_n$  is greater than  $(7.4641)^n$ ?

### 3. Strong Edge-Subgraphs – se

A strong edge-subgraph, E, of  $M_n$  is an edge-subgraph with no isolated vertices. If an edge-subgraph of  $C_{2n}$  is strong then its non-edges must be non-consecutive, cyclic. Thus,  $se(n,0) = \mathcal{L}(2n)$ .

Similarly, the join  $[u, u_2, \ldots, u_r, v)$  of E is strong if it contains no isolated vertices. For this join to be strong we only require that the vertices  $u_2, \ldots, u_r$ , are not isolated because u is incident to a diagonal.

Let  $\beta(r)$  be the number of strong joins for a join of size r. For  $k \ge 1$ , the number of strong edge-subgraphs which come from K of Fig. 3 is  $\prod_{j=1}^{k} \beta(x_j)^2$ .

Thus, (cf., (1)),

$$se(n,k) = \frac{n}{k} \sum_{\underline{x}} \prod_{j=1}^{k} \beta(x_j)^2.$$

Now  $\beta(1) = 2$ ,  $\beta(2) = 3$ , and  $\beta(r) = \beta(r-1) + \beta(r-2)$  for  $r \ge 3$ , see p. 45 of [3]. The recurrence for  $\beta(r)^2$  is  $\beta(r)^2 = 2\beta(r-1)^2 + 2\beta(r-2)^2 - \beta(r-3)^2$ .

Let 
$$B(x) = \sum_{r \ge 1} \beta(r)^2 x^r = \frac{x(4+x-x^2)}{(1+x)(1-3x+x^2)}.$$

Lemma 3.1.

$$se(n,k) = \begin{cases} \mathcal{L}(2n), & k = 0; \\ [x^n]\frac{n}{k}B(x)^k, & k \ge 1. \end{cases}$$

By similar reasoning to that used in  $\S2$ , we have

$$se(n) = \mathcal{L}(2n) + [x^n] \frac{x(4+2x+3x^2-4x^3+x^4)}{(1+x)(1-3x+x^2)(1-6x-3x^2+2x^3)}$$

Theorem 3.2.

$$se(n) \approx (6.4188)^n + (-0.8056)^n + (0.3867)^n - (-1)^n.$$

The first few terms in the sequence  $\{se(n) : n \ge 2\}$  are  $\{41, 265, 1697, 10897, \ldots\}$ .

Let  $H_n$  be an arbitrary cubic bipartite graph on 2n vertices, then, necessarily, the number of vertices in each part is n. Let one part contain the vertices  $v_1, \ldots, v_n$ . Now each  $v_i$  is incident to 3 edges. A strong edge-subgraph of  $H_n$  must contain at least one of these edges i.e., it must contain one of the  $2^3 - 1 = 7$  non-empty subsets of edges incident to  $v_i$ . Hence, the number of strong edge-subgraphs of  $H_n$  is at most  $7^n$ .

Now, for n odd, the graph  $M_n$  is bipartite, and, for large n, contains approximately  $(6.4188)^n$  strong edge-subgraphs. Hence, we may ask the following question: For odd n, does there exist a family of cubic bipartite graphs,  $G_n$ , on 2n vertices, such that, for some  $n_0$  and all odd  $n > n_0$ , the number of strong edge-subgraphs of  $G_n$  is greater than  $(6.4188)^n$ ?

### 4. Strong Forests -sf

A strong forest, F, of  $M_n$  is a forest with no isolated vertices. By comparison with Lemma 2.1, we have:

**Lemma 4.1.** Let F be a (n, k)-graph. Then F is a strong forest of  $M_n$  if and only if

- (i) F is join-pair incomplete;
- (ii) F has  $\leq k 1$  complete joins; and
- (iii) each incomplete join of F is a strong incomplete join.

For k = 0, the number of strong forests of  $M_n$  equals the number of strong edgesubgraphs of  $C_{2n}$  less 1,  $(C_{2n}$  itself). Thus,  $sf(n,0) = \mathcal{L}(2n) - 1$ .

Let  $\gamma(r)$  be the number of strong incomplete joins for a join of size r.

By similar reasoning to that used in  $\S2$ , the number of strong forests which come from K of Fig. 3 is given by

$$\prod_{j=1}^k \gamma(x_j) \cdot \bigg\{ \prod_{j=1}^k (\gamma(x_j) + 2) - 2^k \bigg\}.$$

Thus, (cf., (5)),

$$sf(n,k) = \frac{n}{k} \bigg\{ \sum_{\underline{x}} \prod_{j=1}^{k} (\gamma(x_j)^2 + 2\gamma(x_j)) - \sum_{\underline{x}} \prod_{j=1}^{k} (2\gamma(x_j)) \bigg\}.$$
 (6)

Now  $\gamma(1) = 1$ ,  $\gamma(2) = 2$ , and  $\gamma(r) = \gamma(r-1) + \gamma(r-2) + 1$  for  $r \ge 3$ .

Let 
$$C_1(x) = \sum_{r \ge 1} (\gamma(r)^2 + 2\gamma(r))x^r = \frac{x(3-x)}{(1-x^2)(1-3x+x^2)}$$
 and  
 $C_2(x) = \sum_{r \ge 1} 2\gamma(r)x^r = \frac{2x}{(1-x)(1-x-x^2)}$ .

Again, using Theorem C(i) and (6), we have: Lemma 4.2.

$$sf(n,k) = \begin{cases} \mathcal{L}(2n) - 1, & k = 0;\\ [x^n]\frac{n}{k} \{C_1(x)^k - C_2(x)^k\}, & k \ge 1. \end{cases}$$

$$sf(n) = \mathcal{L}(2n) - 1 + [x^n] \frac{x(1 - 48x^3 + 48x^4 - 4x^5 - 3x^6 + 8x^7 - 8x^8 + 2x^9)}{(1 + x)(1 - 3x + x^2)(1 - x - x^2)(1 - 4x + x^3)(1 - 6x + x^2 + 3x^3 - x^4)}.$$

Theorem 4.3.

$$sf(n) \approx (5.7400)^n + (-0.7340)^n + (0.5953)^n + (0.3986)^n + ((1+\sqrt{5})/2)^n + ((1-\sqrt{5})/2)^n - (-0.4728)^n - (3.9354)^n - (0.5374)^n - (-1)^n - 1.$$

The first few terms in the sequence  $\{sf(n) : n \ge 2\}$  are  $\{19, 132, 851, 5298, \ldots\}$ .

### 5. Single-Component Edge-Subgraphs – sce

A single-component edge-subgraph, E, of  $M_n$  is an edge-subgraph with exactly one non-trivial connected component, i.e., exactly one connected component with one or more edges, all other components being isolated vertices. Clearly sce(n, 0) = 2n(2n - 1) + 1.

If [u, v) has size r then the number of possible incomplete joins is  $\frac{r(r+1)}{2}$ .

Let E be a single-component edge-subgraph which comes from K of Fig. 3, and let  $\overline{E}$ denote the graph obtained from E by switching all complete joins from the left-hand side of 11' to the right-hand side. From Theorem A(ii) we see that  $\overline{E}$  has  $\geq k-1$  complete joins to the right-hand side of 11'.

Let E have (k-1) + s complete joins in total, where  $0 \le s \le k+1$ . Then two cases can occur:

(a)  $\overline{E}$  has exactly k-1 complete joins to the right-hand side of 11'. Let the *i*-th join on the right-hand side of 11' in  $\overline{E}$  be incomplete, for some i, where  $1 \leq i \leq k$ , then its opposite join is also incomplete, (or otherwise case (b) occurs). So  $\overline{E}$  has s complete joins to the left-hand side of 11', and E has exactly s complete join-pairs; here  $0 \le s \le k-1$ .

Let Z be a (k-1-s)-subset of  $[k] \setminus \{i\}$  and let  $\prod_{j \in \emptyset} \frac{x_j(x_j+1)}{2} = 1$ .

Then, the number of such E is given by

$$\sum_{i=1}^{k} \left\{ \sum_{s=0}^{k-1} \left( \sum_{Z \subseteq [k] \setminus \{i\}} \prod_{j \in Z} \frac{x_j(x_j+1)}{2} \cdot 2^{|Z|} \right) \left( \frac{x_i(x_i+1)}{2} \right)^2 \right\}$$
$$= \frac{1}{4} \sum_{i=1}^{k} \left\{ \prod_{j=1}^{k} (x_j^2 + x_j + 1)(x_i^2 + x_i)^2 \right\}.$$

(b)  $\overline{E}$  has exactly k complete joins to the right-hand side of 11', so  $\overline{E}$  has s-1 complete joins to the left-hand side of 11', and E has exactly s-1 complete join-pairs; here  $1 \le s \le k+1$ . Let Z be a (k - (s - 1))-subset of [k]. Hence, the number of such E is given by

$$\sum_{s=1}^{k+1} \left( \sum_{Z \subseteq [k]} \prod_{j \in Z} \frac{x_j(x_j+1)}{2} \cdot 2^{|Z|} \right) = \prod_{j=1}^k (x_j^2 + x_j + 1).$$

Lemma 5.1.

$$sce(n,k) = \frac{n}{4k} \sum_{\underline{x}} \left( \sum_{i=1}^{k} \left\{ \prod_{\substack{j=1\\j\neq i}}^{k} (x_j^2 + x_j + 1) (x_i^2 + x_i)^2 \right\} \right) + \frac{n}{k} \sum_{\underline{x}} \prod_{j=1}^{k} (x_j^2 + x_j + 1).$$
(7)

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Let  

$$D_1(x) = \sum_{r \ge 1} (r^2 + r + 1)x^r = \frac{x(3 - 2x + x^2)}{(1 - x)^3}; D_2(x) = \sum_{r \ge 1} (r^2 + r)^2 x^r = \frac{4x(1 + 4x + x^2)}{(1 - x)^5}.$$
  
Lemma 5.2.

$$sce(n,k) = \begin{cases} 2n(2n-1)+1, & k=0;\\ [x^n] \left\{ \frac{n}{4} D_1(x)^{k-1} D_2(x) + \frac{n}{k} D_1(x)^k \right\}, & k \ge 1. \end{cases}$$

This gives us

$$sce(n) = 2n(2n-1) + 1 + [x^n] \frac{x(4-13x+x^2-44x^3+54x^4-25x^5+x^6-2x^7)}{(1-x)^3(1-6x+5x^2-2x^3)^2} .$$

Partial fractions and De Moivre's theorem give: Theorem 5.3.

$$sce(n) \approx (0.6612n + 1)(5.0958)^n + (2.1430)(0.6264)^n n \sin(0.7646n) + n^2 - (0.6612n - 2)(0.6264)^n \cos(0.7646n) - 2n - 2.$$

The first few terms in the sequence  $\{sce(n) : n \ge 2\}$  are  $\{60, 397, 2464, 14809, \ldots\}$ .

### 6. Single-Component Forests – scf

A single-component forest, F, of  $M_n$  is a forest which contains exactly one non-trivial connected component, all other components being isolated vertices.

An incomplete join [u, v) of F must have at most two non-trivial components, each component must contain an edge incident to either u or v.

Thus, using Theorem A(iii), we have:

**Lemma 6.1.** Let F be a (n, k)-graph. Then F is a single-component forest of  $M_n$  if and only if

- (i) F is join-pair incomplete;
- (ii) F has exactly k-1 complete joins; and
- (iii) each incomplete join [u, v) of F has at most two non-trivial components, each component containing an edge incident to either u or v.

For k = 0, we have scf(n, 0) = 2n(2n - 1).

Again, if [u, v) has size r, then the number of possible incomplete joins is  $\frac{r(r+1)}{2}$ .

For  $k \ge 1$  let K be the (n, k)-graph of Fig. 3. Following §2 with s = k - 1, the number of single-component forests which can be formed from K is given by

$$\left(2^{k-1}\prod_{j=1}^{k}\frac{x_j(x_j+1)}{2}\right)\cdot\sum_{i=1}^{k}\frac{x_i(x_i+1)}{2} = \frac{1}{4}\sum_{i=1}^{k}\left\{\prod_{\substack{j=1\\j\neq i}}^{k}(x_j^2+x_j)(x_i^2+x_i)^2\right\}.$$
(8)

So

$$scf(n,k) = \frac{n}{4k} \sum_{\underline{x}} \left( \sum_{i=1}^{k} \left\{ \prod_{\substack{j=1\\j \neq i}}^{k} (x_j^2 + x_j) (x_i^2 + x_i)^2 \right\} \right).$$

Let  $E(x) = \sum_{r \ge 1} (r^2 + r) x^r = \frac{2x}{(1-x)^3}$  and  $D_2(x)$  be as in §5.

Lemma 6.2.

$$scf(n,k) = \begin{cases} 2n(2n-1), & k = 0; \\ [x^n]\frac{n}{4}E(x)^{k-1}D_2(x), & k \ge 1. \end{cases}$$

Thus

$$scf(n) = 2n(2n-1) + [x^n] \frac{x(1+9x-30x^2-20x^3+27x^4-9x^5-2x^6)}{(1-x)^3(1-5x+3x^2-x^3)^2}.$$

#### Theorem 6.3.

$$scf(n) \approx 0.8757n(4.3652)^n - 1.5432n(0.4786)^n \cos(0.8458n + 0.9674) + n^2 - 2n.$$

The first few terms in the sequence  $\{scf(n) : n \ge 2\}$  are  $\{34, 222, 1280, 6955, \ldots\}$ . The sequence  $\{scf(n, n)\} = \{n2^{n-1}\}$  appears as M3444 in [9].

# 7. Strong Single-Component Edge-Subgraphs – ssce

A strong single-component edge-subgraph is a strong edge-subgraph which contains exactly one non-trivial connected component. Clearly, ssce(n,0) = 2n + 1.

If E, an (n, k)-graph, is a strong single-component edge-subgraph of  $M_n$  then every incomplete join of E is the complete join less one edge; hence an incomplete join of size rhas r possibilities.

Following (7) from §5 we have:

### Lemma 7.1.

$$ssce(n,k) = \frac{n}{k} \sum_{\underline{x}} \left( \sum_{i=1}^{k} \left\{ \prod_{\substack{j=1\\ j \neq i}}^{k} (2x_j + 1) x_i^2 \right\} \right) + \frac{n}{k} \sum_{\underline{x}} \prod_{j=1}^{k} (2x_j + 1).$$

Let 
$$F_1(x) = \sum_{r \ge 1} (2r+1)x^r = \frac{x(3-x)}{(1-x)^2}$$
 and  $F_2(x) = \sum_{r \ge 1} r^2 x^r = \frac{x(1+x)}{(1-x)^3}$ 

Lemma 7.2.

$$ssce(n,k) = \begin{cases} 2n+1, & k=0\\ [x^n] \left\{ nF_1(x)^{k-1}F_2(x) + \frac{n}{k}F_1(x)^k \right\}, & k \ge 1. \end{cases}$$

Thus

$$ssce(n) = 2n + 1 + [x^n] \frac{x(4 - 15x + 2x^2 + 5x^3)}{(1 - x)^2(1 - 5x + 2x^2)^2} .$$

### Theorem 7.3.

$$ssce(n) = \left(\left\{\frac{17-\sqrt{17}}{34}\right\}n+1\right)\left(\frac{5+\sqrt{17}}{2}\right)^n + \left(\left\{\frac{17+\sqrt{17}}{34}\right\}n+1\right)\left(\frac{5-\sqrt{17}}{2}\right)^n + n-1.$$

The first few terms in the sequence  $\{ssce(n) : n \ge 2\}$  are  $\{38, 205, 1092, 5719, \ldots\}$ .

#### 8. Spanning Trees -st

If a (n, k)-graph is a spanning tree of  $M_n$  then, as in §7 above, an incomplete join must be the complete join less one edge.

**Lemma 8.1.** Let F be a (n, k)-graph. Then F is a spanning tree of  $M_n$  if and only if (i) F is join-pair incomplete;

- (ii) F has exactly k-1 complete joins; and
- (iii) each incomplete join of F is the complete join less one edge.

For k = 0 we have st(n, 0) = 2n. The number of possible incomplete joins for a join of size r is r. So, for  $k \ge 1$ , cf., (8) from §6, the number of spanning trees which can be formed from K of Fig. 3 is

$$\left(2^{k-1}\prod_{j=1}^{k} x_j\right) \cdot \sum_{i=1}^{k} x_i = n \, 2^{k-1} \prod_{j=1}^{k} x_j \; .$$

Then

$$st(n,k) = \frac{n^2 2^{k-1}}{k} \sum_{\underline{x}} \prod_{j=1}^k x_j \; .$$

Letting 
$$G(x) = \sum_{r \ge 1} r x^r = \frac{x}{(1-x)^2}$$
, we have:

Lemma 8.2.

$$st(n,k) = \begin{cases} 2n, & k = 0;\\ [x^n] \frac{n^2 2^{k-1}}{k} G(x)^k, & k \ge 1. \end{cases}$$

So

$$st(n) = 2n + [x^n] \frac{x(1+2x-10x^2+2x^3+x^4)}{(1-x)^2(1-4x+x^2)^2} .$$

Theorem 8.3.

$$st(n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n] + n.$$

The first few terms in the sequence  $\{st(n) : n \ge 2\}$  are  $\{16, 81, 392, 1815, \ldots\}$ .

According to p. 42 of Biggs [1] there are two known methods to compute the number of spanning trees of  $M_n$ . The first is the Matrix Tree Theorem, see pp. 39–40 of [1]; and the second is a recursive method mentioned at the end of §2; see Biggs, Damerell, and Sands [2], and Sedlacek [8]. This, then, provides a third method.

#### 9. Restricted Edge-Subgraphs – re

A restricted edge-subgraph of  $M_n$  is an edge-subgraph with maximum degree 2, i.e., every component is an isolated vertex, a path, or a cycle.

It is straightforward to show that an edge-subgraph, E, of  $M_n$  is restricted if and only if it is the complement in  $M_n$  of a strong edge-subgraph, (§3). Clearly, the number of complements of strong edge-subgraphs is equal to the number of strong edge-subgraphs, hence re(n) = se(n). Thus:

### Theorem 9.1.

$$re(n) \approx (6.4188)^n + (-0.8056)^n + (0.3867)^n - (-1)^n.$$

### 10. Restricted Forests -rf

In a restricted forest every component is an isolated vertex or a path. Unfortunately, we are unable to use this counting technique to compute the exact value of rf(n). However, we can use it to obtain the following theorem, the details are omitted.

**Theorem 10.1.** For large n,

$$(4.8820)^n < rf(n) < (6.4188)^n.$$

The first few terms in the sequence  $\{rf(n) : n \ge 2\}$  are  $\{34, 241, 1582, \ldots\}$ , which suggests that the value of rf(n) is about  $(6.3)^n$ .

In some of the following sections we will need knowledge of cycles and unions of cycles in  $M_n$ .

Clearly, each union of cycle(s) in  $M_n$  has either k = 0 or  $k \ge 1$  diagonals, so

- (i) k = 0: we have one cycle of  $M_n$ , i.e.,  $C_{2n}$ ;
- (ii)  $k \ge 1$ : to transform the empty (n, k)-graph K of Fig. 3 into a union of cycle(s) and isolated vertices without increasing the number of diagonals, we must complete every second join beginning at  $X_1$  or at  $X_2$ . Either way will result in a single cycle if k is odd; or a union of  $(\frac{k}{2})$  cycles, each containing 2 diagonals, if k is even.

These constructions determine all cycles and union of cycles of  $M_n$ .

### 11. Union of Cycles -uc

An edge-subgraph of  $M_n$  is a *union of cycles* if each component is a cycle or an isolated vertex.

Clearly uc(n,0) = 2. For  $k \ge 1$ , by the comments above, each K of Fig. 3 determines 2 such edge-subgraphs, so  $uc(n,k) = \frac{n}{k} \sum_{\underline{x}} 2 = \binom{n}{k} 2$ .

### Theorem 11.1.

$$uc(n) = 2^{n+1}.$$

### 12. Cycles – c

A cycle of  $M_n$  is an edge-subgraph with exactly one cycle, all other components being isolated vertices. We have: c(n,0) = 1, c(n,2) = 2, and c(n,k) = 0 for k even and  $\geq 4$ ; and c(n,k) = 2 for k odd. Hence

		I

$$c(n) = 1 + 2\binom{n}{2} + \sum_{k \text{ odd}} 2\binom{n}{k}.$$

Theorem 12.1.

$$c(n) = 2^n + n^2 - n + 1.$$

This formula appears in Entringer and Slater [4]. The first few terms in the sequence  $\{c(n): n \ge 2\}$  are  $\{7, 15, 29, 53, \ldots\}$ .

#### 13. Hamiltonian Cycles -hc

Clearly hc(n,0) = 1. If K of Fig. 3 determines the 2-composition 1, n-1 of n then K gives rise to two Hamiltonian cycles for n = 2 and one if  $n \ge 3$ . Hence, hc(2,2) = 2 and  $hc(n,2) = 1 \cdot n = n$  if  $n \ge 3$ ; this deals with even k. For odd k, the graph K must determine the k-composition  $\underbrace{1 \cdots 1}_{k}$  of n, which gives two more Hamiltonian cycles, hence

### k = n.

Theorem 13.1.

$$hc(n) = \begin{cases} n+1, & n \text{ even} \\ n+3, & n \text{ odd.} \end{cases}$$

#### 14. Paths -p

Let F be a (n, k)-graph which is a path of  $M_n$ , with first vertex  $v_f$  and last vertex  $v_\ell$ ,  $v_f \neq v_\ell$ ; as usual all other components of F are isolated vertices. We will abuse notation and denote by F the non-trivial connected component of F.

Now F is acyclic and connected, and so, from Theorem A(iii), it is join-pair incomplete and has exactly k - 1 complete joins. Hence F has exactly one join-pair,  $X_1X'_1$ , in which both joins are incomplete. All other join-pairs have exactly one join complete and the other incomplete.

#### Fig. 4a and 4b here.

Let  $k \ge 4$ . Consider the join-pairs  $X_2X'_2$  and  $X_kX'_k$  to the right and left of  $X_1X'_1$ , as indicated in Fig. 4; exactly one join in each is complete. There are two cases to consider: (a) Suppose that both  $X_2$  and  $X_k$  are complete, see Fig. 4a. Without loss of generality  $v_f \in X'_k \cup X_1$ , (i.e.,  $v_f$  is connected to 1); for, suppose not, then at least one of  $X'_k$  or  $X_1$ is complete, a contradiction. Similarly  $v_\ell \in X'_1 \cup X'_2$ .

Hence any join between  $X_2$  and  $X_k$ , and between  $X'_2$  and  $X'_k$ , is either complete or empty because it cannot contain  $v_f$  or  $v_\ell$ . Thus, any join-pair  $X_jX'_j$  for  $3 \le j \le k-1$  has one join complete and the other empty, this forces k to be even.

Since  $v_f$  is connected to 1 there are  $(x_k + x_1 - 1)$  vertices where  $v_f$  can be placed (the edges between  $v_f$  and 1 are then included as part of F), and  $v_\ell$  can be placed with a choice of  $(x_1 + x_2 - 1)$  places. Each placement can then be extended uniquely to form a path. Similarly if both  $X'_2$  and  $X'_k$  are complete. Also, the unique join-pair with both joins incomplete can be any of the k join-pairs  $X_j X'_j$  for  $1 \le j \le k$ .

Hence, the total number of paths for this case is

$$k \cdot \frac{n}{k} \sum_{\underline{x}} 2(x_1 + x_2 - 1)(x_1 + x_k - 1) = 10n \binom{n}{k+1} + 2n \binom{n-1}{k-1}.$$

(b) Suppose that both  $X_2$  and  $X'_k$  are complete, see Fig. 4b; then k is odd and  $\geq 3$ . Without loss of generality  $v_f \in X'_1 \cup X'_2$  and  $v_\ell \in X_k \cup X'_1$ .

If  $v_f \in X'_2$  (with  $x_2$  places) then the number of places for  $v_\ell$  is  $(x_1 + x_k - 1)$ , yielding  $x_2(x_1 + x_k - 1)$  paths.

If  $v_f \in X'_1$ , the number of places for  $v_\ell$  is  $\sum_{t=1}^{x_1-1} (x_1 + x_k - 1 - t)$ , where t is the number f edges between  $w_\ell$  and w'

of edges between  $v_f$  and  $v'_2$ .

From the previous two paragraphs and the comments in (a) above, the number of paths for this case is

$$k \cdot \frac{n}{k} \sum_{\underline{x}} 2\left(\frac{x_1^2}{2} + x_1x_2 + x_1x_k + x_2x_k - \frac{3}{2}x_1 - x_2 - x_k + 1\right) = 8n\binom{n}{k+1} + 2n\binom{n-1}{k-1}.$$

The formula for p(n, 2) may be incorporated into the formula for p(n, k) derived in (a) above.

Lemma 14.1.

$$p(n,k) = \begin{cases} 2n(2n-1), & k = 0;\\ n(3n^2 - 3n + 1), & k = 1;\\ 10n\binom{n}{k+1} + 2n\binom{n-1}{k-1}, & k \text{ even and } \ge 2;\\ 8n\binom{n}{k+1} + 2n\binom{n-1}{k-1}, & k \text{ odd and } \ge 3. \end{cases}$$

Theorem 14.2.

$$p(n) = 10n2^n - n^3 - 5n^2 - 11n.$$

The first few terms in the sequence  $\{p(n) : n \ge 2\}$  are  $\{30, 135, 452, 1295, \ldots\}$ . The sequence  $\{p(n, 1)\} = \{n(3n^2 - 3n + 1)\}$  is M4933 in [9].

#### **15.** Hamiltonian Paths -hp

In this section let  $\underline{x} = x_1 \cdots x_k = x_1, \dots, x_k$ .

Again, let F be a (n, k)-graph which is a Hamiltonian path for  $M_n$ , then using the same reasoning as in §14, there are two cases to consider:

(a) Both  $X_2$  and  $X_k$  are complete. Since F is a Hamiltonian path then K of Fig. 4a determines the composition  $\underline{y} = x_1, x_2, \underline{1, \ldots, 1}, n - x_1 - x_2 - k + 3$ , or otherwise F will

$$k-3$$

contain isolated vertices. Furthermore, k is even and  $\geq 4$ , and at least one of  $x_1 = 1$  or  $x_k = 1$ , and at least one of  $x_1 = 1$  or  $x_2 = 1$ , or, again, there will be isolated vertices in F. (i) If  $x_1 = 1$  then  $\underline{y} = 1, x_2, \underbrace{1, \ldots, 1}_{k=1}, n-x_2-k+2$ , where  $1 \leq x_2 \leq n-k+1$ . Hence, cf., §14,

the total number of Hamiltonian paths for this case is  $k \cdot \frac{n}{k} \sum_{y} 2 = 2n(n-k+1)$ .

(ii) If  $x_1 > 1$  then  $x_k = x_2 = 1$ , and  $\underline{y} = n - k + 1, \underbrace{1, \ldots, 1}_{k-1}$ , and the total number of

Hamiltonian paths is 2n.

(b) Both  $X_2$  and  $X'_k$  are complete. Then k is odd and  $\geq 3$ , and the relevant composition determined by Fig. 4b is again  $\underline{y} = 1, x_2, \underbrace{1, \ldots, 1}_{k-3}, n-x_2-k+2$ , which yields 2n(n-k+1)

Hamiltonian paths.

Again, the formula for hp(n, 2) may be incorporated into the formula for hp(n, k) derived in (a) above.

Lemma 15.1.

$$hp(n,k) = \begin{cases} 2n, & k = 0, 1, \text{ and } k = n; \\ 2n(n-k+2), & k \text{ even and } 2 \le k \le n-1; \\ 2n(n-k+1), & k \text{ odd and } 3 \le k \le n-1. \end{cases}$$

Theorem 15.2.

$$hp(n) = \begin{cases} n^3 + 2n, & n \text{ even};\\ n^3 + 3n, & n \text{ odd.} \end{cases}$$

The first few terms in the sequence  $\{hp(n) : n \ge 2\}$  are  $\{12, 36, 72, 140, \ldots\}$ .

#### 16. No-Leaf Edge-Subgraphs – nle

Call a vertex in a graph a *leaf* if it has degree 1. A no-leaf edge-subgraph of  $M_n$  is an edge-subgraph with no leaves, i.e., each of its vertices has degree 0, 2, or 3. Clearly nle(n,0) = 2. If E, an (n,k)-graph, is a no-leaf edge-subgraph then its joins must be empty or complete; moreover, its empty joins must be non-consecutive, cyclic. So the number of such E which come from an arbitrary K of Fig. 3 is  $\mathcal{L}(2k)$ .

Hence

$$nle(n,k) = \frac{n}{k} \sum_{\underline{x}} \mathcal{L}(2k) = \binom{n}{k} \mathcal{L}(2k).$$

Theorem 16.1.

$$nle(n) = \left(\frac{5+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{2}\right)^n.$$

The first few numbers in the sequence  $\{nle(n) : n \ge 2\}$  are  $\{15, 50, 175, 625, \ldots\}$ .

#### 17. Matchings -m

A matching, E, of  $M_n$  is a collection of disjoint edges and isolated vertices. So  $m(n,0) = \mathcal{L}(2n)$ .

Let  $\delta(r)$  be the number possibilities for a join of E of size r. Then  $\delta(1) = \delta(2) = 1$ , and  $\delta(r) = \delta(r-1) + \delta(r-2)$  for  $r \ge 3$ .

By exact analogy with §3, the number of matchings which come from K of Fig. 3 is  $\prod_{j=1}^{k} \delta(x_j)^2$ . This gives us

$$m(n) = \mathcal{L}(2n) + [x^n] \frac{x(1 - 2x + 4x^2 - 2x^3 + x^4)}{(1 + x)(1 - 3x + x^2)(1 - 3x - x^2 + x^3)}.$$

Theorem 17.1.

$$m(n) \approx (3.2143)^n + (-0.6751)^n + (0.4608)^n - (-1)^n.$$

The first few numbers in the sequence  $\{m(n) : n \ge 2\}$  are  $\{10, 34, 106, 344, \ldots\}$ , see Table 1 in Hosoya and Harary [7].

Let  $H_n$  be an arbitrary cubic bipartite graph on 2n vertices, then, by a similar argument to that used at the end of §3, the graph  $H_n$  contains at most  $4^n$  matchings. Hence: For odd n, does there exist a family of cubic bipartite graphs,  $G_n$ , on 2n vertices, such that, for some  $n_0$  and all odd  $n > n_0$ , the number of matchings of  $G_n$  is greater than  $(3.2143)^n$ ?

### **18.** One-Factors - of Two-Factors - tf

A one-factor, E, of  $M_n$  is a collection of n disjoint edges, i.e., E is regular of degree one. For example, see Fig. 1b, let  $B_n$  be the one-factor containing outside edge 12 and every second outside-edge, and  $C_n$  the one-factor containing the outside edges not in  $B_n$ .

An *edge-pair* is a pair of opposite outside edges, i.e., the pair  $e_j = \{j(j+1), j'(j+1)'\}$ for some j with  $1 \le j \le n-1$ , or the pair  $e_n = \{n1', n'1\}$ . Let us identify [n] with the set of edge-pairs,  $\mathcal{E}_n = \{e_1, \ldots, e_n\}$ , in the obvious way. Recall the definition of an ncc set of [n], so of  $\mathcal{E}_n$ , and the number of ncc t-sets of  $[n], \psi(n, t)$ , from the Preliminaries.

An *edge-pair one-factor* is a one-factor in which all outside edges occur in edge-pairs. A *t-edge-pair one-factor* is an edge-pair one-factor which contains t edge-pairs.

Let F be a *t*-edge-pair one-factor of  $M_n$ , then its *t* edge-pairs form a ncc *t*-set of  $\mathcal{E}_n$ . Conversely, the edge-pairs corresponding to a ncc *t*-set of  $\mathcal{E}_n$  can be uniquely extended to a *t*-edge-pair one-factor by adding on the diagonals ii' for each vertex *i* not already covered. Hence, the number of *t*-edge-pair one-factors of  $M_n$  is  $\psi(n,t) = \frac{n}{n-t} \binom{n-t}{t}$ , where  $0 \le t \le \lfloor \frac{n}{2} \rfloor$ .

Here we have not used our composition technique, however, consider the following: Call a k-composition of  $n, \underline{x} = x_1 x_2 \cdots x_k$ , odd if each  $x_j$  is odd. By adding 1 to each such  $x_j$ , we can set up a bijection between odd k-compositions of n and k-compositions of  $\frac{n+k}{2}$ ; hence, the number of odd k-compositions of n is  $\left(\frac{n+k}{2}-1\right)$ . Now, an (n,k)-graph, E, can be extended to a unique one-factor if and only if each of its joins is odd. Let  $\epsilon(r)$  denote the number of possibilities for a join of E of size r, then  $\epsilon(r) = 0$  if r is even and  $\epsilon(r) = 1$  if r is odd. Also, note that an edge-pair one-factor with k diagonals has  $\frac{n-k}{2}$  edge-pairs. Hence

$$of(n,k) = \frac{n}{k} \sum_{\underline{x}} \prod_{j=1}^{k} \epsilon(x_j)^2 = \frac{n}{k} \binom{\frac{n+k}{2}-1}{k-1} = \frac{2n}{n+k} \binom{\frac{n+k}{2}}{\frac{n-k}{2}} = \psi(n, \frac{n-k}{2}), \text{ as required.}$$

It is straightforward to prove that most one-factors of  $M_n$  are edge-pair one-factors: **Lemma 18.1.** All one-factors of  $M_n$  are edge-pair one-factors, except for  $B_n$  and  $C_n$ when n is odd.

Hence, for odd n,  $of(n) = \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \psi(n,t) + 2$ , where the +2 corresponds to the non-edgepair one-factors  $B_n$  and  $C_n$ ; we do not need the +2 for even n.

The numbers of(n) are closely related to the Lucas numbers, see the comments in the Preliminaries.

Now, a two-factor of  $M_n$  is the complement in  $M_n$  of a one-factor, and vice versa, so: **Theorem 18.2.** 

$$of(n) = tf(n) = \begin{cases} \mathcal{L}(n), & n \text{ even};\\ \mathcal{L}(n) + 2, & n \text{ odd.} \end{cases}$$

This formula appears in [7]. The first few terms in the sequence  $\{of(n) : n \ge 2\}$  are  $\{3, 6, 7, 13, \ldots\}$ .

### **19.** One-Factorizations -oF

In this section we do not use our composition technique. Our results give us a combinatorial interpretation of a sequence which occurs on p. 603 of Guy [5], it appears to be the first combinatorial interpretation of this sequence.

A one-factorization of  $M_n$  is a triple of one-factors which partition the edges of  $M_n$ . For example, see Fig. 1b, let  $A_n$  be the one-factor of  $M_n$  consisting of its n diagonals, and  $B_n$  and  $C_n$  be as in §18, then  $\{A_n, B_n, C_n\}$  is a one-factorization of  $M_n$ .

An *edge-pair cycle* is a cycle in which all outside edges occur in edge-pairs, each edgepair cycle is an even cycle. An *edge-pair two-factor* is a two-factor in which each component is an edge-pair cycle.

As noted earlier, if we remove a one-factor from  $M_n$  we obtain a two-factor, moreover: **Lemma 19.1.** Let F be an edge-pair one-factor of  $M_n$ . Then, for  $t \ge 1$ , F is a t-edge-pair one-factor if and only if  $M_n - F$  is an edge-pair two-factor with t components.

**Proof** Assume that F is a t-edge-pair one-factor where  $t \ge 1$ .

Without loss of generality, let two consecutive edge-pairs of F be  $\{c(c+1), c'(c+1)'\}$ and  $\{d(d+1), d'(d+1)'\}$ , see Fig. 5, with obvious notational changes if d = n. Then  $c+1, c+2, \ldots, d-1, d, d', (d-1)', \ldots, (c+2)', (c+1)', c+1$ , shown with an arrowed broken line in Fig. 5, is an even edge-pair cycle of length 2(d-c) which lies in  $M_n - F$ . Similarly, there is an even edge-pair cycle "between" any pair of consecutive edge-pairs of F, and there are t such pairs of edge-pairs, and so t such cycles, which cover all vertices of  $M_n$ . Hence,  $M_n - F$  is a edge-pair two-factor with t components. Conversely, if X is an edge-pair two-factor with t components, then, clearly,  $M_n - X$  is an edge-pair one-factor. Suppose  $M_n - X$  has  $s \neq t$  edge-pairs. Then, by above,  $X = M_n - (M_n - X)$  has s components, a contradiction. Hence,  $M_n - X$  is a t-edge-pair one-factor.

### Fig. 5 here.

An *edge-pair one-factorization* of  $M_n$  is a one-factorization in which all one-factors are edge-pair one-factors. It is straightforward to prove the following, (cf., Lemma 18.1):

**Lemma 19.2.** All one-factorizations of  $M_n$  are edge-pair one-factorizations, except for  $\{A_n, B_n, C_n\}$  when n is odd.

Let the number of edge-pair one-factorizations of  $M_n$  be eoF(n).

Again, let F be a t-edge-pair one-factor where  $t \ge 1$ . Now, an even cycle has 2 one-factors, thus  $M_n - F$ , which contains t even cycles, has  $2^t$  one-factors.

Let  $\{F, F', F''\}$  be an edge-pair one-factorization of  $M_n$  which contains F. How many such one-factorizations are there? Such a one-factorization can be formed by choosing Ffirst, then there are  $2^t$  ways in which to choose F', and the remaining edges then form F''. However, this one-factorization could have been formed by choosing F'' second and leaving F'. Hence, the total number of edge-pair one-factorizations which contain F is  $\frac{1}{2} \cdot 2^t = 2^{t-1}$ .

Recall that the number of t-edge-pair one-factors of  $M_n$  is  $\psi(n,t)$ . Hence, for even  $\lfloor \frac{n}{2} \rfloor$ 

*n*, we have  $3eoF(n) = \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \psi(n,t) 2^{t-1} + 1$ , where the +1 corresponds to the 0-edge-pair

one-factor  $A_n$ ; we do not need the +1 for odd *n*. This gives: eoF(2) = eoF(3) = 1 and eoF(n) = eoF(n-1) + 2eoF(n-2) for  $n \ge 4$ . So  $eoF(n) = \frac{1}{3}\{2^{n-1} + (-1)^n\}$ .

Now oF(n) = eoF(n) if n is even, and oF(n) = eoF(n) + 1 if n is odd via Lemma 19.2. Hence:

### Theorem 19.3.

$$oF(n) = \begin{cases} \frac{1}{3}(2^{n-1}+1), & n \text{ even};\\ \frac{1}{3}(2^{n-1}+2), & n \text{ odd}. \end{cases}$$

The first few terms in the sequence  $\{oF(n) : n \ge 2\}$  are  $\{1, 2, 3, 6, \ldots\}$ . This sequence is M0788 in [9], and appears at the bottom of p. 603 in Guy [5].

### **20.** Perfect One-Factorizations – *poF*

A perfect one-factorization of  $M_n$  is a one-factorization in which the union of any 2 one-factors is a Hamiltonian cycle.

If n = 2 or if n is odd and  $\geq 5$  then, via Lemma 19.1,  $\{A_n, B_n, C_n\}$  is the only perfect one-factorization of  $M_n$ . If n = 3 there is one more perfect one-factorization, there are none for n even and  $\geq 4$ .

#### Theorem 20.1.

$$poF(n) = \begin{cases} 1, & n = 2 \text{ or } n \text{ is odd and } \ge 5; \\ 2, & n = 3; \\ 0, & n \text{ even and } \ge 4. \end{cases}$$

### 21. Elementary Edge-Subgraphs – ee

An elementary edge-subgraph, E, of  $M_n$  is an edge-subgraph in which every component is regular of degree 0, 1, or 2; i.e., every component is an isolated vertex, a single edge, or a cycle. Clearly  $ee(n, 0) = \mathcal{L}(2n) + 1$ .

See K of Fig. 3, and let  $k \ge 2$ . In order to extend K to an elementary edge-subgraph we can first form cycles, then add on disjoint edges. Now, from §12, cycles of  $M_n$  can contain either two diagonals, or k diagonals for odd k.

For any k, our extended K, call it E, may contain cycles with two diagonals. These cycles are formed by choosing both joins in an arbitrary join-pair to be complete. These join-pairs must be a ncc (sub)set of  $\mathcal{X}_k = \{X_1 X'_1, \ldots, X_k X'_k\}$ , denote this set by N. Those joins of E which are incomplete are either empty or a matching. Hence, such a join of size r has  $\delta(r)$  possibilities where  $\delta(r)$  is defined as in §17. Conversely, any ncc set  $N \subset \mathcal{X}_k$  gives rise to |N| cycles by completing the join-pairs corresponding to the elements of N; this graph can then be extended to elementary edge-subgraphs by adding disjoint, outside edges. Let us identify  $\mathcal{X}_k$  with [k] in the obvious manner. Hence, the number of elementary edge-subgraphs which come from K is  $\sum_{N \subset [k]} \prod_{j \notin N} \delta(x_j)^2$ .

If k is odd, we may have a cycle with k diagonals. Here either join  $X_1$  or  $X_2$  of E is complete and then every second join is complete. Again, incomplete joins have  $\delta(r)$  possibilities, hence K gives rise to  $2\prod_{i=1}^{k} \delta(x_i)$  elementary edge-subgraphs.

Theorem 21.1.

$$ee(n,k) = \begin{cases} \mathcal{L}(2n) + 1, & k = 0; \\ \frac{n}{k} \sum_{\underline{x}} \left\{ \sum_{N \subset [k] \\ N \text{ ncc set}} \prod_{j \notin N} \delta(x_j)^2 \right\} + \frac{2n}{k} \sum_{\underline{x}} \prod_{j=1}^k \delta(x_j), & k \text{ odd}; \\ \frac{n}{k} \sum_{\underline{x}} \left\{ \sum_{N \subset [k] \\ N \text{ ncc set}} \prod_{j \notin N} \delta(x_j)^2 \right\}, & k \text{ even.} \end{cases}$$

It does not seem possible to compute a concise expression for ee(n). The first few terms in the sequence  $\{ee(n) : n \ge 2\}$  are  $\{17, 58, 181, 602, \ldots\}$ .

#### 22. Strong Elementary Edge-Subgraphs – see

A strong elementary edge-subgraph (sometimes called sesquivalent) of  $M_n$  is an elementary edge-subgraph which is strong. An even join of K of Fig. 3 cannot be made strong using disjoint edges; an odd join can be made strong in one way.

The formulas for see(n, k) are as in §21 except that see(n, 0) = 3, and  $\delta(r)$  must be replaced by  $\epsilon(r)$  where  $\epsilon(r) = 0$  if r is even, and  $\epsilon(r) = 1$  if r is odd, ( $\epsilon(r)$  as in §18).

Again, there does not seem to be a more precise expression for see(n).

The first few terms in the sequence  $\{see(n) : n \geq 2\}$  are  $\{6, 21, 26, 81, ...\}$ . The sequence  $\{see(n, n-1)\} = \{(n/\sqrt{5})\{((1+\sqrt{5})/2)^{n-2} - ((1-\sqrt{5})/2)^{n-2}\}\}$ , appears as M2362 in [9], Generalized Lucas Numbers.

Finally, the degree of a vertex in an edge-subgraph of  $M_n$  is 0, 1, 2, or 3. Hence, we may choose any subset of  $\{0, 1, 2, 3\}$  and count the subgraphs of  $M_n$  whose vertex degrees are equal to this subset. For example, in §17, we counted matchings which corresponds to the subset  $\{0, 1\}$ ; also, by taking complements in  $M_n$ , we counted the edge-subgraphs corresponding to the subset  $\{3-0, 3-1\} = \{2, 3\}$ . Barring trivial cases, we have considered all subsets except  $\{1, 2\}$ , which is our final section.

### 23. Strong Restricted Edge-Subgraphs – sre

A strong restricted edge-subgraph of  $M_n$  is an edge-subgraph in which the degree of each vertex is 1 or 2, i.e., every component is a path with at least one edge, or a cycle.

We are unable to use this counting technique to compute the value of sre(n) exactly. However, we have the following symmetric relationship, since the class of strong restricted edge-subgraphs is closed under complements in  $M_n$ : sre(n,k) = sre(n,n-k). We also have:

**Theorem 23.1.** For large n,

$$(3.2143)^n < sre(n) < (6.4188)^n.$$

The first few terms in the sequence  $\{sre(n) : n \ge 2\}$  are  $\{18, 102, 418, 2006, \ldots\}$ , which suggests that the value of sre(n) is about  $(4.6)^n$ .

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