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Multivariate Matching Polynomials of Cyclically Labelled Graphs

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Keywords

Matching, Matching Polynomial, Labelled, Graph, Recurrence, MacMahon's Master Theorem

Abstract

We consider the matching polynomials of graphs whose edges have been cyclically labelled with the ordered set of t labels $\{x_1, \ldots, x_t\}$.

We first work with the cyclically labelled path, with first edge label x_i, followed by N full cycles of labels $\{x_1, \ldots, x_t\}$, and last edge label x_j. Let $\Phi_{i,Nt+j}$ denote the matching polynomial of this path. It satisfies the (τ, Δ)-recurrence: $\Phi_{i,Nt+j} = \tau\, \Phi_{i,(N-1)t+j} - \Delta\, \Phi_{i,(N-2)t+j}$, where τ is the sum of all non-consecutive cyclic monomials in the variables $\{x_1, \ldots, x_t\}$ and $\Delta = (-1)^t\, x_1 \cdots x_t$. A combinatorial/algebraic proof and a matrix proof of this fact are given. Let G_N denote the first fundamental solution to the (τ, Δ)-recurrence. We express G_N (i) as a cyclic binomial using the Symmetric Representation of a matrix, (ii) in terms of Chebyshev polynomials of the second kind in the variables τ and Δ, and (iii) as a quotient of two matching polynomials. We extend our results from paths to cycles and rooted trees.

Introduction

The matching polynomial of a graph is defined in Farrell [1]. Often in pure mathematics and combinatorics it is interesting to consider cyclic structures, *eg.*, cyclic groups, cyclic designs, and circulant graphs. Here we consider the (multivariate) matching polynomial of a graph whose edges have been cyclically labelled.

We concentrate mainly on paths, cycles and trees. To cyclically label a path with the ordered set of t labels $\{x_1, \ldots, x_t\}$, label the first edge with any x_i, the second with x_{i+1}, and so on until label x_t has been used, then start with x_1, then $x_2, \ldots, x_t$, then x_1 again $\ldots$, repeating cyclically until all edges have been labelled, with the last edge receiving label x_j. Suppose that N full cycles of labels $\{x_1, \ldots, x_t\}$ have been used. Call the matching polynomial of this labelled path $\Phi_{i,Nt+j}$. We show, for a fixed i and j, that $\Phi_{i,Nt+j}$ satisfies the following recurrence, the (τ, Δ)-recurrence:

$$\Phi_{i,Nt+j} = \tau\, \Phi_{i,(N-1)t+j} - \Delta\, \Phi_{i,(N-2)t+j},$$

where τ is the sum of all non-consecutive cyclic monomials in the variables $\{x_1, \ldots, x_t\}$ (see Section 1), and $\Delta = (-1)^t\, x_1 \cdots x_t$. We give two different proofs of this fact. The first one is a combinatorial/algebraic proof in Section 2 that uses the following Theorem concerning decomposing the matching polynomial $\mathcal{M}(G, \mathbf{x})$ of a graph.

Theorem Let G be a labelled graph, H a subgraph of G, and M_H a matching of H, then

$$\mathcal{M}(G, \mathbf{x}) = \sum_{M_H} M_H(\mathbf{x})\, \mathcal{M}(G - H - \overline{M}_H, \mathbf{x}),$$

where the summation is over every matching M_H of H. The second proof (Section 3) uses a matrix formulation of the recurrences that we develop.

Let G_N denote the first fundamental solution to the (τ, Δ)-recurrence; three different expressions for G_N are given in Section 4. The first expression is a sum of cyclic binomials and uses the Symmetric Representation of matrices from Section 3; the second involves Chebyshev polynomials of the second kind in the variables τ and Δ; and the third is a quotient of two matching polynomials, see Theorem 4.5.

In Section 5 we extend our results from paths to cycles and rooted trees; we find explicit forms for the matching polynomial of a cyclically labelled cycle, and indicate how to find the matching polynomial of a cyclically labelled rooted tree, again using the decomposition Theorem stated above.

Many examples are given throughout the paper.

1 The multivariate matching polynomial of a graph, its decomposition; non-consecutive and non-consecutive cyclic functions

For a fixed $t \geq 1$ we use multi-index notations: $\mathbf{k} = (k_1, \ldots, k_t)$, where each $k_s \geq 0$, $\mathbf{0} = (0, \ldots, 0)$, and variables $\mathbf{x} = (x_1, \ldots, x_t)$. The total degree of $\mathbf{k}$ is denoted by $|\mathbf{k}| = k_1 + \cdots + k_t$.

Let G be a finite simple graph with vertex set $V(G)$ where $|V(G)| \geq 1$, and edge set $E(G)$. We label these edges from the t commutative variables $\{x_1, \ldots, x_t\}$, exactly one label per edge. A *matching* of G is a collection of edges, no two of which have a vertex in common. A $\mathbf{k}$-*matching* of G is a matching with exactly k_s edges with label x_s, for each s with $1 \leq s \leq t$. If M_G is a $\mathbf{k}$-matching of G we define its *weight* to be

$$M_G(\mathbf{x}) = x_1^{k_1} \cdots x_t^{k_t}.$$

The empty matching of G, which contains no edges, is denoted by $M_\emptyset$; it is the unique $\mathbf{0}$-matching and its weight is $M_\emptyset(\mathbf{x}) = 1$.

Define the *multivariate matching polynomial*, or simply, the *matching polynomial*, of G, by

$$\mathcal{M}(G, \mathbf{x}) = \sum_{M_G} M_G(\mathbf{x}),$$

where the summation is over every matching M_G of G.

Denote the number of $\mathbf{k}$-matchings of G by $a(G, \mathbf{k})$. Then an alternative definition of the multivariate matching polynomial of G is

$$\mathcal{M}(G, \mathbf{x}) = \sum_{(k_1, \ldots, k_t)} a(G, \mathbf{k})\, x_1^{k_1} \cdots x_t^{k_t}.$$

The multivariate matching polynomial is a natural extension of the matching polynomial of Farrell [1]. Indeed, here with $t = 1$ and in [1] with $w_1 = 1$ and $w_2 = x_1$, the polynomials are identical.

Let P_1 be the graph with one vertex and no edges, *i.e.*, an isolated vertex; we define $\mathcal{M}(P_1, \mathbf{x}) = 1$. Now suppose $G' = G \cup nP_1$, where $n \geq 1$, *i.e.*, G'

is the disjoint union of G and n isolated vertices, then we define $\mathcal{M}(G',\mathbf{x}) = \mathcal{M}(G,\mathbf{x})$.

For any edge $e \in E(G)$, let $\overline{e}$ denote the set of edges that are incident to e; and for any subgraph H of G, let $\overline{H} = \cup_{e\in E(H)}\overline{e}$. Define $\overline{M}_\emptyset = \emptyset$. Also let $G - H$ be the graph obtained from G when all the *edges* of H are removed, so $G - H$ has the same vertex set as G.

Now let H be a fixed subgraph of G and let M_H be a matching of H. In the following theorem we express $\mathcal{M}(G,\mathbf{x})$ as a sum of terms, each term containing the weight of a fixed matching, $M_H(\mathbf{x})$, of H; we call this *decomposing* $\mathcal{M}(G,\mathbf{x})$ *at* H.

Theorem 1.1 *Let G be a graph labelled as above, H a fixed subgraph of G, and M_H a matching of H. Then*

$$\mathcal{M}(G,\mathbf{x}) = \sum_{M_H} M_H(\mathbf{x})\,\mathcal{M}(G - H - \overline{M}_H,\mathbf{x}), \tag{1}$$

where the summation is over every matching M_H of H.

Proof. Let M_G be a matching of G which induces a (fixed) matching M_H on H, *i.e.*, M_G contains exactly M_H and no other edges from H. Then $M_G(\mathbf{x}) = M_H(\mathbf{x})\,M(\mathbf{x})$ where M is a matching of G with no edges in H, and also with no edges in $\overline{M}_H$ or else M_G would not be a matching. Hence, M is a matching of $G - H - \overline{M}_H$, *i.e.*, $M(\mathbf{x})$ is a term of $\mathcal{M}(G - H - \overline{M}_H,\mathbf{x})$. So $M_H(\mathbf{x})\,\mathcal{M}(G - H - \overline{M}_H,\mathbf{x})$ is the sum of the weights of all the matchings in G which induce M_H on H.

Now every matching in G induces some matching on H, so we may sum over all matchings in H to give (1). ∎

Theorem 1.1 extends known facts about matching polynomials, *eg.*, see Theorem 1 of Farrell [1] for the case where H is a single edge. We have the corresponding:

Corollary 1.2 *Let G be a graph labelled as above, and let $H = e$ labelled with x be an edge of G. Then*

$$\mathcal{M}(G,\mathbf{x}) = \mathcal{M}(G-e,\mathbf{x}) + x\,\mathcal{M}(G - e - \overline{e},\mathbf{x}). \tag{2}$$

Proof. The result comes from (1) since $H = e$ has just two matchings: the empty matching $M_\emptyset$ with weight $M_\emptyset(\mathbf{x}) = 1$, and the matching e with weight $M_e(\mathbf{x}) = x$. ■

Notation Throughout this paper we use P_m to denote the path with m vertices and $m-1$ edges.

Fix i and j where $1 \le i \le j \le t$. Consider the path P_{j-i+2} with its $j-i+1$ edges labelled from the ordered set $\{x_i, \dots, x_j\}$, the first edge receiving label x_i, and the last x_j; see Fig. 1.

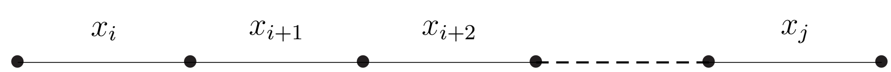


Fig. 1: The labelled path P_{j-i+2} with matching polynomial $\phi_{i,j}$.

The pair $x_s x_{s+1}$ for any fixed s with $i \le s \le j-1$ is called a *consecutive* pair. A monomial from the ordered set $\{x_i, \dots, x_j\}$ that contains no consecutive pairs is a *non-consecutive monomial*, a *nc*-monomial. Note that the empty monomial is a *nc*-monomial that we denote by 1.

Let $\phi_{i,j}$ be the sum of all *nc*-monomials in the ordered variables $\{x_i, \dots, x_j\}$. Then $\phi_{i,j} = \mathcal{M}(P_{j-i+2}, \mathbf{x})$ is the matching polynomial of the labelled path P_{j-i+2}. We call the functions $\phi_{i,j}$ *elementary non-consecutive functions*, and for any $i \ge 1$ define the initial values

$$\phi_{i,i-2} = \phi_{i,i-1} = 1. \tag{3}$$

These initial values ensure that the following recurrence is valid for any j with $i \le j \le t$.

Theorem 1.3 *For a fixed i and j with $1 \le i \le j \le t$ and the initial values in (3), we have*

$$\phi_{i,j} = \phi_{i,j-1} + x_j\, \phi_{i,j-2}. \tag{4}$$

Proof. Let e be the rightmost edge of $G = P_{j-i+2}$ shown in Fig. 1, and apply (2). ■

Example 1 For arbitrary i we have

$$\begin{aligned}
&\phi_{i,i} = 1 + x_i, \quad \phi_{i,i+1} = 1 + x_i + x_{i+1},\\
&\phi_{i,i+2} = 1 + x_i + x_{i+1} + x_{i+2} + x_i x_{i+2},\\
&\phi_{i,i+3} = 1 + x_i + x_{i+1} + x_{i+2} + x_{i+3} + x_i x_{i+2} + x_i x_{i+3} + x_{i+1} x_{i+3}.
\end{aligned}$$

Example 2 For arbitrary i, putting $j = i - 1$ and $j = i - 2$ in Recurrence (4) and using (3) give

$$\phi_{i,i-3} = 0 \quad \text{and} \quad \phi_{i,i-4} = \frac{1}{x_{i-2}}.$$

In the second equation, if $i = 1$ we replace x_{-1} by x_{t-1}, and if $i = 2$ we replace x_0 by x_t.

Consider Recurrence (4). It is convenient to work with a basis of solutions to this recurrence. Denote the first fundamental solution by $f_{i,j}$ and the second by $g_{i,j}$, with initial values

$$f_{i,i-2} = 0, f_{i,i-1} = 1 \quad \text{and} \quad g_{i,i-2} = 1, g_{i,i-1} = 0. \tag{5}$$

So

$$\phi_{i,i-2} = f_{i,i-2} + g_{i,i-2} \quad \text{and} \quad \phi_{i,i-1} = f_{i,i-1} + g_{i,i-1}.$$

Now, from Recurrence (4) and strong induction on j, we have (6) below for all j with $i \le j \le t$

$$\phi_{i,j} = f_{i,j} + g_{i,j}. \tag{6}$$

$$\phi_{i,j} = \phi_{i+1,j} + x_i \, \phi_{i+2,j}. \tag{7}$$

Equation (7) comes from decomposing $\phi_{i,j}$ at the leftmost edge of P_{j-i+2}, whose label is x_i, *i.e.*, decomposing $\phi_{i,j}$ at x_i; see Corollary 1.2. These two equations suggest that the fundamental solutions are given by

$$f_{i,j} = \phi_{i+1,j} \quad \text{and} \quad g_{i,j} = x_i \, \phi_{i+2,j}.$$

This is indeed the case:

Lemma 1.4 *For any j with $i \le j \le t$ we have*

(i) $f_{i,j} = \phi_{i+1,j}$,

(ii) $g_{i,j} = x_i \, \phi_{i+2,j}$.

Proof. We need only prove (i) because of (6) and (7) above.

From (5) we have $f_{i,i-2} = 0$ and from Example 2 we have $\phi_{i+1,i-2} = 0$; thus $f_{i,i-2} = \phi_{i+1,i-2}$. Similarly, from (5) and (3), we have $f_{i,i-1} = \phi_{i+1,i-1}$. So both $f_{i,j}$ and $\phi_{i+1,j}$ have the same initial values at $j = i-2$ and $j = i-1$ and they both satisfy Recurrence (4), so they are equal for any j with $i \leq j \leq t$. ∎

Thus we know combinatorially what the two fundamental solutions to Recurrence (4) are. The first, $f_{i,j}$, is the matching polynomial of the path shown in Fig. 2(a); the second, $g_{i,j}$, is $x_i\ \times$ the matching polynomial of the path in Fig. 2(b).

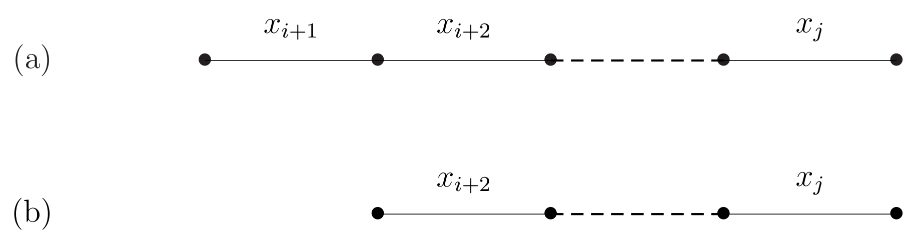


Fig. 2 (a) The labelled path with matching polynomial $f_{i,j}$.
(b) The labelled path with matching polynomial $\dfrac{g_{i,j}}{x_i}$.

Example 3 For arbitrary i we have

$$\begin{array}{ll} f_{i,i} = 1, & g_{i,i} = x_i, \\ f_{i,i+1} = 1 + x_{i+1}, & g_{i,i+1} = x_i, \\ f_{i,i+2} = 1 + x_{i+1} + x_{i+2}, & g_{i,i+2} = x_i + x_i x_{i+2}. \end{array}$$

Now arrange the variables $\{x_i, \ldots, x_j\}$ clockwise around a circle. Thus x_i and x_j are consecutive. Call a pair $x_s x_{s'}$ *consecutive cyclic* if x_s and $x_{s'}$ are consecutive on this circle. Call a monomial from $\{x_i, \ldots, x_j\}$ a *non-consecutive cyclic* monomial — *ncc*-monomial — if it contains no consecutive cyclic pairs. The empty monomial is a *ncc*-monomial that we denote by 1.

Let $\tau_{i,j}$ be the sum of all *ncc*-monomials in the variables $\{x_i, \ldots, x_j\}$. Then, for $j \geq i+2$, $\tau_{i,j} = \mathcal{M}(C_{j-i+1}, \mathbf{x})$ is the matching polynomial of the labelled cycle C_{j-i+1} with $j-i+1$ edges and $j-i+1$ vertices, shown in Fig. 3; the cycle starts at the large vertex, and proceeds clockwise.

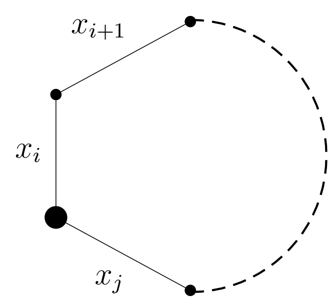


Fig. 3: The labelled cycle C_{j-i+1} with matching polynomial $\tau_{i,j}$.

For initial values let

$$\tau_{i,i-1} = 2, \quad \tau_{i,i} = 1, \quad \text{and} \quad \tau_{i,i+1} = 1 + x_i + x_{i+1}. \tag{8}$$

Lemma 1.5 *For any j with $i \le j \le t$ we have*
(i) $\tau_{i,j} = f_{i,j} + g_{i,j-1}$,
(ii) $\phi_{i,j} - \tau_{i,j} = x_i x_j \, \phi_{i+2,j-2}$.

Proof. (i) We check this equality at $j = i$ and $j = i+1$ using (5), Example 3, and (8). For $j \ge i+2$ we decompose $\tau_{i,j}$ at x_i yielding $\tau_{i,j} = \phi_{i+1,j} + x_i \, \phi_{i+2,j-1}$, which gives (i) via Lemma 1.4.
(ii) We check at $j = i$ and $j = i+1$ using Examples 1 and 2, and (8). For $j \ge i+2$ the difference $\phi_{i,j} - \tau_{i,j}$ consists of all nc-monomials that contain the consecutive cyclic pair $x_i x_j$; clearly this is $x_i x_j \times$ the sum of all nc-monomials on $\{x_{i+2}, \ldots, x_{j-2}\}$, *i.e.*, $x_i x_j \, \phi_{i+2,j-2}$. ∎

Example 4 For arbitrary i we have

$$\tau_{i,i+2} = 1 + x_i + x_{i+1} + x_{i+2},$$
$$\tau_{i,i+3} = 1 + x_i + x_{i+1} + x_{i+2} + x_{i+3} + x_i x_{i+2} + x_{i+1} x_{i+3}.$$

2 Cyclically labelled paths; $\Phi_{i,Nt+j}$ and the (τ, Δ)-recurrence

Consider a path P and the ordered set of t labels $\{x_1, \ldots, x_t\}$. For a fixed i, where $1 \le i \le t$, and moving from left to right, label the first edge of P with x_i, the second with x_{i+1}, and so on until label x_t has been used; so the $(t-i+1)$-th edge receives label x_t. Then label edge $t-i+2$ with x_1, and edge $t-i+3$ with x_2, and so on ..., labelling cyclically with $\{x_1, \ldots, x_t\}$ until all edges have been labelled. Let the last edge receive label x_j, where $1 \le j \le t$. Suppose that $N \ge 0$ full cycles of labels $\{x_1, \ldots, x_t\}$ have been used beginning at edge $t-i+2$. Then if $j = t$ we call this path $P(i, Nt)$, or if $1 \le j < t$ we call it $P(i, Nt+j)$. This labelling is a *cyclic labelling*. The cyclically labelled path $P(i, Nt+j)$ is shown in Fig. 4. Let $\Phi_{i,Nt+j}(\mathbf{x}) = \Phi_{i,Nt+j}$ denote the matching polynomial of the path $P(i, Nt+j)$.

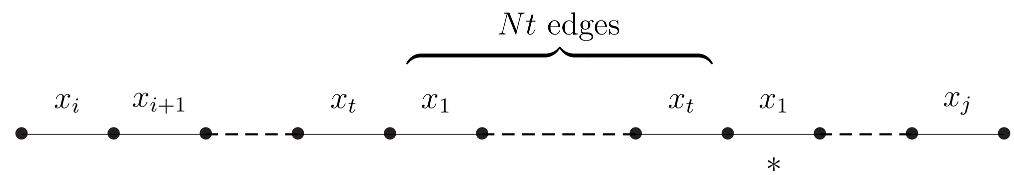


Fig. 4: The cyclically labelled path $P(i, Nt+j)$ with matching polynomial $\Phi_{i,Nt+j}$.

We define the initial conditions for $\Phi_{i,Nt+j}$ as

$$\begin{array}{lll} & N = -1: & \Phi_{i,-t+j} = \phi_{i,j}, \quad \text{for all } j \text{ with } 0 \le j \le t, \\ \text{also} & N = 0: & \Phi_{i,0t+j} = \Phi_{i,j}. \end{array} \tag{9}$$

In order to find $\phi_{i,j}$ if $j < i$ we use the initial values for $\phi_{i,j}$ from (3) and push back Recurrence (4), as shown in Example 2.

Now $\Phi_{i,Nt+j}$ satisfies the same recurrence as that of $\phi_{i,j}$, Recurrence (4); the proof is similar, noting that x_0 must be replaced by x_t, and considering $Nt-1$ as $(N-1)t+t-1$, etc.

Lemma 2.1 *For any $N \ge -1$ and j with $0 \le j \le t$ we have*

$$\Phi_{i,Nt+j} = \Phi_{i,Nt+j-1} + x_j \, \Phi_{i,Nt+j-2}. \tag{10}$$

■

Notation For $i = 1$ we write $\phi_{i,j} = \phi_{1,j} = \phi_j$ and $\phi_t = \phi$, also $\tau_{1,j} = \tau_j$ and $\tau_t = \tau$, and $f_{1,j} = f_j$, etc. Also let $\Delta = (-1)^t x_1 \cdots x_t$.

Lemma 2.2 *For any $N \geq 0$ and any j with $0 \leq j \leq t$ we have*

$$\Phi_{i,Nt+j} = \Phi_{i,Nt}\, f_j + \Phi_{i,Nt-1}\, g_j. \tag{11}$$

Proof. With $N = 0$ and $j = 0$ Equation (11) is true using the initial values $f_0 = 1$ and $g_0 = 0$ of (5) with $i = 1$. Otherwise, consider the path $P(i, Nt + j)$ of Fig. 4 and decompose its matching polynomial, $\Phi_{i,Nt+j}$, at the edge labelled x_1 marked with a $*$. This gives

$$\begin{aligned} \Phi_{i,Nt+j} &= \Phi_{i,Nt}\, \phi_{2,j} + x_1\, \Phi_{i,Nt-1}\, \phi_{3,j} \\ &= \Phi_{i,Nt}\, f_j + \Phi_{i,Nt-1}\, g_j, \end{aligned}$$

using Lemma 1.4. ∎

Now we define the second order $(\tau, \Delta)-$*recurrence*

$$\Theta_N = \tau\, \Theta_{N-1} - \Delta\, \Theta_{N-2}. \tag{12}$$

Let $G_N(\mathbf{x}) = G_N$ denote the first fundamental solution to this recurrence. We will evaluate G_N in Section 4.

In Theorem 2.4 below we show that, for a fixed i and j, $\Phi_{i,Nt+j}$ satisfies the (τ, Δ)-recurrence. First:

Lemma 2.3 *For any $N \geq 1$ we have*

$$\begin{aligned} &(i)\ \Phi_{i,Nt-1}\, f_t - \Phi_{i,Nt}\, f_{t-1} = \Delta\, \Phi_{i,(N-1)t-1}, \\ &(ii)\ \Phi_{i,Nt-1}\, g_t - \Phi_{i,Nt}\, g_{t-1} = -\Delta\, \Phi_{i,(N-1)t}. \end{aligned} \tag{13}$$

Proof. (i) Using Recurrence (4) on f_t and on $\Phi_{i,Nt}$ (see Lemma 2.1), the left-hand side of (13) becomes

$$\Phi_{i,Nt-1}\, \{f_{t-1} + x_t\, f_{t-2}\} - \{\Phi_{i,Nt-1} + x_t\, \Phi_{i,Nt-2}\}\, f_{t-1} = -x_t\, \{\Phi_{i,Nt-2}\, f_{t-1} - \Phi_{i,Nt-1}\, f_{t-2}\}.$$

The second factor in the right-hand side of this equation is the left-hand side of (13) with subscripts shifted down by 1. After t such iterations the left-hand side of (13) becomes

$$(-x_t)\,(-x_{t-1}) \ldots (-x_1)\{\Phi_{i,(N-1)t-1}\, f_0 - \Phi_{i,(N-1)t}\, f_{-1}\} = \Delta\, \Phi_{i,(N-1)t-1},$$

using the initial values $f_0 = 1$ and $f_{-1} = 0$. The proof of (ii) is similar. ∎

Now a main result: $\Phi_{i,Nt+j}$ satisfies the (τ, Δ)-recurrence.

Theorem 2.4 *For any $N \geq 1$, and any fixed i with $1 \leq i \leq t$, and any fixed j with $0 \leq j \leq t$, we have*

$$\Phi_{i,Nt+j} = \tau\,\Phi_{i,(N-1)t+j} - \Delta\,\Phi_{i,(N-2)t+j}. \tag{14}$$

Proof. Due to Recurrences (4) and (10) we need only show that (14) is true when $j = t$ and $t-1$. It will then be true for all j with $0 \leq j \leq t$ by pushing back Recurrence (10).

With $N \geq 1$ and $j = t$, Equation (11) gives

$$\begin{aligned}
\Phi_{i,Nt+t} &= \Phi_{i,Nt}\,f_t + \Phi_{i,Nt-1}\,g_t \\
&= \Phi_{i,Nt}\,f_t + \Phi_{i,Nt-1}\,g_t + \Phi_{i,Nt}\,g_{t-1} - \Phi_{i,Nt}\,g_{t-1} \\
&= \Phi_{i,Nt}\,f_t + \Phi_{i,Nt}\,g_{t-1} + \Phi_{i,Nt-1}\,g_t - \Phi_{i,Nt}\,g_{t-1} \\
&= \tau\,\Phi_{i,Nt} - \Delta\,\Phi_{i,(N-1)t}, \\
&= \tau\,\Phi_{i,(N-1)t+t} - \Delta\,\Phi_{i,(N-2)t+t},
\end{aligned}$$

using $\tau = \tau_t = f_t + g_{t-1}$ from Lemma 1.5(i), and Lemma 2.3(ii) at the fourth line. For $j = t-1$ the proof is similar using Lemma 2.3(i). ∎

3 Matrix formulation of recurrences

Here we use matrices to give another proof that $\Phi_{i,Nt+j}$ satisfies the (τ, Δ)-recurrence, and prepare for the evaluation of G_N in Section 4.

Recall from Section 1 that $f_{i,j}$ and $g_{i,j}$ are the 2 fundamental solutions to Recurrence (4). Now define the matrix

$$X_{i,j} = \begin{pmatrix} g_{i,j-1} & f_{i,j-1} \\ g_{i,j} & f_{i,j} \end{pmatrix}.$$

Then the recurrences for $f_{i,j}$ and $g_{i,j}$ can be written as:

$$X_{i,j} = \begin{pmatrix} 0 & 1 \\ x_j & 1 \end{pmatrix} \begin{pmatrix} g_{i,j-2} & f_{i,j-2} \\ g_{i,j-1} & f_{i,j-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x_j & 1 \end{pmatrix} X_{i,j-1}. \tag{15}$$

Consistent with (5) we have $X_{i,i-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, the 2×2 identity matrix. Thus, for $j \geq i$, we have

$$X_{i,j} = \begin{pmatrix} 0 & 1 \\ x_j & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_{j-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ x_i & 1 \end{pmatrix}. \tag{16}$$

Let $\mathbf{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{e} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and let $\langle \cdot\,, \cdot \rangle$ denote the usual inner product. Then for $j \geq i$, and using (6),

$$\phi_{i,j} = \langle X_{i,j}\mathbf{1}, \mathbf{e} \rangle. \tag{17}$$

As before if $i = 1$ we let $X_{1,j} = X_j$ and if $j = t$ we let $X = X_t$, in particular,

$$X = \begin{pmatrix} g_{t-1} & f_{t-1} \\ g_t & f_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x_t & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_{t-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ x_1 & 1 \end{pmatrix}. \tag{18}$$

For $N \geq 0$, from (10) we may also write

$$\begin{pmatrix} \Phi_{i,Nt+j-1} \\ \Phi_{i,Nt+j} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x_j & 1 \end{pmatrix} \begin{pmatrix} \Phi_{i,Nt+j-2} \\ \Phi_{i,Nt+j-1} \end{pmatrix},$$

and then repeated use of (15) gives

$$\Phi_{i,Nt+j} = \langle X_j X^N X_{i,t}\mathbf{1}, \mathbf{e} \rangle. \tag{19}$$

Now using (16) and (18) we see that $X_j X^{-1} X_{i,t} = X_{i,j}$. So, using (17) and (9), we have

$$\langle X_j X^{-1} X_{i,t} \mathbf{1}, \mathbf{e} \rangle = \langle X_{i,j} \mathbf{1}, \mathbf{e} \rangle = \phi_{i,j} = \Phi_{i,-t+j},$$

thus (19) is true for $N = -1$ also.

Theorem 3.1 *For $N \geq -1$ we have*

$$\Phi_{i,Nt+j} = \langle X_j X^N X_{i,t} \mathbf{1}, \mathbf{e} \rangle.$$

$\blacksquare$

From Lemma 1.5(i) and (16) we have the following forms for the trace and determinant of matrix $X_{i,j}$

$$\mathrm{tr}(X_{i,j}) = \tau_{i,j} \qquad \text{and} \qquad \det(X_{i,j}) = (-1)^{j-i+1} x_i \cdots x_j.$$

In particular, for matrix X from (18), we have

$$\mathrm{tr}(X) = \tau \qquad \text{and} \qquad \det(X) = \Delta. \tag{20}$$

Now let Z be any invertible 2×2 matrix with trace $\mathrm{tr}(Z)$ and determinant $\det(Z)$, and let T denote transpose. Then the Cayley-Hamilton theorem says that $Z^2 = \mathrm{tr}(Z)\, Z - \det(Z)\, I$, so $Z^N = \mathrm{tr}(Z)\, Z^{N-1} - \det(Z)\, Z^{N-2}$, for $N \geq 1$. Let $\mathbf{u}$ and $\mathbf{v} \in \mathbf{R}^2$ and, for $N \geq -1$, define $\Psi_N = \langle Z^N \mathbf{u}, \mathbf{v} \rangle$. Then

Lemma 3.2 *For $N \geq 1$, Ψ_N satisfies the recurrence*

$$\Psi_N = \mathrm{tr}(Z)\, \Psi_{N-1} - \det(Z)\, \Psi_{N-2},$$

with initial conditions $\Psi_{-1} = \langle Z^{-1} \mathbf{u}, \mathbf{v} \rangle$ and $\Psi_0 = \langle Z^0 \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$.

$\blacksquare$

Now for $N \geq -1$,

$$\Phi_{i,Nt+j} = \langle X_j X^N X_{i,t} \mathbf{1}, \mathbf{e} \rangle = \langle X^N X_{i,t} \mathbf{1}, X_j^{\mathrm{T}} \mathbf{e} \rangle.$$

So, for $N \geq 1$, Lemma 3.2 with $Z = X$, $\mathbf{u} = X_{i,t} \mathbf{1}$, and $\mathbf{v} = X_j^{\mathrm{T}} \mathbf{e}$, and (20), gives,

$$\Phi_{i,Nt+j} = \tau\, \Phi_{i,(N-1)t+j} - \Delta\, \Phi_{i,(N-2)t+j}.$$

This is a second proof that $\Phi_{i,Nt+j}$ satisfies the (τ, Δ)-recurrence.

4 The Symmetric Representation, MacMahon's Master Theorem, three expressions for G_N

Consider polynomials in the variables $u_1, \ldots, u_d$. We will work with the vector space whose basis elements are the homogeneous polynomials of degree N in these variables, *i.e.*, with

$$\{u_1^{n_1} \cdots u_d^{n_d} \,|\, n_1 + \cdots + n_d = N, \text{each } n_\ell \geq 0\},$$

this vector space has dimension $\binom{N+d-1}{N}$.

The symmetric representation of a $d \times d$ matrix $A = (a_{\ell\ell'})$ is the action on polynomials induced by:

$$u_1^{n_1} \cdots u_d^{n_d} \to v_1^{n_1} \cdots v_d^{n_d},$$

where

$$v_\ell = \sum_{\ell'} a_{\ell\ell'} u_{\ell'}$$

or, more compactly, $v = Au$. That is, define the matrix element $\left\langle \begin{matrix} m_1, \ldots, m_d \\ n_1, \ldots, n_d \end{matrix} \right\rangle_A$ to be the coefficient of $u_1^{n_1} \cdots u_d^{n_d}$ in $v_1^{m_1} \cdots v_d^{m_d}$. Then, for a fixed $(m_1, \ldots, m_d)$, we have

$$v_1^{m_1} \cdots v_d^{m_d} = \sum_{(n_1, \ldots, n_d)} \left\langle \begin{matrix} m_1, \ldots, m_d \\ n_1, \ldots, n_d \end{matrix} \right\rangle_A u_1^{n_1} \cdots u_d^{n_d}. \tag{21}$$

Observe that the total degree $N = |n| = \sum n_\ell = |m| = \sum m_\ell$, i.e., homogeneity of degree N is preserved. We use multi-indices: $m = (m_1, \ldots, m_d)$ and $n = (n_1, \ldots, n_d)$. Then, for a fixed m, (21) becomes

$$v^m = \sum_n \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_A u^n.$$

Successive application of B then A shows that this is a homomorphism of the multiplicative semi-group of square $d \times d$ matrices into the multiplicative semi-group of square $\binom{N+d-1}{N} \times \binom{N+d-1}{N}$ matrices.

Proposition 4.1 *Matrix elements satisfy the homomorphism property*

$$\left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_{AB} = \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_A \left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle_B.$$

Proof. Let $v = (AB)u$ and $w = Bu$. Then,

$$\begin{aligned} v^m &= \sum_n \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_{AB} u^n \\ &= \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_A w^k \\ &= \sum_n \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_A \left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle_B u^n. \end{aligned}$$

■

Definition Fix the degree $N = \sum n_\ell = \sum m_\ell$. Define $\mathrm{tr}^N_{\mathrm{Sym}}(A)$, the *symmetric trace* of A in degree N, as the sum of the diagonal elements $\left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_A$, *i.e.*,

$$\mathrm{tr}^N_{\mathrm{Sym}}(A) = \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_A .$$

Equality such as $\mathrm{tr}_{\mathrm{Sym}}(A) = \mathrm{tr}_{\mathrm{Sym}}(B)$ means that the symmetric traces are equal in every degree $N \geq 0$.

Remark The action defined here on polynomials is equivalent to the action on symmetric tensor powers, see Fulton and Harris [2], pp. 472-5.

Now it is straightforward to see directly (cf. the diagonal case shown in the Corollary below) that if A is upper-triangular, with eigenvalues $\lambda_1, \ldots, \lambda_d$, then $\mathrm{tr}^N_{\mathrm{Sym}}(A) = h_N(\lambda_1, \ldots, \lambda_d)$, the N^{th} homogeneous symmetric function. The homomorphism property, Proposition 4.1, shows that $\mathrm{tr}^N_{\mathrm{Sym}}(AB) = \mathrm{tr}^N_{\mathrm{Sym}}(BA)$, and that similar matrices have the same trace. Again by the homomorphism property, if two $d \times d$ matrices are similar, $A = MBM^{-1}$, then that relation extends to their respective symmetric representations in every degree. Recall that any matrix is similar to an upper-triangular one with the same eigenvalues. Thus,

Theorem 4.2 Symmetric Trace Theorem (see pp. 51-2 of Springer [5]). *We have*

$$\frac{1}{\det(I - cA)} = \sum_{N=0}^{\infty} c^N \mathrm{tr}^N_{\mathrm{Sym}}(A).$$

Proof. With λ_ℓ denoting the eigenvalues of A,

$$\frac{1}{\det(I - cA)} = \prod_\ell \frac{1}{1 - c\lambda_\ell}$$

$$
\begin{aligned}
&= \sum_{N=0}^{\infty} c^N h_N(\lambda_1, \ldots, \lambda_d) \\
&= \sum_{N=0}^{\infty} c^N \mathrm{tr}^N_{\mathrm{Sym}}(A).
\end{aligned}
$$

■

As a Corollary we have MacMahon's Master Theorem, which we express in the above terminology.

Corollary 4.3 MacMahon's Master Theorem.
The diagonal matrix element $\left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_A$ *is the coefficient of* $u^m = u_1^{m_1} \cdots u_d^{m_d}$ *in the expansion of* $\det(I - UA)^{-1}$ *where* $U = diag(u_1, \ldots, u_d)$ *is the diagonal matrix with entries* $u_1, \ldots, u_d$ *on the diagonal.*

Proof. From Theorem 4.2, with $c = 1$, we want to calculate the symmetric trace of UA. By the homomorphism property,

$$
\begin{aligned}
\mathrm{tr}^N_{\mathrm{Sym}}(UA) &= \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_{UA} \\
&= \sum_m \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_U \left\langle \begin{matrix} k \\ m \end{matrix} \right\rangle_A .
\end{aligned}
$$

Now, with $v = Uw$ and $v_\ell = u_\ell w_\ell$, then

$$
v^m = (u_1 w_1)^{m_1} \cdots (u_d w_d)^{m_d} = u^m w^m = \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_U w^k,
$$

i.e.,

$$
\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_U = u_1^{m_1} \cdots u_d^{m_d} \delta_{k_1 m_1} \cdots \delta_{k_d m_d}
$$

so that

$$
\mathrm{tr}^N_{\mathrm{Sym}}(UA) = \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_A u^m.
$$

■

Now we restrict ourselves to $d = 2$, and return to the (τ, Δ)-recurrence.

Recall, from (18), the 2×2 matrix

$$\begin{aligned} X &= \begin{pmatrix} 0 & 1 \\ x_t & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_{t-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ x_1 & 1 \end{pmatrix} \\ &= \xi_t \xi_{t-1} \cdots \xi_1, \end{aligned}$$

where $\xi_s = \begin{pmatrix} 0 & 1 \\ x_s & 1 \end{pmatrix}$ for $1 \le s \le t$. Let us modify ξ_s slightly by defining $\alpha_s = \begin{pmatrix} 0 & 1 \\ x_s & a_s \end{pmatrix}$ for $1 \le s \le t$, and calling

$$\begin{aligned} \overline{X} &= \begin{pmatrix} 0 & 1 \\ x_t & a_t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_{t-1} & a_{t-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ x_1 & a_1 \end{pmatrix} \\ &= \alpha_t \alpha_{t-1} \cdots \alpha_1. \end{aligned}$$

Let

$$\operatorname{tr}(\overline{X}) = \overline{\tau} \quad \text{and} \quad \det(\overline{X}) = \overline{\Delta},$$

and let $\overline{G}_N$ be the first fundamental solution to the $(\overline{\tau}, \overline{\Delta})$-recurrence:

$$\Theta_N = \overline{\tau}\, \Theta_{N-1} - \overline{\Delta}\, \Theta_{N-2}. \tag{22}$$

Then

$$\begin{aligned} \sum_{N=0}^{\infty} c^N \overline{G}_N &= \frac{1}{1 - \overline{\tau} c + \overline{\Delta} c^2} \\ &= \frac{1}{\det(I - c\overline{X})} \\ &= \sum_{N=0}^{\infty} c^N \operatorname{tr}^N_{\mathrm{Sym}}(\overline{X}). \end{aligned}$$

So

$$\overline{G}_N = \operatorname{tr}^N_{\mathrm{Sym}}(\overline{X}) = \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_{\overline{X}} = \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_{\alpha_t \alpha_{t-1} \cdots \alpha_1}.$$

We need to calculate the symmetric trace of $\overline{X}$ and so identify $\overline{G}_N$. By the homomorphism property, we need only find the matrix elements for each matrix α_s, multiply together and take the trace.

For $\alpha_s = \begin{pmatrix} 0 & 1 \\ x_s & a_s \end{pmatrix}$ the mapping induced on polynomials is

$$v_1 = u_2\,, \quad v_2 = x_s\, u_1 + a_s\, u_2. \tag{23}$$

For any integer $N \geq 0$, the expansion of $v_1^m v_2^{N-m}$ in powers of u_1 and u_2 is of the form

$$v_1^m v_2^{N-m} = \sum_n \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_{\alpha_s} u_1^n u_2^{N-n}, \tag{24}$$

with the notation for the matrix elements abbreviated accordingly. From (23) and (24), the binomial theorem yields

$$\left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_{\alpha_s} = \binom{N-m}{n} x_s^n a_s^{N-m-n}.$$

For example, when $t = 3$, the product $\overline{X} = \alpha_3\alpha_2\alpha_1$ gives the matrix elements, for homogeneity of degree N,

$$\begin{aligned} \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_{\overline{X}} &= \sum_{(k_2,k_3)} \left\langle \begin{matrix} m \\ k_3 \end{matrix} \right\rangle_{\alpha_3} \left\langle \begin{matrix} k_3 \\ k_2 \end{matrix} \right\rangle_{\alpha_2} \left\langle \begin{matrix} k_2 \\ n \end{matrix} \right\rangle_{\alpha_1} \\ &= \sum_{(k_2,k_3)} \binom{N-m}{k_3}\binom{N-k_3}{k_2}\binom{N-k_2}{n} x_1^n x_2^{k_2} x_3^{k_3}\, a_1^{N-k_2-n} a_2^{N-k_3-k_2} a_3^{N-k_3-m}. \end{aligned}$$

Thus, the symmetric trace $\operatorname{tr}_{\mathrm{Sym}}^N(\overline{X}) = \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_{\overline{X}}$ is

$$\sum_{(k_1,k_2,k_3)} \binom{N-k_2}{k_1}\binom{N-k_3}{k_2}\binom{N-k_1}{k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3}\, a_1^{N-k_1-k_2} a_2^{N-k_2-k_3} a_3^{N-k_3-k_1},$$

a *cyclic binomial*. In general, for a product of arbitrary length, the symmetric trace is given by the corresponding cyclic binomial.

Recall the recurrence

$$S_N(x) = 2x\, S_{N-1}(x) - S_{N-2}(x), \tag{25}$$

for $N \geq 1$. The Chebyshev polynomials of the first kind, $T_N = T_N(x)$, are solutions of this recurrence with initial conditions $T_{-1} = x$ and $T_0 = 1$, and the Chebyshev polynomials of the second kind, $U_N = U_N(x)$, are solutions with $U_{-1} = 0$ and $U_0 = 1$.

Combining these observations yields the main identities:

Theorem 4.4 *Let $\overline{X} = \alpha_t\alpha_{t-1}\cdots\alpha_1$, with $\alpha_s = \begin{pmatrix} 0 & 1 \\ x_s & a_s \end{pmatrix}$ for $1 \le s \le t$, and let $\overline{\tau} = \mathrm{tr}(\overline{X})$ and $\overline{\Delta} = \det(\overline{X})$. Let $\overline{G}_N$ denote the first fundamental solution to the $(\overline{\tau}, \overline{\Delta})$-recurrence (22).*

Then we have the **cyclic binomial identity**

$$\begin{aligned}
\overline{G}_N &= \sum_{(k_1,\ldots,k_t)} \binom{N-k_2}{k_1}\binom{N-k_3}{k_2}\cdots\binom{N-k_1}{k_t} x_1^{k_1}x_2^{k_2}\cdots x_t^{k_t}\, a_1^{N-k_1-k_2}a_2^{N-k_2-k_3}\cdots a_t^{N-k_t-k_1} \\
&= \overline{\Delta}^{N/2} U_N\left(\frac{\overline{\tau}}{2\sqrt{\overline{\Delta}}}\right) \\
&= \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N-k}{k} \overline{\tau}^{N-2k}(-\overline{\Delta})^k,
\end{aligned}$$

where U_N denotes the Chebyshev polynomial of the second kind.

Proof. The first equality follows by computing the symmetric trace for arbitrary t as indicated above. The second follows by induction on N using initial conditions $\overline{G}_{-1} = 0$ and $\overline{G}_0 = 1$, the $(\overline{\tau}, \overline{\Delta})$-recurrence (22) and the Chebyshev recurrence (25). The third follows from the second by the Symmetric Trace Theorem applied to $\overline{X} = \begin{pmatrix} 0 & 1 \\ -\overline{\Delta} & \overline{\tau} \end{pmatrix}$, the shift matrix for the $(\overline{\tau}, \overline{\Delta})$-recurrence. ∎

Note that $G_{-1} = 0$ and $G_0 = 1$, so $G_1 = \tau$ using the (τ, Δ)-recurrence. This also follows directly from the condition $k_{s-1} + k_s \le 1$ for non-zero terms in the cyclic binomial summation above. Note also that setting all $a_s = 1$ above gives explicit expressions for G_N.

Example 5 Here $N = 2$ and $t = 3$. Let $A^{Sym(N)}$ denote the symmetric representation in degree N of the matrix A. From the above we have

$$\begin{aligned}
G_2 &= \sum_{(k_1,k_2,k_3)} \binom{2-k_2}{k_1}\binom{2-k_3}{k_2}\binom{2-k_1}{k_3} x_1^{k_1}x_2^{k_2}x_3^{k_3} \\
&= 1 + 2x_1 + 2x_2 + 2x_3 + x_1^2 + 2x_1x_2 + 2x_1x_3 + x_2^2 + 2x_2x_3 + x_3^2 + x_1x_2x_3.
\end{aligned}$$

Also $d=2$, so $\binom{N+d-1}{N}=3$, and $\xi_s=\begin{pmatrix}0 & 1\\ x_s & 1\end{pmatrix}$ for $1\le i\le 3$, thus

$$X=\xi_3\xi_2\xi_1=\begin{pmatrix}x_1 & x_2+1\\ x_1x_3+x_1 & x_2+x_3+1\end{pmatrix}.$$

Now ${\xi_s}^{Sym(2)}=\begin{pmatrix}0 & 0 & 1\\ 0 & x_s & 1\\ x_s^2 & 2x_s & 1\end{pmatrix}$ for $1\le s\le 3$, and so

$$\begin{aligned}X^{Sym(2)} &= {\xi_3}^{Sym(2)}{\xi_2}^{Sym(2)}{\xi_1}^{Sym(2)}\\ &= \begin{pmatrix} x_1^2 & 2x_1x_2+2x_1 & x_2^2+2x_2+1\\ x_1^2x_3+x_1^2 & \begin{matrix}x_1x_2x_3+2x_1x_2\\ +2x_1x_3+2x_1\end{matrix} & \begin{matrix}x_2^2+x_2x_3+2x_2\\ +x_3+1\end{matrix}\\ x_1^2x_3^2+2x_1^2x_3+x_1^2 & \begin{matrix}2x_1x_2x_3+2x_1x_3^2+2x_1x_2\\ +4x_1x_3+2x_1\end{matrix} & \begin{matrix}x_3^2+2x_2x_3+x_2^2\\ +2x_2+2x_3+1\end{matrix}\end{pmatrix}.\end{aligned}$$

We check that $G_2=\mathrm{tr}(X^{Sym(2)})$, as indicated above.

We now give an expression for G_N as a quotient of two matching polynomials; this requires (29) from the next section.

Theorem 4.5 *For $N\ge 0$ we have*

$$G_N=\frac{\Phi_{1,Nt-2}}{\phi_{t-2}}.$$

Proof. Equation (29) is

$$\Phi_{i,Nt+j}=\Phi_{i,j}\,G_N-\Delta\,\phi_{i,j}\,G_{N-1}, \tag{26}$$

and from Example 2 we have $\phi_{i,i-3}=0$. So (26) with $j=i-3$ gives

$$G_N=\frac{\Phi_{i,Nt+i-3}}{\Phi_{i,i-3}}=\frac{\Phi_{1,Nt-2}}{\phi_{t-2}}, \tag{27}$$

the second equality comes from putting $i=1$ in the first and then using (9) in the denominator. ∎

Finally, consider the Fibonacci sequence $\{F_m \,|\, m \geq 1\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$. It is straightforward to show that the number of matchings in the path P_m with $m-1$ edges is F_{m+1}. Now $\Phi_{1,Nt-2}$ is the matching polynomial of the path $P(1, Nt-2)$ which has $(N+1)t-2$ edges and so has $F_{(N+1)t}$ matchings. Similarly, the path whose matching polynomial is ϕ_{t-2} has F_t matchings. Now, evaluating (27) above with $N = N-1$ and $x_s = 1$ for all $1 \leq s \leq t$, gives $F_t | F_{Nt}$, a well-known result on Fibonacci numbers, see pp. 148-9, Hardy and Wright [4]. Furthermore, we have

$$\frac{F_{(N+1)t}}{F_t} = \sum_{(k_1,\ldots,k_t)} \binom{N-k_2}{k_1}\binom{N-k_3}{k_2}\cdots\binom{N-k_1}{k_t}.$$

5 Examples: Paths, Cycles, Trees

In this Section we express the matching polynomial of some well-known graphs in terms of the fundamental solutions to the (τ, Δ)-recurrence (12).

G_N is the first fundamental solution to the (τ, Δ)-recurrence, so the initial values for G_N are

$$G_{-2} = \frac{-1}{\Delta}, \quad G_{-1} = 0, \quad G_0 = 1, \quad (\text{and} \quad G_1 = \tau). \tag{28}$$

The second fundamental solution is $-\Delta G_{N-1}$.

5.1 Paths

$\Phi_{i,Nt+j}$ satisfies the (τ, Δ)-recurrence whose fundamental solutions are G_N and $-\Delta G_{N-1}$, thus $\Phi_{i,Nt+j} = a\, G_N + b\,(-\Delta G_{N-1})$ for some a and b. The initial conditions for $\Phi_{i,Nt+j}$ from (9) and for G_N from (28) give $a = \Phi_{i,j}$ and $b = \Phi_{i,-t+j} = \phi_{i,j}$. Hence for $N \geq -1$,

$$\Phi_{i,Nt+j} = \Phi_{i,j}\, G_N - \Delta\, \phi_{i,j}\, G_{N-1}. \tag{29}$$

Example 6 Here $i = 2$ and $t = 3$,

$N = -1$ $\quad \phi_{2,2} = 1 + x_2$,
$N = 0$ $\quad \phi_{2,3} = 1 + x_2 + x_3$,
$N = 0$ $\quad \Phi_{2,1} = 1 + x_1 + x_2 + x_3 + x_1x_2$,
$N = 0$ $\quad \Phi_{2,2} = 1 + x_1 + 2x_2 + x_3 + x_1x_2 + x_2^2 + x_2x_3$,
$N = 1$ $\quad \Phi_{2,3} = 1 + x_1 + 2x_2 + 2x_3 + x_1x_2 + x_1x_3 + x_2^2 + 2x_2x_3 + x_3^2 + x_1x_2x_3$.

For $N \geq 1$ let $P_{Nt+j+1} = P(1,(N-1)t+j)$ be the path with $Nt+j+1$ vertices and $Nt+j$ edges, cyclically labelled starting with label x_1. Let $\mathcal{P}_{Nt+j+1}(\mathbf{x}) = \mathcal{P}_{Nt+j+1} = \Phi_{1,(N-1)t+j}$ be its matching polynomial. With this notation any subscript on a P, $\mathcal{P}$, C, or $\mathcal{C}$ refers to the number of vertices in the appropriate graph.

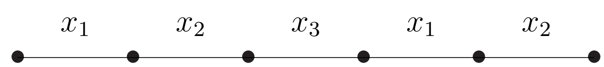
Theorem 5.1 *For any $N \geq 1$ we have*

$$\begin{aligned} (i) \quad \mathcal{P}_{Nt+j+1} &= \Phi_{1,j}\, G_{N-1} - \Delta\, \phi_j\, G_{N-2}, \\ (ii) \quad \mathcal{P}_{Nt+1} &= G_N + (\phi - \tau)\, G_{N-1}. \end{aligned}$$

Proof. The proof of (i) is clear using (29) with $i = 1$ and $N = N - 1$. So (i) with $j = 0$ gives $\mathcal{P}_{Nt+1} = \Phi_{1,0}\, G_{N-1} - \Delta\, \phi_0\, G_{N-2}$. But $\Phi_{1,0} = \Phi_{1,-t+t} = \phi_{1,t} = \phi_t = \phi$ and $\phi_0 = \phi_{1,0} = 1$, and then using the (τ, Δ)-recurrence for G_N gives (ii). ∎

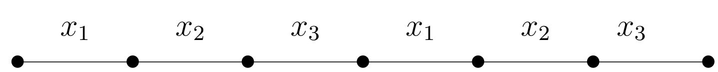
Example 7 Here $t = 3$,

$P_{1\cdot 3+2+1}$



$\mathcal{P}_{1\cdot 3+2+1} = 1 + 2x_1 + 2x_2 + x_3 + x_1^2 + 2x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_1x_2x_3.$

$P_{2\cdot 3+1}$



$\mathcal{P}_{2\cdot 3+1} = 1 + 2x_1 + 2x_2 + 2x_3 + x_1^2 + 2x_1x_2 + 3x_1x_3 + x_2^2 + 2x_2x_3 + x_3^2 + x_1^2x_3 + 2x_1x_2x_3 + x_1x_3^2.$

5.2 Cycles

Now we identify the first and the last vertices of the path $P(i, Nt+j)$ to form the cyclically labelled cycle $C(i, Nt+j)$ with matching polynomial $\Gamma_{i,Nt+j}(\mathbf{x}) = \Gamma_{i,Nt+j}$.

By decomposing $\Gamma_{i,Nt+j}$ at the 'first' edge labelled x_i we see that, *cf.* (29),

$$\begin{aligned}
\Gamma_{i,Nt+j} &= \Phi_{i+1,Nt+j} + x_i\,\Phi_{i+2,Nt+j-1},\\
&= \Phi_{i+1,j}\,G_N - \Delta\,\phi_{i+1,j}\,G_{N-1} + x_i\{\Phi_{i+2,j-1}\,G_N - \Delta\,\phi_{i+2,j-1}\,G_{N-1}\},\\
&= \{\Phi_{i+1,j} + x_i\,\Phi_{i+2,j-1}\}\,G_N - \Delta\,\{\phi_{i+1,j} + x_i\,\phi_{i+2,j-1}\}\,G_{N-1},\\
&= \Gamma_{i,j}\,G_N - \Delta\,\tau_{i,j}\,G_{N-1}, \qquad (30)
\end{aligned}$$

using (29) at the second line, and decomposing $\Gamma_{i,j}$ and $\tau_{i,j}$ at the first edge x_i at the fourth line. Also, defining $\Gamma_{i,-t+j} = \tau_{i,j}$ ensures that (30) is true for all $N \geq -1$.

Example 8 Here $i = 2$ and $t = 3$ again,

$N = -1$ $\quad \tau_{2,2} = 1$,
$N = 0$ $\quad \tau_{2,3} = 1 + x_2 + x_3$,
$N = 0$ $\quad \Gamma_{2,1} = 1 + x_1 + x_2 + x_3$,
$N = 0$ $\quad \Gamma_{2,2} = 1 + x_1 + 2x_2 + x_3 + x_1x_2 + x_2x_3$,
$N = 1$ $\quad \Gamma_{2,3} = 1 + x_1 + 2x_2 + 2x_3 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$.

Let $C_{Nt+j} = C(1, (N-1)t+j)$ be the cycle with $Nt+j$ vertices and $Nt+j$ edges in which labelling has started with x_1, and let $\mathcal{C}_{Nt+j}(\mathbf{x}) = \mathcal{C}_{Nt+j} = \Gamma_{1,(N-1)t+j}$ be its matching polynomial. Compare with Theorem 5.1,

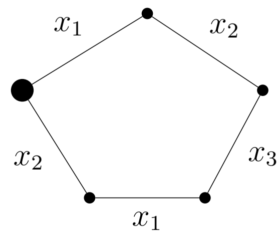
Theorem 5.2 *For any $N \geq 1$ we have*

$$\begin{aligned}
(i) \quad \mathcal{C}_{Nt+j} &= \Gamma_{1,j}\,G_{N-1} - \Delta\,\tau_j\,G_{N-2},\\
(ii) \quad \mathcal{C}_{Nt} &= G_N - \Delta\,G_{N-2}.
\end{aligned}$$

Proof. The proof of (i) is clear from (30). Part (i) with $j = 0$ gives (ii), using $\Gamma_{1,0} = \tau$, and $\tau_0 = 2$ from (8). ∎

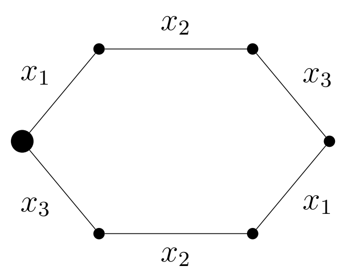
Example 9 Here $t = 3$ again, the cycle starts at the large vertex and proceeds clockwise,

$C_{1\cdot 3+2}$



$\mathcal{C}_{1\cdot 3+2} = 1 + 2x_1 + 2x_2 + x_3 + x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3$

$C_{2\cdot 3}$



$\mathcal{C}_{2\cdot 3} = 1 + 2x_1 + 2x_2 + 2x_3 + x_1^2 + 2x_1x_2 + 2x_1x_3 + x_2^2 + 2x_2x_3 + x_3^2 + 2x_1x_2x_3.$

For a fixed $t \geq 1$ write $\widehat{\mathcal{P}}_N = \mathcal{P}_{Nt+1}$ and $\widehat{\mathcal{C}}_N = \mathcal{C}_{Nt}$. We now express G_N, $\widehat{\mathcal{P}}_N$, and $\widehat{\mathcal{C}}_N$ in terms of Chebyshev polynomials.

It is well-known that, in one variable x, the matching polynomial of the path P_{2m} is related to U_{2m} as follows

$$\mathcal{M}(P_{2m}, x) = (-1)^m x^m\, U_{2m}\left(\frac{i}{2\sqrt{x}}\right),$$

and, for P_{2m-1} we have

$$\mathcal{M}(P_{2m-1}, x) = (-1)^m x^m \left[U_{2m}\left(\frac{i}{2\sqrt{x}}\right) + U_{2m-2}\left(\frac{i}{2\sqrt{x}}\right)\right],$$

where $i = \sqrt{-1}$. Also, for the matching polynomials $\mathcal{M}(C_{2m})$ and $\mathcal{M}(C_{2m-1})$ of the cycles C_{2m} and C_{2m-1} there are similar formulas but with a factor of 2 on the right-hand side where U is replaced by T. See Theorem 3 of Godsil and Gutman [3], and Theorems 9 and 11 of Farrell [1].

Now Theorem 4.4 modified for G_N gives

$$G_N = \Delta^{N/2} U_N \left(\frac{\tau}{2\sqrt{\Delta}} \right). \tag{31}$$

Formulas for $\widehat{\mathcal{P}}_N$ and $\widehat{\mathcal{C}}_N$ in terms of U_N and T_N are given below, where the variable t is suppressed.

Theorem 5.3 *For any $N \geq 1$ we have*

$$(i) \quad \widehat{\mathcal{P}}_N = \Delta^{N/2} \left\{ U_N \left(\frac{\tau}{2\sqrt{\Delta}} \right) + \left(\frac{\phi - \tau}{\sqrt{\Delta}} \right) U_{N-1} \left(\frac{\tau}{2\sqrt{\Delta}} \right) \right\},$$
$$(ii) \quad \widehat{\mathcal{C}}_N = 2\Delta^{N/2} T_N \left(\frac{\tau}{2\sqrt{\Delta}} \right).$$

Proof. (i) This follows from Theorem 5.1(ii) and (31).
(ii) From Theorem 5.2(ii) we have $\widehat{\mathcal{C}}_N = G_N - \Delta G_{N-2}$, and now the well-known relation $2T_N = U_N - U_{N-2}$ between the two types of Chebyshev polynomials and (31) gives the result. ■

Expressions for G_N, $\widehat{\mathcal{P}}_N$, and $\widehat{\mathcal{C}}_N$ for $N = 0$, 1, 2, 3, and 4 are given below

$G_0 = 1$	$\widehat{\mathcal{P}}_0 = 1$	$\widehat{\mathcal{C}}_0 = 2$
$G_1 = \tau$	$\widehat{\mathcal{P}}_1 = \phi$	$\widehat{\mathcal{C}}_1 = \tau$
$G_2 = \tau^2 - \Delta$	$\widehat{\mathcal{P}}_2 = \phi\tau - \Delta$	$\widehat{\mathcal{C}}_2 = \tau^2 - 2\Delta$
$G_3 = \tau^3 - 2\tau\Delta$	$\widehat{\mathcal{P}}_3 = \phi\tau^2 - \phi\Delta - \tau\Delta$	$\widehat{\mathcal{C}}_3 = \tau^3 - 3\tau\Delta$
$G_4 = \tau^4 - 3\tau^2\Delta + \Delta^2$	$\widehat{\mathcal{P}}_4 = \phi\tau^3 - 2\phi\tau\Delta - \tau^2\Delta + \Delta^2$	$\widehat{\mathcal{C}}_4 = \tau^4 - 4\tau^2\Delta + 2\Delta^2$.

5.3 Trees

Here we consider cyclically labelled trees.

First let us extend the definition of a cyclically labelled path to include the path of Fig. 1, and the graph P_1 with one vertex and no edges.

A tree is a connected simple graph with no cycles, and a rooted tree is a tree in which some vertex of degree 1 has been specified to be the root, r. Given any rooted tree, let us label its edges by first labelling the edge incident to r with x_i. Then label all edges incident to this edge with x_{i+1}, then label all edges incident to these edges with x_{i+2}, and so on until label x_t has been used. Then label with the ordered set $\{x_1, \ldots, x_t\}$ in a similar manner to before, repeating cyclically until all edges have been labelled,..., and so on. Let T denote such a cyclically labelled tree, see Fig. 5 for an example with $i = 2$ and $t = 3$.

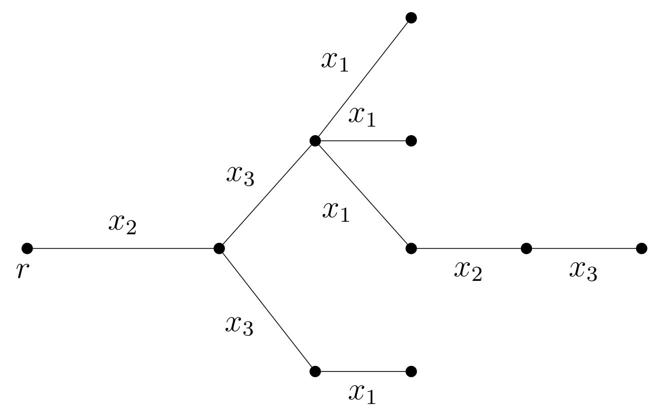


Fig. 5: A cyclically labelled tree with $i = 2$ and $t = 3$.

We may draw any such T with r as the leftmost vertex. Then we place the other vertices of T from 'left to right' according to their distance from r, *i.e.*, if a vertex v_1 is at distance d_1 from r and vertex v_2 is at distance d_2 from r where $d_2 > d_1$, then v_2 is placed to the right of v_1.

Paths in T are of two types: (I) A path that always moves from left to right (a path that always moves from right to left can be thought of one that always moves from left to right): such a path is clearly cyclically labelled; or (II) a path that moves first from right to left and then from left to right; such a path must pass through at least one vertex of degree ≥ 3, *i.e.*, a vertex where T 'branches'.

Let V denote the set of vertices of degree ≥ 3 in T, and let $v \in V$ be arbitrary of degree $\deg(v)$. Vertex v has 1 edge to its left and $\deg(v) - 1 \geq 2$ edges to its right. Let H_v be the subgraph of T that consists of the 'last' $\deg(v) - 2 \geq 1$ edges as we rotate clockwise around v. Thus H_v is the star $K_{1,\deg(v)-2}$ centered at v. Set $H = \cup_{v \in V} H_v$.

Lemma 5.4 *The forest $T - H$ is a union of cyclically labelled paths.*

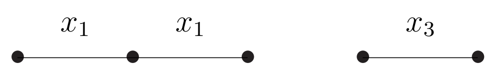
Proof. We show that $T - H$ does not contain a path of type (II). Suppose it does contain a path of type (II), then this path must pass through some vertex $v \in V$. So 2 edges incident to v and to the right of v lie in this path and so lie in $T - H$, a contradiction because $T - H$ contains only 1 edge incident to v and to the right of v. Thus $T - H$ is a union of paths of type (I), each of which is a cyclically labelled path. ∎

Thus $T - H$ is a union of cyclically labelled paths, and so $T - H - \overline{M}_H$ is also, for every matching M_H of H. We know the matching polynomial of any cyclically labelled path, so we can decompose the matching polynomial of T, $\mathcal{M}(T, \mathbf{x})$, at H, according to Theorem 1.1,

$$\mathcal{M}(T, \mathbf{x}) = \sum_{M_H} M_H(\mathbf{x})\, \mathcal{M}(T - H - \overline{M}_H, \mathbf{x}),$$

where the summation is over every matching M_H of H.

Example 10 See Fig. 5.

Here $H =$ 

H has 6 matchings with weights: 1, x_1, x_1, x_3, x_1x_3, and x_1x_3. Thus there are 6 terms in the decomposition, and $\mathcal{M}(T, \mathbf{x})$ is the sum of the following 6 terms:

$$1.\phi_1\phi_{2,3}\Phi_{2,1}+x_1.\phi_1\phi_{2,2}\phi_{2,3}+x_1.\phi_1\phi_{2,2}\phi_{3,3}+x_3.\phi_1\phi_{2,3}+x_1x_3.\phi_{2,3}+x_1x_3.\phi_{3,3}$$
$$= 1 + 4x_1 + 2x_2 + 3x_3 + 3x_1^2 + 7x_1x_2 + 8x_1x_3 + x_2^2 + 3x_2x_3 + 2x_3^2$$
$$+5x_1^2x_2 + 3x_1^2x_3 + 3x_1x_2^2 + 7x_1x_2x_3 + 4x_1x_3^2 + 2x_1^2x_2^2 + 3x_1^2x_2x_3.$$

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