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Issa Amadou Tall
Southern Illinois University Carbondale, itall@math.siu.edu

Witold Respondek
INSA de Rouen

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FEEDBACK CLASSIFICATION OF NONLINEAR SINGLE-INPUT CONTROL SYSTEMS WITH CONTROLLABLE LINEARIZATION: NORMAL FORMS, CANONICAL FORMS, AND INVARIANTS*

ISSA AMADOU TALL† AND WITOLD RESPONDEK†

Abstract. We study the feedback group action on single-input nonlinear control systems. We follow an approach of Kang and Krener based on analyzing, step by step, the action of homogeneous transformations on the homogeneous part of the same degree of the system. We construct a dual normal form and dual invariants with respect to those obtained by Kang. We also propose a canonical form and a dual canonical form and show that two systems are equivalent via a formal feedback if and only if their canonical forms (resp., their dual canonical forms) coincide. We give an explicit construction of transformations bringing the system to its normal, dual normal, canonical, and dual canonical forms. We illustrate our results by simple examples on $\mathbb{R}^3$ and $\mathbb{R}^4$.

Key words. feedback equivalence, normal forms, canonical forms, invariants

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1. Introduction.

The problem of transforming the nonlinear control single-input system

$$\Sigma \ : \ \dot{\xi} = f(\xi) + g(\xi)u$$

by a feedback transformation of the form

$$\Gamma \ : \ \begin{array}{l} x = \phi(\xi), \\ u = \alpha(\xi) + \beta(\xi)v \end{array}$$

to a simpler form has been extensively studied during the last twenty years. The transformation Γ brings Σ to the system

$$\tilde{\Sigma} \ : \ \dot{x} = \tilde{f}(x) + \tilde{g}(x)v,$$

whose dynamics are given by

$$\begin{aligned} \tilde{f} &= \phi_*(f + g\alpha), \\ \tilde{g} &= \phi_*(g\beta), \end{aligned}$$

where for any vector field f and any diffeomorphism ϕ we denote

$$(\phi_* f)(x) = d\phi(\phi^{-1}(x)) \cdot f(\phi^{-1}(x)).$$

A natural question to ask is whether we can find a transformation Γ such that the

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†Institut National des Sciences Appliquées de Rouen, Laboratoire de Mathématiques de l'INSA, Pl. Emile Blondel, 76 131 Mont Saint Aignan Cedex, France (tall@insa-rouen.fr, wresp@insa-rouen.fr). The first author worked on this paper while on leave from the Department of Mathematics, Université Cheikh Anta Diop, Dakar, Senegal.

transformed system $\tilde{\Sigma}$ is linear, that is, whether we can linearize the system Σ via feedback. Necessary and sufficient geometric conditions for this to be the case have been given in [13] and [18]. Those conditions, except for the planar case, turn out to be restrictive, and a natural problem that arises is to find normal forms for nonlinearizable systems. Although natural, this problem is very involved and has been extensively studied during the last twenty years. Four basic methods have been proposed for studying feedback equivalence problems. The first method is based on the theory of singularities of vector fields and distributions, and their invariants, and using this method a large variety of feedback classification problems have been solved; see, e.g., [4], [7], [14], [15], [18], [19], [27], [29], [32], [38]. The second approach, proposed by Gardner [9], uses Cartan's method of equivalence [6] and describes the geometry of feedback equivalence [10], [11], [12], [28]. The third method, inspired by the Hamiltonian formalism for optimal control problems, was developed by Bonnard [3], [4] and Jakubczyk [16], [17] and has led to a very nice description of feedback invariants in terms of singular extremals. Finally, a very fruitful approach was proposed by Kang and Krener [26] and then followed by Kang [21], [22]. Their idea, which is closely related with Poincaré's classical technique for linearization of dynamical systems (see, e.g., [1]), is to analyze the system Σ and the feedback transformation Γ step by step and, as a consequence, to produce a simpler equivalent system $\tilde{\Sigma}$ also step by step.

Our paper is deeply inspired by those of Kang and Krener [26], [21] and can be considered as a completion of their results. In [21], Kang constructed a normal form for single-input nonlinear control systems with controllable linearization using successively homogeneous feedback transformations, and he proved that the homogeneous terms of a given degree of his normal form are unique under homogeneous feedback transformations of the same degree. He also showed that a nonlinear system can admit different normal forms under feedback resulting from the action of lower order terms of the feedback transformation on higher order terms of the system. The main goal of our paper is to propose a canonical form for the class of single-input systems with controllable linearization and to prove that two systems are equivalent, via a formal feedback, if and only if their canonical forms coincide.

In [26] Kang and Krener constructed two normal forms for the quadratic part of a single-input system. In the first normal form, all components of the linear part of the control vector field are annihilated and all nonremovable quadratic nonlinearities are grouped in the drift; in the second normal form, all quadratic terms of the drift are annihilated and all nonremovable nonlinearities are present in the control vector field. Kang normal form is a generalization, for higher order terms, of the first normal form. In this paper, we generalize the second one and produce a dual normal form for higher order terms. We also construct dual invariants of homogeneous feedback transformations. They contain the same information, as Kang invariants, encoded in a different way. We also give a dual canonical form and prove that two systems are equivalent, via formal feedback, if and only if their dual canonical forms coincide.

The third aim of the paper is to construct explicit homogeneous feedback transformations which bring the homogeneous part of the system of the same degree into its normal, or dual normal, form. For any fixed degree, our transformations are easily computable via differentiation and integration of polynomials. A successive application of those transformations gives formal feedbacks that bring any system to its normal form, dual normal form, canonical form, and dual canonical form.

The theory of normal forms initialized and developed by Kang and Krener [26] and Kang [21], [22] and continued in the present paper (and in [33], [34]) has proved to be very useful in analyzing structural properties of nonlinear control systems. It has been used to study bifurcations of nonlinear systems [23], [24], [25], has led to a complete description of symmetries around equilibrium [30], [31], and has allowed us to characterize systems equivalent to feedforward forms [35], [36], [37].

The paper is organized as follows. In section 2 we will introduce, following [21] and [26], homogeneous feedback transformations. We give a normal form obtained by Kang and discuss invariants of homogeneous transformations, also obtained by him. We provide an explicit construction of transformations bringing the system to Kang normal form. In section 3 we construct a canonical form and give one of our main results stating that two control systems are feedback equivalent if and only if their canonical forms coincide. Proofs of results presented in sections 2 and 3 are given in section 4.

Section 5 dualizes the main results of section 2: we give a dual normal form, explicitly construct transformations bringing the system to that form, and define dual invariants of homogeneous transformations. Similarly to normal forms, a given system can admit different dual normal forms. In section 6 we thus dualize the results of section 3 by constructing a dual canonical form and proving that two control systems are feedback equivalent if and only if their dual canonical forms coincide. Section 7 contains proofs of results presented in sections 5 and 6. Throughout the paper, we illustrate our results by simple examples on $\mathbb{R}^3$ and $\mathbb{R}^4$.

2. Normal form and m-invariants. All objects, that is, functions, maps, vector fields, control systems, etc., are considered in a neighborhood of $0 \in \mathbb{R}^n$ and assumed to be C^∞-smooth. Let h be a smooth $\mathbb{R}$-valued function. By

$$h(x) = h^{[0]}(x) + h^{[1]}(x) + h^{[2]}(x) + \cdots = \sum_{m=0}^{\infty} h^{[m]}(x)$$

we denote its Taylor series expansion at $0 \in \mathbb{R}^n$, where $h^{[m]}(x)$ stands for a homogeneous polynomial of degree m.

Similarly, for a map ϕ of an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ (resp., for a vector field f on an open subset of $\mathbb{R}^n$) we will denote by $\phi^{[m]}$ (resp., by $f^{[m]}$) the homogeneous term of degree m of its Taylor series expansion at $0 \in \mathbb{R}^n$, that is, each component $\phi_j^{[m]}$ of $\phi^{[m]}$ (resp., $f_j^{[m]}$ of $f^{[m]}$) is a homogeneous polynomial of degree m in x.

We will denote by $H^{[m]}(x)$ the space of homogeneous polynomials of degree m of the variables $x_1, \ldots, x_n$ and by $H^{\geq m}(x)$ the space of formal power series of the variables $x_1, \ldots, x_n$ starting from terms of degree m.

Analogously, we will denote by $R^{[m]}(x)$ the space of homogeneous vector fields whose components are in $H^{[m]}(x)$ and by $R^{\geq m}(x)$ the space of vector fields formal power series whose components are in $H^{\geq m}(x)$.

Consider the Taylor series expansion of the system Σ given by

$$\Sigma^\infty \ : \ \dot{\xi} = F\xi + Gu + \sum_{m=2}^{\infty} \left(f^{[m]}(\xi) + g^{[m-1]}(\xi)u \right), \tag{2.1}$$

where $F = \frac{\partial f}{\partial \xi}(0)$ and $G = g(0)$. We will assume throughout the paper that $f(0) = 0$ and $g(0) \neq 0$.

Consider also the Taylor series expansion Γ^∞ of the feedback transformation Γ

given by

$$\Gamma^\infty \ : \ \begin{aligned} x &= T\xi + \sum_{m=2}^{\infty} \phi^{[m]}(\xi), \\ u &= K\xi + Lv + \sum_{m=2}^{\infty} \left(\alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v\right), \end{aligned} \tag{2.2}$$

where T is an invertible matrix and $L \neq 0$. Let us analyze the action of Γ^∞ on the system Σ^∞ step by step.

To start with, consider the linear system

$$\dot{\xi} = F\xi + Gu.$$

Throughout the paper we will assume that it is controllable. It can be thus transformed by a linear feedback transformation of the form

$$\Gamma^1 : \begin{aligned} x &= T\xi, \\ u &= K\xi + Lv \end{aligned}$$

to the Brunovský canonical form (A, B); see, e.g., [20]. Assuming that the linear part (F, G), of the system Σ^∞ given by (2.1), has been transformed to the Brunovský canonical form (A, B), we follow an idea of Kang and Krener [26], [21] and apply successively a series of transformations

$$\Gamma^m : \begin{aligned} x &= \xi + \phi^{[m]}(\xi), \\ u &= v + \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v \end{aligned} \tag{2.3}$$

for $m = 2, 3, \ldots$. A feedback transformation defined as a series of successive compositions of Γ^m, $m = 1, 2, \ldots$, will also be denoted by Γ^∞ because, as a formal power series, it is of the form (2.2). We will not address the problem of convergence and will call such a series of successive compositions a *formal feedback transformation*.

Observe that each transformation Γ^m for $m \geq 2$ leaves invariant all homogeneous terms of degree smaller than m of the system Σ^∞, and we will call Γ^m a *homogeneous feedback transformation of degree m*. We will study the action of Γ^m on the following *homogeneous system*:

$$\Sigma^{[m]} \ : \ \dot{\xi} = A\xi + Bu + f^{[m]}(\xi) + g^{[m-1]}(\xi)u. \tag{2.4}$$

Consider another homogeneous system, $\tilde{\Sigma}^{[m]}$, given by

$$\tilde{\Sigma}^{[m]} \ : \ \dot{x} = Ax + Bv + \tilde{f}^{[m]}(x) + \tilde{g}^{[m-1]}(x)v. \tag{2.5}$$

We will say that the homogeneous system $\Sigma^{[m]}$ is feedback equivalent to the homogeneous system $\tilde{\Sigma}^{[m]}$ if there exists a homogeneous feedback transformation of the form (2.3), which brings $\Sigma^{[m]}$ into $\tilde{\Sigma}^{[m]}$ modulo terms in $R^{\geq m+1}(x, v)$.

Notation. Because of various normal forms and various transformations that are used throughout the paper, we will keep the following notation. We will denote, respectively, by $\Sigma^{[m]}$ and Σ^{∞} the following systems:

$$\Sigma^{[m]} \quad : \quad \dot{\xi} = A\xi + Bu + f^{[m]}(\xi) + g^{[m-1]}(\xi)u,$$

$$\Sigma^{\infty} \quad : \quad \dot{\xi} = A\xi + Bu + \sum_{k=2}^{\infty} \left(f^{[k]}(\xi) + g^{[k-1]}(\xi)u \right).$$

The systems $\Sigma^{[m]}$ and Σ^{∞} will stand for the systems under consideration. Their state vector will be denoted by ξ and their control by u. The system $\Sigma^{[m]}$ (resp., the system Σ^{∞}) transformed via feedback will be denoted by $\tilde{\Sigma}^{[m]}$ (resp., by $\tilde{\Sigma}^{\infty}$). Its state vector will be denoted by x, its control by v, and the vector fields, defining its dynamics, by $\tilde{f}^{[k]}$ and $\tilde{g}^{[k-1]}$. Feedback equivalence of homogeneous systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$ will be established via a smooth feedback, that is, precisely, via a homogeneous feedback Γ^m. On the other hand, feedback equivalence of systems Σ^{∞} and $\tilde{\Sigma}^{\infty}$ will be established via a formal feedback Γ^{∞}.

We will introduce two kinds of normal forms, Kang normal forms and dual normal forms, as well as canonical forms and dual canonical forms. The "bar" symbol will correspond to the vector field $\bar{f}^{[m]}$ defining the Kang normal forms $\Sigma_{NF}^{[m]}$ and Σ_{NF}^{∞} and the canonical form Σ_{CF}^{∞} as well as to the vector field $\bar{g}^{[m-1]}$ defining the dual normal forms $\Sigma_{DNF}^{[m]}$ and Σ_{DNF}^{∞} and the dual canonical form Σ_{DCF}^{∞}. Analogously, the m-invariants (resp., dual m-invariants) of the system $\Sigma^{[m]}$ will be denoted by $a^{[m]j,i+2}$ (resp., by $b_j^{[m-1]}$) and the m-invariants (resp., dual m-invariants) of the normal form $\Sigma_{NF}^{[m]}$ (resp., dual normal form $\Sigma_{DNF}^{[m]}$) by $\bar{a}^{[m]j,i+2}$ (resp., by $\bar{b}_j^{[m-1]}$).

The starting point is the following result, proved by Kang [21].

PROPOSITION 1. *The homogeneous feedback transformation Γ^m, defined by* (2.3), *brings the system $\Sigma^{[m]}$, given by* (2.4), *into $\tilde{\Sigma}^{[m]}$, given by* (2.5), *if and only if the relations*

$$\left\{ \begin{aligned} L_{A\xi}\phi_j^{[m]}(\xi) - \phi_{j+1}^{[m]}(\xi) &= \tilde{f}_j^{[m]}(\xi) - f_j^{[m]}(\xi), \\ L_B\phi_j^{[m]}(\xi) &= \tilde{g}_j^{[m-1]}(\xi) - g_j^{[m-1]}(\xi), \\ L_{A\xi}\phi_n^{[m]}(\xi) + \alpha^{[m]}(\xi) &= \tilde{f}_n^{[m]}(\xi) - f_n^{[m]}(\xi), \\ L_B\phi_n^{[m]}(\xi) + \beta^{[m-1]}(\xi) &= \tilde{g}_n^{[m-1]}(\xi) - g_n^{[m-1]}(\xi) \end{aligned} \right. \tag{2.6}$$

hold for any $1 \le j \le n-1$, where $\phi_j^{[m]}$ are the components of $\phi^{[m]}$.

This proposition represents the essence of the method developed by Kang and Krener and used in our paper. The problem of studying the feedback equivalence of two systems Σ and $\tilde{\Sigma}$ requires, in general, solving a system of first order partial differential equations. On the other hand, if we perform the analysis step by step, then the problem of establishing the feedback equivalence of two systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$ reduces to solving the algebraic system (2.6).

Using the above proposition, Kang [21] proved the following result.

THEOREM 1. *The homogeneous system $\Sigma^{[m]}$ can be transformed, via a homogeneous feedback transformation Γ^m, into the normal form*

$$
\Sigma_{NF}^{[m]} \; : \; \begin{cases} \dot{x}_1 = x_2 + \sum\limits_{i=3}^{n} x_i^2 P_{1,i}^{[m-2]}(x_1, \dots, x_i), \\ \quad \vdots \\ \dot{x}_j = x_{j+1} + \sum\limits_{i=j+2}^{n} x_i^2 P_{j,i}^{[m-2]}(x_1, \dots, x_i), \\ \quad \vdots \\ \dot{x}_{n-2} = x_{n-1} + x_n^2 P_{n-2,n}^{[m-2]}(x_1, \dots, x_n), \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = v, \end{cases} \tag{2.7}
$$

where $P_{j,i}^{[m-2]}(x_1, \dots, x_i)$ are homogeneous polynomials of degree $m-2$ depending on the indicated variables.

In order to construct invariants of homogeneous feedback transformations, let us define

$$
X_i^{m-1}(\xi) = (-1)^i ad^i_{A\xi + f^{[m]}(\xi)}(B + g^{[m-1]}(\xi))
$$

and let $X_i^{[m-1]}$ be its homogeneous part of degree $m-1$. By π_i we will denote the projection on the subspace

$$
W_i = \{\xi \in R^n : \;\; \xi_{i+1} = \cdots = \xi_n = 0\}, \tag{2.8}
$$

that is,

$$
\pi_i(\xi) = (\xi_1, \dots, \xi_i, 0, \dots, 0).
$$

Following Kang [21], we denote by $a^{[m]j,i+2}(\xi)$ the homogeneous part of degree $m-2$ of the polynomials

$$
CA^{j-1}\left[X_i^{m-1}, X_{i+1}^{m-1}\right](\pi_{n-i}(\xi)),
$$

where $C = (1, 0, \dots, 0)^T \in \mathbb{R}^n$ and $(j,i) \in \Delta \subset \mathbb{N} \times \mathbb{N}$, defined by

$$
\Delta = \{(j,i) \in \mathbb{N} \times \mathbb{N} : \; 1 \le j \le n-2, \; 0 \le i \le n-j-2\}.
$$

The homogeneous polynomials $a^{[m]j,i+2}$ for $(j,i) \in \Delta$ will be called *m-invariants* of $\Sigma^{[m]}$.

The following result of Kang [21] asserts that m-invariants $a^{[m]j,i+2}$ for $(j,i) \in \Delta$ are complete invariants of homogeneous feedback and, moreover, illustrates their meaning for the homogeneous normal form $\Sigma_{NF}^{[m]}$.

Consider two homogeneous systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$ and let

$$
\{\, a^{[m]j,i+2} : \; (j,i) \in \Delta \,\}
$$

and

$$
\{\, \tilde{a}^{[m]j,i+2} : \; (j,i) \in \Delta \,\}
$$

denote, respectively, their m-invariants. The following theorem was proved by Kang [21].

THEOREM 2. *The m-invariants have the following properties:*

(i) *Two homogeneous systems* $\Sigma^{[m]}$ *and* $\tilde{\Sigma}^{[m]}$ *are equivalent via a homogeneous feedback transformation* Γ^m *if and only if*

$$a^{[m]j,i+2} = \tilde{a}^{[m]j,i+2}$$

for any $(j,i) \in \Delta$.

(ii) *The* m*-invariants* $\bar{a}^{[m]j,i+2}$ *of the homogeneous normal form* $\Sigma_{NF}^{[m]}$*, defined by* (2.7)*, are given by*

$$\bar{a}^{[m]j,i+2}(x) = \frac{\partial^2}{\partial x_{n-i}^2} x_{n-i}^2 P_{j,n-i}^{[m-2]}(x_1, \dots, x_{n-i}) \tag{2.9}$$

for any $(j,i) \in \Delta$.

Our first aim is to find explicitly feedback transformations bringing the homogeneous system $\Sigma^{[m]}$ to its normal form $\Sigma_{NF}^{[m]}$. Define the homogeneous polynomials $\psi_{j,i}^{[m-1]}(\xi)$ by setting $\psi_{j,0}^{[m-1]}(\xi) = \psi_{1,1}^{[m-1]}(\xi) = 0$,

$$\psi_{j,i}^{[m-1]}(\xi) = -CA^{j-1}\left(ad_{A\xi}^{n-i} g^{[m-1]} + \sum_{t=1}^{n-i} (-1)^t ad_{A\xi}^{t-1} ad_{A^{n-i-t}B} f^{[m]} \right) \tag{2.10}$$

if $1 \le j < i \le n$ and

$$\begin{aligned} \psi_{j,i}^{[m-1]}(\xi) = {} & L_{A^{n-i}B} f_{j-1}^{[m]}(\pi_i(\xi)) + L_{A\xi} \psi_{j-1,i}^{[m-1]}(\pi_i(\xi)) \\ & + \psi_{j-1,i-1}^{[m-1]}(\pi_{i-1}(\xi)) + \int_0^{\xi_i} L_{A^{n-i+1}B} \psi_{j-1,i}^{[m-1]}(\pi_i(\xi)) d\xi_i \end{aligned} \tag{2.11}$$

if $1 \le i \le j$, where $\psi_{j,i}^{[m-1]}(\pi_i(\xi))$ is the restriction of $\psi_{j,i}^{[m-1]}(\xi)$ to the submanifold W_i. Define the components $\phi_j^{[m]}$ of $\phi^{[m]}$ for $1 \le j \le n$ and the feedback $(\alpha^{[m]}, \beta^{[m-1]})$ by

$$\begin{aligned} \phi_j^{[m]}(\xi) &= \sum_{i=1}^{n} \int_0^{\xi_i} \psi_{j,i}^{[m-1]}(\pi_i(\xi)) d\xi_i, \qquad 1 \le j \le n-1, \\ \phi_n^{[m]}(\xi) &= f_{n-1}^{[m]}(\xi) + L_{A\xi} \phi_{n-1}^{[m]}(\xi), \\ \alpha^{[m]}(\xi) &= -\left(f_n^{[m]}(\xi) + L_{A\xi} \phi_n^{[m]}(\xi) \right), \\ \beta^{[m-1]}(\xi) &= -\left(g_n^{[m-1]}(\xi) + L_B \phi_n^{[m]}(\xi) \right). \end{aligned} \tag{2.12}$$

We have the following result.

THEOREM 3. *The homogeneous feedback transformation*

$$\Gamma^m \ : \quad \begin{aligned} x &= \xi + \phi^{[m]}(\xi), \\ u &= v + \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi) v, \end{aligned}$$

where $\alpha^{[m]}$, $\beta^{[m-1]}$, *and the components* $\phi_j^{[m]}$ *of* $\phi^{[m]}$ *are defined by* (2.12)*, brings the homogeneous system* $\Sigma^{[m]}$ *into its normal form* $\Sigma_{NF}^{[m]}$ *given by* (2.7).

Proof of Theorem 3. Denote by

$$\tilde{\Sigma}^{[m]} \ : \ \dot{x} = Ax + Bv + \tilde{f}^{[m]}(x) + \tilde{g}^{[m-1]}(x) v$$

the system $\Sigma^{[m]}$ transformed via the feedback transformation Γ^m defined by (2.12).

From the expressions of $\alpha^{[m]}(\xi)$ and $\beta^{[m-1]}(\xi)$ given by (2.12) and the last two equations of (2.6), we get

$$\tilde{f}_n^{[m]}(x) = 0 \qquad \text{and} \qquad \tilde{g}_n^{[m-1]}(x) = 0.$$

Plugging $\phi_j^{[m]}$, defined by (2.12), into the second equation of (2.6) gives

$$\psi_{j,n}^{[m-1]}(x) = \tilde{g}_j^{[m-1]}(x) - g_j^{[m-1]}(x),$$

which, by (2.10), implies $\tilde{g}_j^{[m-1]}(x) = 0$ for $1 \le j \le n-1$. Now we consider the first equation of (2.6). From the expression of $\phi_n^{[m]}$ we get $\tilde{f}_{n-1}^{[m]}(x) = 0$, and for any $1 \le i \le n$, we obtain by differentiating

$$\frac{\partial \tilde{f}_j^{[m]}}{\partial x_i} = \frac{\partial f_j^{[m]}}{\partial x_i} + L_{Ax}\frac{\partial \phi_j^{[m]}}{\partial x_i} + \frac{\partial \phi_j^{[m]}}{\partial x_{i-1}} - \frac{\partial \phi_{j+1}^{[m]}}{\partial x_i}. \tag{2.13}$$

In the above formula, the term $\frac{\partial \phi_j^{[m]}}{\partial x_{i-1}}$ is not present in the case $i = 1$.

If $i \ge j+1$, we get

$$\begin{aligned}\frac{\partial \tilde{f}_j^{[m]}}{\partial x_i}(\pi_{i-1}(x)) &= \left(\frac{\partial f_j^{[m]}}{\partial x_i} + L_{Ax}\frac{\partial \phi_j^{[m]}}{\partial x_i} + \frac{\partial \phi_j^{[m]}}{\partial x_{i-1}} - \frac{\partial \phi_{j+1}^{[m]}}{\partial x_i}\right)(\pi_{i-1}(x)) \\ &= \frac{\partial f_j^{[m]}}{\partial x_i}(\pi_{i-1}(x)) + L_{Ax}\psi_{j,i}^{[m-1]}(\pi_{i-1}(x)) \\ &\quad + \psi_{j,i-1}^{[m-1]}(\pi_{i-1}(x)) - \psi_{j+1,i}^{[m-1]}(\pi_{i-1}(x)).\end{aligned}$$

Hence, by an induction argument, we obtain

$$\frac{\partial f_j^{[m]}}{\partial x_i}(\pi_{i-1}(x)) + L_{Ax}\psi_{j,i}^{[m-1]}(\pi_{i-1}(x)) + \psi_{j,i-1}^{[m-1]}(\pi_{i-1}(x)) - \psi_{j+1,i}^{[m-1]}(\pi_{i-1}(x)) = 0$$

and, finally, we get

$$\frac{\partial \tilde{f}_j^{[m]}}{\partial x_i}(\pi_{i-1}(x)) = 0. \tag{2.14}$$

If $1 \le i \le j$, then, using (2.12) and (2.13), we obtain

$$\begin{aligned}\frac{\partial \tilde{f}_j^{[m]}}{\partial x_i}(\pi_i(x)) &= \left(\frac{\partial f_j^{[m]}}{\partial x_i} + L_{Ax}\frac{\partial \phi_j^{[m]}}{\partial x_i} + \frac{\partial \phi_j^{[m]}}{\partial x_{i-1}} - \frac{\partial \phi_{j+1}^{[m]}}{\partial x_i}\right)(\pi_i(x)) \\ &= \frac{\partial f_j^{[m]}}{\partial x_i}(\pi_i(x)) + L_{Ax}\psi_{j,i}^{[m-1]}(\pi_i(x)) + \psi_{j,i-1}^{[m-1]}(\pi_{i-1}(x)) \\ &\quad + \int_0^{x_i} \frac{\partial \psi_{j,i}^{[m-1]}(\pi_i(x))}{\partial x_{i-1}} dx_i - \psi_{j+1,i}^{[m-1]}(\pi_i(x)).\end{aligned}$$

Using the expression (2.11), it follows that

$$\frac{\partial \tilde{f}_j^{[m]}}{\partial x_i}(\pi_i(x)) = 0. \tag{2.15}$$

From the relations (2.14) and (2.15), we conclude that

$$\tilde{f}_j^{[m]}(x) = \sum_{i=j+2}^{n} x_i^2 P_{j,i}^{[m-2]}(x_1, \dots, x_i),$$

which proves that $\tilde{\Sigma}^{[m]}$ is a normal form satisfying (2.7). Thus the system $\Sigma^{[m]}$ given by (2.4) is feedback equivalent to the normal form $\Sigma_{NF}^{[m]}$ given by (2.7). □

Example 1. To illustrate results of this section, we consider the system $\Sigma^{[m]}$, given by (2.4) on $\mathbb{R}^3$. Theorem 1 implies that the system $\Sigma^{[m]}$ is equivalent, via a homogeneous feedback transformation Γ^m defined by (2.12), to its normal form $\Sigma_{NF}^{[m]}$ (see (2.7))

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^2 P^{[m-2]}(x_1, x_2, x_3), \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= v, \end{aligned}$$

where $P^{[m-2]}(x_1, x_2, x_3)$ is a homogeneous polynomial of degree $m-2$ of the variables x_1, x_2, x_3. □

We would like now to discuss the interest of Theorem 3. As we have already mentioned, Poincaré's method allows us to replace a partial differential equation by solving successively linear algebraic equations defined by the homological equation (2.6); see [26] and [21], and Proposition 1. The solvability of this equation was proved in [26] and [21], while Theorem 3 provides an explicit solution (in the form of the transformations (2.12), which are easily computable via differentiation and integration of homogeneous polynomials) to the homological equation. As a consequence, for any given control system, Theorem 3 gives transformations bringing the homogeneous part of the system to its normal form. For example, if the system is feedback linearizable, up to order $m_0 - 1$ (see [27]), then a diffeomorphism and a feedback compensating all nonlinearities of degree lower than m_0 can be calculated explicitly without solving partial differential equations. More generally, by a successive application of transformations given by (2.12) we can bring the system, without solving partial differential equations, to its normal form given in Theorem 4 below.

Consider the system Σ^∞ of the form (2.1) and recall that we assume the linear part (F, G) to be controllable. Apply successively to Σ^∞ a series of transformations Γ^m, $m = 1, 2, \dots$, such that each Γ^m brings $\Sigma^{[m]}$ to its normal form $\Sigma_{NF}^{[m]}$; for instance we can take a series of transformations defined by (2.12). Successive repeating of Theorem 1 gives the following result of Kang [21].

THEOREM 4. *There exists a formal feedback transformation* Γ^∞ *which brings the* *system Σ^∞ to a normal form Σ^∞_{NF} given by*

$$
(2.16) \qquad \Sigma^\infty_{NF} \; : \; \begin{cases} \dot x_1 = x_2 + \sum_{i=3}^{n} x_i^2 P_{1,i}(x_1, \dots, x_i), \\ \quad \vdots \\ \dot x_j = x_{j+1} + \sum_{i=j+2}^{n} x_i^2 P_{j,i}(x_1, \dots, x_i), \\ \quad \vdots \\ \dot x_{n-2} = x_{n-1} + x_n^2 P_{n-2,n}(x_1, \dots, x_n), \\ \dot x_{n-1} = x_n, \\ \dot x_n = v, \end{cases}
$$

where $P_{j,i}(x_1, \dots, x_i)$ are formal power series depending on the indicated variables.

Example 2. Consider a system Σ defined on $\mathbb{R}^3$ whose linear part is controllable. Theorem 4 implies that the system Σ is equivalent, via a formal feedback transformation Γ^∞, to its normal form Σ^∞_{NF}

$$
\begin{aligned} \dot x_1 &= x_2 + x_3^2 P(x_1, x_2, x_3), \\ \dot x_2 &= x_3, \\ \dot x_3 &= v, \end{aligned}
$$

where $P(x_1, x_2, x_3)$ is a formal power series of the variables x_1, x_2, x_3. $\square$

3. Canonical form. As proved by Kang and recalled in Theorem 2, the normal form $\Sigma^{[m]}_{NF}$ is unique under homogeneous feedback transformation Γ^m. The normal form Σ^∞_{NF} is constructed by a successive application of homogeneous transformations Γ^m for $m \geq 1$ which bring the corresponding homogeneous systems $\Sigma^{[m]}$ into their normal forms $\Sigma^{[m]}_{NF}$. Therefore a natural and fundamental question which arises is whether the system Σ^∞ can admit two different normal forms, that is, whether the normal forms given by Theorem 4 are in fact canonical forms under a general formal feedback transformations of the form Γ^∞. It turns out that a given system can admit different normal forms, as the following example of Kang [21] shows. The main reason for the nonuniqueness of the normal form Σ^∞_{NF} is that, although the normal form $\Sigma^{[m]}_{NF}$ is unique, homogeneous feedback transformation Γ^m bringing $\Sigma^{[m]}$ into $\Sigma^{[m]}_{NF}$ is not. It is this small group of homogeneous feedback transformations of order m that preserve $\Sigma^{[m]}_{NF}$ (described by Proposition 2 below), which causes the nonuniqueness of Σ^∞_{NF}.

The aim of this section is thus to construct a canonical form for Σ^∞ under feedback transformation Γ^∞.

Example 3. Consider the system

$$
(3.1) \qquad \begin{aligned} \dot\xi_1 &= \xi_2 + \xi_3^2 - 2\xi_1\xi_3^2, \\ \dot\xi_2 &= \xi_3, \\ \dot\xi_3 &= u \end{aligned}
$$

on $\mathbb{R}^3$. Clearly, this system is in Kang normal form (compare with Theorem 4). The feedback transformation

$$\Gamma^{\leq 3} : \begin{aligned} x_1 &= \xi_1 - \xi_1^2 - \tfrac{4}{3}\xi_2^3, \\ x_2 &= \xi_2 - 2\xi_1\xi_2, \\ x_3 &= \xi_3 - 2(\xi_2^2 + \xi_1\xi_3) - 2\xi_2\xi_3^2, \\ u &= v + 6\xi_2\xi_3 + 12\xi_1\xi_2\xi_3 - 4\xi_3^3 + 2(\xi_1 + 2\xi_1^2 + 2\xi_2\xi_3)v \end{aligned}$$

brings the system (3.1) into the form

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= v \end{aligned}$$

modulo terms in $R^{\geq 4}(x, v)$. Applying successively homogeneous feedback transformations Γ^m given, for any $m \geq 4$, by (2.12), we transform the above system into the normal form

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^2 + x_3^2 P(x), \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= v, \end{aligned} \tag{3.2}$$

where P is a formal power series whose 1-jet at $0 \in \mathbb{R}^3$ vanishes. The systems (3.1) and (3.2) are in their normal forms and, moreover, feedback equivalent, but the latter system does not contain any term of degree 3. As a consequence, the normal form Σ_{NF}^{∞} is not unique under formal feedback transformations. $\square$

Consider the system Σ^{∞} of the form (2.1). Since its linear part (F, G) is assumed to be controllable, we bring it, via a linear transformation and linear feedback, to the Brunovský canonical form (A, B). Let the first homogeneous term of Σ^{∞} which cannot be annihilated by a feedback transformation be of degree m_0. As proved by Krener [27], the degree m_0 is given by the largest integer p such that all distributions $\mathcal{D}^k = \operatorname{span}\{g, \ldots, ad_f^{k-1} g\}$ for $1 \leq k \leq n-1$ are involutive modulo terms of order $p - 2$. We can thus, due to Theorems 1 and 2, assume that, after applying a suitable feedback, Σ^{∞} takes the form

$$\dot{\xi} = A\xi + Bu + \bar{f}^{[m_0]}(\xi) + \sum_{m=m_0+1}^{\infty} \left(f^{[m]}(\xi) + g^{[m-1]}(\xi)u \right),$$

where (A, B) is in Brunovský canonical form and the first nonvanishing homogeneous vector field $\bar{f}^{[m_0]}$ is of the form

$$\bar{f}_j^{[m_0]}(\xi) = \begin{cases} \displaystyle\sum_{i=j+2}^{n} \xi_i^2 P_{j,i}^{[m_0-2]}(\xi_1, \ldots, \xi_i), & 1 \leq j \leq n-2, \\ 0, & n-1 \leq j \leq n. \end{cases}$$

Let $(i_1, \ldots, i_{n-s})$, where $i_1 + \cdots + i_{n-s} = m_0$ and $i_{n-s} \geq 2$, be the largest, in the lexicographic ordering, $(n-s)$-tuple of nonnegative integers such that for some $1 \leq j \leq n-2$, we have

$$\frac{\partial^{m_0} \bar{f}_j^{[m_0]}}{\partial \xi_1^{i_1} \cdots \partial \xi_{n-s}^{i_{n-s}}} \neq 0.$$

Define

$$j^* = \sup \left\{ j = 1, \dots, n-2 \ : \ \frac{\partial^{m_0} \bar{f}_j^{[m_0]}}{\partial \xi_1^{i_1} \cdots \partial \xi_{n-s}^{i_{n-s}}} \neq 0 \right\}.$$

We have the following result.

THEOREM 5. *The system* Σ^∞ *given by* (2.1) *is equivalent by a formal feedback* Γ^∞ *to a system of the form*

$$\Sigma_{CF}^\infty \ : \ \dot{x} = Ax + Bv + \sum_{m=m_0}^{\infty} \bar{f}^{[m]}(x), \tag{3.3}$$

where, for any $m \geq m_0$,

$$\bar{f}_j^{[m]}(x) = \begin{cases} \displaystyle\sum_{i=j+2}^{n} x_i^2 P_{j,i}^{[m-2]}(x_1, \dots, x_i), & 1 \leq j \leq n-2, \\ 0, & n-1 \leq j \leq n; \end{cases} \tag{3.4}$$

additionally, we have

$$\frac{\partial^{m_0} \bar{f}_{j^*}^{[m_0]}}{\partial x_1^{i_1} \cdots \partial x_{n-s}^{i_{n-s}}} = \pm 1 \tag{3.5}$$

and, moreover, for any $m \geq m_0 + 1$,

$$\frac{\partial^{m_0} \bar{f}_{j^*}^{[m]}}{\partial x_1^{i_1} \cdots \partial x_{n-s}^{i_{n-s}}}(x_1, 0, \dots, 0) = 0. \tag{3.6}$$

The form Σ_{CF}^∞ satisfying (3.4), (3.5), and (3.6) will be called the *canonical form* of Σ^∞. The name is justified by the following.

THEOREM 6. *Two systems* Σ_1^∞ *and* Σ_2^∞ *are formally feedback equivalent if and only if their canonical forms* $\Sigma_{1,CF}^\infty$ *and* $\Sigma_{2,CF}^\infty$ *coincide.*

Proofs of Theorems 5 and 6 are given in section 4.

Kang [21], generalizing [26], proved that any system Σ^∞ can be brought by a formal feedback into the normal form (3.3) for which (3.4) is satisfied. He also observed that his normal forms are not unique; see Example 3. Our results, Theorems 5 and 6, complete his study. We show that for each degree m of homogeneity we can use a one-dimensional subgroup of feedback transformations which preserves the "triangular" structure of (3.4) and at the same time allows us to normalize one higher order term. The form of (3.5) and (3.6) is a result of this normalization. These one-dimensional subgroups of feedback transformations are given by the following proposition.

PROPOSITION 2. *The transformation* Γ^m *given by* (2.3) *leaves invariant the system* $\Sigma^{[m]}$ *defined by* (2.4) *if and only if*

$$\begin{aligned} \phi_j^{[m]} &= a_m L_{A\xi}^{j-1} \xi_1^m, \qquad 1 \leq j \leq n, \\ \alpha^{[m]} &= -a_m L_{A\xi}^{n} \xi_1^m, \\ \beta^{[m-1]} &= -a_m L_B L_{A\xi}^{n-1} \xi_1^m, \end{aligned} \tag{3.7}$$

where a_m is an arbitrary real parameter.

Proof of Proposition 2. Observe that, following Proposition 1, the transformation Γ^m leaves invariant the system $\Sigma^{[m]}$ if and only if it satisfies the following system of equations:

$$\begin{cases} L_{A\xi}\phi_j^{[m]} - \phi_{j+1}^{[m]}(\xi) = 0, & 1 \le j \le n-1, \\ L_B\phi_j^{[m]} = 0, & 1 \le j \le n-1, \\ L_{A\xi}\phi_n^{[m]} + \alpha^{[m]}(\xi) = 0, & \\ L_B\phi_n^{[m]} + \beta^{[m-1]}(\xi) = 0. & \end{cases}$$

In order to solve the above system, let us remark, using the second equation of the system, that for any j such that $1 \le j \le n-1$, the component $\phi_j^{[m]}$ does not depend to the variable ξ_n. Putting $j = n-2$ into the first equation, we get

$$\frac{\partial\phi_{n-2}^{[m]}}{\partial\xi_1}\xi_2 + \cdots + \frac{\partial\phi_{n-2}^{[m]}}{\partial\xi_{n-1}}\xi_n = \phi_{n-1}^{[m]}.$$

Since $\phi_{n-1}^{[m]}$ and $\phi_{n-2}^{[m]}$ do not depend on the variable ξ_n, we conclude that $\phi_{n-2}^{[m]}$ does not depend on the variable ξ_{n-1}. An inductive argument shows that $\phi_1^{[m]}$ depends only on the variable ξ_1, that is, $\phi_1^{[m]}(\xi) = a_m\xi_1^m$. Now, all equations of (3.7) follow easily. $\square$

Theorem 5 establishes an equivalence of the system Σ^∞ with its canonical form Σ_{CF}^∞ via a formal feedback. Its direct corollary yields the following result for equivalence under a smooth feedback of the form

$$\Gamma \ : \quad \begin{array}{l} x = \phi(\xi), \\ u = \alpha(\xi) + \beta(\xi)v, \end{array}$$

up to an arbitrary order.

COROLLARY 1. *Consider a smooth control system*

$$\Sigma \ : \ \dot{\xi} = f(\xi) + g(\xi)u.$$

For any positive integer k we have the following:

(i) *There exists a smooth feedback Γ transforming Σ, locally around $0 \in \mathbb{R}^n$, into its canonical form $\Sigma_{CF}^{\le k}$ given by*

$$\Sigma_{CF}^{\le k} \ : \ \dot{x} = Ax + Bv + \sum_{m=m_0}^{k} \bar{f}^{[m]}(x),$$

modulo $O(x,v)^{k+1}$, where $\bar{f}^{[m]}(x)$, for any $m_0 \le m \le k$, satisfies (3.4), (3.5), (3.6).

(ii) *Feedback equivalence of Σ and $\Sigma_{CF}^{\le k}$, modulo $O(x,v)^{k+1}$, can be established via a polynomial feedback transformation $\Gamma^{\le k}$ of degree k.*

(iii) *Two smooth systems Σ_1 and Σ_2 are feedback equivalent modulo terms of order $O(x,v)^{k+1}$ if and only if their canonical forms $\Sigma_{1,CF}^{\le k}$ and $\Sigma_{2,CF}^{\le k}$ coincide.*

This corollary follows directly from Theorem 5 and its proof, given in section 4, which provides explicit polynomial transformations (4.4)–(4.5) bringing, step by step, the system into its canonical form.

We will illustrate results of this section by two examples.

Example 4. Let us reconsider the system Σ given by Example 2. It is feedback equivalent to the normal form

$$\begin{aligned}\dot{x}_1 &= x_2 + x_3^2 P(x_1, x_2, x_3),\\ \dot{x}_2 &= x_3,\\ \dot{x}_3 &= v,\end{aligned}$$

where $P(x_1, x_2, x_3)$ is a formal power series. Assume, for simplicity, that $m_0 = 2$, which is equivalent to the following generic condition: g, $ad_f g$, and $[g, ad_f g]$ are linearly independent at $0 \in \mathbb{R}^3$. This implies that we can express $P = P(x_1, x_2, x_3)$ as

$$P = c + P_1(x_1) + x_2 P_2(x_1, x_2) + x_3 P_3(x_1, x_2, x_3),$$

where $c \neq 0$ and $P_1(0) = 0$. Observe that any $P(x_1, x_2, x_3)$, of the above form, gives a normal form Σ_{NF}^{∞}. In order to get the canonical form Σ_{CF}^{∞}, we use Theorem 5, which ensures the existence of a feedback transformation Γ^{∞} of the form

$$\begin{aligned}\tilde{x} &= \phi(x),\\ v &= \alpha(x) + \beta(x)\tilde{v},\end{aligned}$$

which normalizes the constant c and annihilates the formal power series $P_1(x_1)$. More precisely, Γ^{∞} transforms Σ into its canonical form Σ_{CF}^{∞},

$$\begin{aligned}\dot{\tilde{x}}_1 &= \tilde{x}_2 + \tilde{x}_3^2 \tilde{P}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3),\\ \dot{\tilde{x}}_2 &= \tilde{x}_3,\\ \dot{\tilde{x}}_3 &= \tilde{v},\end{aligned}$$

where the formal power series $\tilde{P}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is of the form

$$\tilde{P}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = 1 + \tilde{x}_2 \tilde{P}_2(\tilde{x}_1, \tilde{x}_2) + \tilde{x}_3 \tilde{P}_3(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3).$$

Now, we give an example of constructing the canonical form for a physical model of variable length pendulum. □

Example 5. Consider the variable length pendulum of Bressan and Rampazzo [5] (see also [2] and [8]). We denote by ξ_1 the length of the pendulum, by ξ_2 its velocity, by ξ_3 the angle with respect to the horizontal, and by ξ_4 the angular velocity. The control $u = \dot{\xi}_4 = \ddot{\xi}_3$ is the angular acceleration. The mass is normalized to 1. The equations are (compare [5] and [8])

$$\begin{aligned}\dot{\xi}_1 &= \xi_2,\\ \dot{\xi}_2 &= -g \sin \xi_3 + \xi_1 \xi_4^2,\\ \dot{\xi}_3 &= \xi_4,\\ \dot{\xi}_4 &= u,\end{aligned}$$

where g denotes the gravity. Notice that if we suppose to control the angular velocity $\xi_4 = \dot{\xi}_3$, which is the case of [5] and [8], then the system is three-dimensional but the control enters nonlinearly.

At any equilibrium point $\xi_0 = (\xi_{10}, \xi_{20}, \xi_{30}, \xi_{40})^T = (\xi_{10}, 0, 0, 0)^T$, the linear part of the system is controllable. Our goal is to produce, for the variable length pendulum, a normal form and the canonical form as well as to answer the question whether the systems corresponding to various values of the gravity constant g are feedback equivalent. In order to get a normal form, put

$$
\begin{aligned}
x_1 &= \xi_1, \\
x_2 &= \xi_2, \\
x_3 &= -g\sin\xi_3, \\
x_4 &= -g\xi_4\cos\xi_3, \\
v &= g\xi_4^2\sin\xi_3 - ug\cos\xi_3.
\end{aligned}
$$

The system becomes

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3 + x_4^2\frac{x_1}{g^2 - x_3^2}, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= v,
\end{aligned}
$$

which gives a normal form. Indeed, we rediscover Σ_{NF}^{∞}, given by (2.16), with $P_{1,3} = 0$, $P_{1,4} = 0$, and

$$
P_{2,4} = \frac{x_1}{g^2 - x_3^2}.
$$

In order to bring the system to its canonical form Σ_{CF}^{∞}, first observe that $m_0 = 3$. Indeed, the function $x_4^2\frac{x_1}{g^2-x_3^2}$ starts with third order terms, which corresponds to the fact that the invariants $a^{[2]j,i+2}$ vanish for any $1 \le j \le 2$ and any $0 \le i \le 2 - j$. The only nonzero component of $f^{[3]}$ is $f_2^{[3]} = x_4^2 P_{2,4}^{[1]}$. Hence $j^* = 2$ and the only, and thus largest, quadruplet (i_1, i_2, i_3, i_4) of nonnegative integers, satisfying $i_1+i_2+i_3+i_4 = 3$ and such that

$$
\frac{\partial^3 f_2^{[3]}}{\partial x_1^{i_1}\cdots\partial x_4^{i_4}} \neq 0,
$$

is $(i_1, i_2, i_3, i_4) = (1, 0, 0, 2)$. In order to normalize $f_2^{[3]}$, put

$$
\begin{aligned}
\tilde{x}_i &= a_1 x_i, \quad 1 \le i \le 4, \\
\tilde{v} &= a_1 v,
\end{aligned}
$$

where $a_1 = 1/g$. We get the following canonical form for the variable length pendulum:

$$
\begin{aligned}
\dot{\tilde{x}}_1 &= \tilde{x}_2, \\
\dot{\tilde{x}}_2 &= \tilde{x}_3 + \tilde{x}_4^2\frac{\tilde{x}_1}{1 - \tilde{x}_3^2}, \\
\dot{\tilde{x}}_3 &= \tilde{x}_4, \\
\dot{\tilde{x}}_4 &= \tilde{v}.
\end{aligned}
$$

Independently of the value of the gravity constant g, all systems are feedback equivalent to each other. $\square$

4. Proofs of Theorems 5 and 6.

Proof of Theorem 5. The proof of this theorem will be done in two steps. In the first step we will deal with terms of degree m_0. Then we will prove the general step by an induction argument.

First step. Let us consider the system Σ^∞ given by (2.1) and let m_0 be the degree of the first nonlinearizable homogeneous part. We can assume that (see Theorems 1 and 2), after applying a suitable feedback transformation, the system Σ^∞ given by (2.1) takes the form

$$\dot{\xi} = A\xi + Bu + \bar{f}^{[m_0]}(\xi) + \sum_{m=m_0+1}^{\infty} \left(f^{[m]}(\xi) + g^{[m-1]}(\xi)u \right), \tag{4.1}$$

where (A, B) is in Brunovský canonical form and the first nonvanishing vector field $\bar{f}^{[m_0]}$ is of the form

$$\bar{f}_j^{[m_0]}(\xi) = \begin{cases} \displaystyle\sum_{i=j+2}^{n} \xi_i^2 P_{j,i}^{[m_0-2]}(\xi_1, \ldots, \xi_i), & 1 \le j \le n-2, \\ 0, & n-1 \le j \le n. \end{cases}$$

Notice that the linear feedback transformation

$$\Gamma^1 \ : \quad \begin{aligned} x &= a_1 \xi, \\ u &= \tfrac{1}{a_1} v, \end{aligned}$$

where $a_1 \in \mathbb{R}$ and $a_1 \neq 0$, brings the system (4.1) into the following one:

$$\dot{x} = Ax + Bv + \frac{1}{a_1^{m_0-1}} \bar{f}^{[m_0]}(x) + \sum_{m=m_0+1}^{\infty} \left(\tilde{f}^{[m]}(x) + \tilde{g}^{[m-1]}(x)v \right).$$

By the definitions of $(i_1, \ldots, i_{n-s})$ and j^*, we have

$$\frac{\partial^{m_0} \bar{f}_{j^*}^{[m_0]}}{\partial x_1^{i_1} \cdots \partial x_{n-s}^{i_{n-s}}} \neq 0,$$

and thus we can suitably choose the parameter a_1 such that

$$\frac{\partial^{m_0} \bar{f}_{j^*}^{[m_0]}}{\partial x_1^{i_1} \cdots \partial x_{n-s}^{i_{n-s}}} = \pm 1.$$

General step. Now, we assume that, for some $l \ge 1$, the system Σ^∞ given by (2.1), takes the form

$$\Sigma^\infty \ : \ \dot{\xi} = A\xi + Bu + \sum_{m=m_0}^{m_0+l-1} \bar{f}^{[m]}(\xi) + f^{[m_0+l]}(\xi) + g^{[m_0+l-1]}(\xi)u + r(\xi, u), \tag{4.2}$$

where $r(\xi, u) \in R^{\ge m_0+l+1}(\xi, u)$ and the vector fields $\bar{f}^{[m]}(\xi)$ for any m such that $m_0 \le m \le m_0 + l - 1$ satisfy the conditions (3.4), (3.5), and (3.6). We will construct a transformation Γ^∞ which preserves all terms of degree smaller than $m_0 + l$ while taking those of degree $m_0 + l$ into the canonical form defined by (3.4) and (3.6).

Consider the following feedback transformation

$$
(4.3) \qquad \Gamma^\infty \; : \; \begin{aligned} x &= \xi + \sum_{m=l+1}^{\infty} \phi^{[m]}(\xi), \\ u &= v + \sum_{m=l+1}^{\infty} \left(\alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v\right), \end{aligned}
$$

where, for any m such that $m_0 \le m \le m_0 + l - 1$, the triplet $(\phi^{[m]}, \alpha^{[m]}, \beta^{[m-1]})$ is given by (3.7) and $\phi^{[m]} = 0$, $\alpha^{[m]} = 0$, and $\beta^{[m-1]} = 0$ for $m \ge m_0 + l + 1$.

The transformation Γ^∞ is actually a polynomial transformation $\Gamma^{\le m_0+l}$ and can be viewed as a composition of a transformation $\Gamma^{\le m_0+l-1}$ and a homogeneous transformation Γ^{m_0+l} defined, respectively, by

$$
(4.4) \qquad \Gamma^{\le m_0+l-1} \; : \; \begin{aligned} y &= \xi + \sum_{m=l+1}^{m_0+l-1} \phi^{[m]}(\xi), \\ u &= w + \sum_{m=l+1}^{m_0+l-1} \left(\alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)w\right) \end{aligned}
$$

and

$$
(4.5) \qquad \Gamma^{m_0+l} \; : \; \begin{aligned} x &= y + \phi^{[m_0+l]}(y), \\ w &= v + \alpha^{[m_0+l]}(y) + \beta^{[m_0+l-1]}(y)v. \end{aligned}
$$

Let us denote by $\tilde{\Sigma}^\infty$ the system Σ^∞, given by (4.2), transformed via $\Gamma^{\le m_0+l-1}$. Since

$$
\bar{f}^{[m_0]}(\xi) = \bar{f}^{[m_0]}(y - \phi^{[l+1]}(y) - \cdots) = \bar{f}^{[m_0]}(y) - \frac{\partial \bar{f}^{[m_0]}}{\partial y}\phi^{[l+1]}(y) + r_1(y),
$$

where $r_1(y) \in R^{\ge m_0+l+1}(y)$ and for any $m \ge m_0 + 1$,

$$
\bar{f}^{[m]}(\xi) = \bar{f}^{[m]}(y - \phi^{[l+1]}(y) - \cdots) = \bar{f}^{[m]}(y) + r_2(y),
$$

where $r_2(y) \in R^{\ge m_0+l+1}(y)$, we get

(4.6)

$$
\tilde{\Sigma}^\infty \; : \; \dot{y} = Ay + Bw + \sum_{m=m_0}^{m_0+l-1} \bar{f}^{[m]}(y) + \tilde{f}^{[m_0+l]}(y) + \tilde{g}^{[m_0+l-1]}(y)w + r_3(y,w),
$$

where $r_3(y,w) \in R^{\ge m_0+l+1}(y,w)$ and

$$
\begin{aligned} \tilde{f}^{[m_0+l]} &= f^{[m_0+l]} + [\bar{f}^{[m_0]}, \phi^{[l+1]}], \\ \tilde{g}^{[m_0+l-1]} &= g^{[m_0+l-1]}. \end{aligned}
$$

Let

$$
\left\{ a^{[m_0+l]j,i+2} \; : \; (j,i) \in \Delta \right\}
$$

and

$$\left\{\tilde{a}^{[m_0+l]j,i+2} \ : \ (j,i) \in \Delta\right\}$$

denote, respectively, the sets of (m_0+l)-invariants associated with the homogeneous parts of degree m_0+l of the systems (4.2) and (4.6). We have

$$\tilde{a}^{[m_0+l]j,i+2} = a^{[m_0+l]j,i+2} + \hat{a}^{[m_0+l]j,i+2}, \tag{4.7}$$

where

$$\begin{aligned}\hat{a}^{[m_0+l]j,i+2} = CA^{j-1} &\left[\sum_{k=0}^{i}(-1)^{i+k}\left(ad_{A^iB}ad_{A\xi}^{i-k}ad_{A^kB}\left[\bar{f}^{[m_0]},\phi^{[l+1]}\right]\right)(\pi_{n-i}(\xi))\right.\\ &\left.+\ \sum_{k=0}^{i-1}(-1)^{i+k}\left(ad_{A^{i+1}B}ad_{A\xi}^{i-k-1}ad_{A^kB}\left[\bar{f}^{[m_0]},\phi^{[l+1]}\right]\right)(\pi_{n-i}(\xi))\right].\end{aligned}$$

Since the identity

$$ad_{A^{n-1}B}^{k}ad_{A\xi}^{i}h = ad_{A\xi}^{i}ad_{A^{n-1}B}^{k}h$$

holds for any vector field h and any $k, i \geq 0$, we get by differentiating

$$\begin{aligned}&L_{A^{n-1}B}^{i_1+l}\hat{a}^{[m_0+l]j,i+2}\\ &= CA^{j-1}\left[\sum_{k=0}^{i}(-1)^{i+k}\left(ad_{A^iB}ad_{A\xi}^{i-k}ad_{A^kB}ad_{A^{n-1}B}^{i_1+l}\left[\bar{f}^{[m_0]},\phi^{[l+1]}\right]\right)(\pi_{n-i}(\xi))\right.\\ &\quad\left.+\sum_{k=0}^{i-1}(-1)^{i+k}\left(ad_{A^{i+1}B}ad_{A\xi}^{i-k-1}ad_{A^kB}ad_{A^{n-1}B}^{i_1+l}\left[\bar{f}^{[m_0]},\phi^{[l+1]}\right]\right)(\pi_{n-i}(\xi))\right].\end{aligned} \tag{4.8}$$

Due to the definition of the $(n-s)$-tuple $(i_1, \ldots, i_{n-s})$, we obtain

$$\begin{aligned}ad_{A^{n-1}B}^{i_1+l}\left[\bar{f}^{[m_0]},\phi^{[l+1]}\right] &= c_1\left[ad_{A^{n-1}B}^{i_1}\bar{f}^{[m_0]}, ad_{A^{n-1}B}^{l}\phi^{[l+1]}\right]\\ &\quad + c_2\left[ad_{A^{n-1}B}^{i_1-1}\bar{f}^{[m_0]}, ad_{A^{n-1}B}^{l+1}\phi^{[l+1]}\right],\end{aligned} \tag{4.9}$$

where c_1 and c_2 are strictly positive integers. From the relations

$$\begin{aligned}ad_{A^{n-1}B}^{l}\phi^{[l+1]} &= a_{l+1}(l+1)!(\xi_1, \xi_2, \ldots, \xi_n)^T,\\ ad_{A^{n-1}B}^{l+1}\phi^{[l+1]} &= a_{l+1}(l+1)!(1, 0, \ldots, 0)^T,\end{aligned}$$

one can easily deduce that identity (4.10) can be rewritten as

$$ad_{A^{n-1}B}^{i_1+l}\left[\bar{f}^{[m_0]},\phi^{[l+1]}\right] = \gamma_l ad_{A^{n-1}B}^{i_1}\bar{f}^{[m_0]},$$

where we set $\gamma_l = -a_{l+1}(l+1)!\,(c_1(m_0 - i_1 + 1) + c_2)$. Plugging the above identity into the formula (4.8), we obtain

$$\begin{aligned}&L_{A^{n-1}B}^{i_1+l}\hat{a}^{[m_0+l]j,i+2}\\ &= \gamma_l CA^{j-1}\left[\sum_{k=0}^{i}(-1)^{i+k}\left(ad_{A^{n-1}B}^{i_1}ad_{A^iB}ad_{A\xi}^{i-k}ad_{A^kB}\bar{f}^{[m_0]}\right)(\pi_{n-i}(\xi))\right.\\ &\quad\left.+\sum_{k=0}^{i-1}(-1)^{i+k}\left(ad_{A^{n-1}B}^{i_1}ad_{A^{i+1}B}ad_{A\xi}^{i-k-1}ad_{A^kB}\bar{f}^{[m_0]}\right)(\pi_{n-i}(\xi))\right].\end{aligned}$$

Since $\bar f^{[m_0]}$ is of the form (3.4), we get for any k such that $0 \le k \le i-1$,

$$ad_{A^kB}\bar f^{[m_0]}(\pi_{n-i}(\xi)) = 0$$

and for any $t \ge 0$,

$$\left(ad^t_{A\xi} ad_{A^kB}\bar f^{[m_0]}\right)(\pi_{n-i}(\xi)) = 0.$$

Thus, we can deduce the relation

$$\frac{\partial^{i_1+l}\hat a^{[m_0+l]j,i+2}}{\partial \xi_1^{i_1+l}} = \gamma_l C A^{j-1}\frac{\partial^{i_1+2}\bar f^{[m_0]}}{\partial\xi_1^{i_1}\partial\xi_{n-i}^2}(\pi_{n-i}(\xi)),$$

which leads, after differentiating and setting $j=j^*$ and $i=s$, to the following one:

$$\frac{\partial^{m_0+l-2}\hat a^{[m_0+l]j^*,s+2}}{\partial\xi_1^{i_1+l}\partial\xi_2^{i_2}\cdots\partial\xi_{n-s}^{i_{n-s}-2}} = \gamma_l\frac{\partial^{m_0}\bar f^{[m_0]}_{j^*}}{\partial\xi_1^{i_1}\partial\xi_2^{i_2}\cdots\partial\xi_{n-s}^{i_{n-s}}}.$$

Differentiating (4.7) and using the above identity, we get

$$\frac{\partial^{m_0+l-2}\tilde a^{[m_0+l]j^*,s+2}}{\partial\xi_1^{i_1+l}\partial\xi_2^{i_2}\cdots\partial\xi_{n-s}^{i_{n-s}-2}} = \frac{\partial^{m_0+l-2}a^{[m_0+l]j^*,s+2}}{\partial\xi_1^{i_1+l}\partial\xi_2^{i_2}\cdots\partial\xi_{n-s}^{i_{n-s}-2}} + \gamma_l\frac{\partial^{m_0}\bar f^{[m_0]}_{j^*}}{\partial\xi_1^{i_1}\partial\xi_2^{i_2}\cdots\partial\xi_{n-s}^{i_{n-s}}}.$$

We can choose suitably the parameter a_{l+1} (recall the definition of γ_l) such that we obtain

$$\frac{\partial^{m_0+l-2}\tilde a^{[m_0+l]j^*,s+2}}{\partial\xi_1^{i_1+l}\partial\xi_2^{i_2}\cdots\partial\xi_{n-i}^{i_{n-i}-2}} = 0.$$

Now, transforming the homogeneous part of degree m_0+l of the system (4.6) to its normal form via a homogeneous transformation Γ^{m_0+l} and taking into account Theorem 2, we bring the system (4.6) into the form

$$\Sigma^\infty \ : \ \dot x = Ax + Bv + \sum_{m=m_0}^{m_0+l}\bar f^{[m]}(x) + r(x,v),$$

where $r(x,v) \in R^{\ge m_0+l+1}(x,v)$ and the vector fields $\bar f^{[m]}$, for any m such that $m_0 \le m \le m_0+l$, satisfy the conditions (3.4), (3.5), and (3.6). This completes the proof of Theorem 5. □

In our proof of Theorem 6, we will use the following result.

LEMMA 1. *A transformation Γ^∞ leaves invariant all terms of degree smaller than m_0+l of the system* (4.2) *if and only if Γ^∞ is of the form*

$$\Gamma^\infty \ : \ \begin{aligned} x &= \xi + \sum_{m=l+1}^{\infty}\phi^{[m]}(\xi), \\ u &= v + \sum_{m=l+1}^{\infty}\left(\alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v\right), \end{aligned} \tag{4.10}$$

where, for any m such that $m_0 \le m \le m_0+l-1$, the triplet $(\phi^{[m]},\alpha^{[m]},\beta^{[m-1]})$ is given by (3.7).

Proof of Lemma 1. We have shown, when proving Theorem 5, that the transformation Γ^∞, defined by (4.10) and (3.7), leaves invariant all terms of degree smaller than m_0+l of the system (4.2).

Conversely, assume that the transformation Γ^∞ leaves invariant all terms of degree smaller than m_0+l of the system (4.2). Without loss of generality, we can take

$$\Gamma^\infty \quad : \quad \begin{aligned} x &= \xi + \sum_{m=k+1}^{\infty} \phi^{[m]}(\xi), \\ u &= v + \sum_{m=k+1}^{\infty} (\alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v), \end{aligned}$$

where $k+1$ denotes the smallest degree among degrees of all nonvanishing components $\phi_j^{[m]}$ of the transformation Γ^∞. There is nothing to prove if $k+1 \geq m_0+l$. We thus focus our attention on the case $k+2 \leq m_0+l$. Since Γ^∞ leaves invariant all terms of degree smaller than m_0+l of the system (4.2), in particular it leaves invariant terms of degree $k+1$, which implies that $(\phi^{[k+1]}, \alpha^{[k+1]}, \beta^{[k]})$ satisfies the condition (3.7). By induction, we show that $(\phi^{[m]}, \alpha^{[m]}, \beta^{[m-1]})$ also satisfies the condition (3.7) for any m such that $k+1 \leq m \leq m_0+k-1$. Thus it remains only to prove that $k \geq l$. Assume this is false; that is, suppose $k \leq l-1$. We can see that the transformation Γ^∞ brings the system (4.2) into the following one:

$$\dot{x} = Ax + Bv + \sum_{m=m_0}^{m_0+k-1} \bar{f}^{[m]}(x) + \tilde{f}^{[m_0+k]}(x) + r(x,v), \tag{4.11}$$

where $r(x,v) \in R^{\geq m_0+k+1}(x,v)$ and the vector field $\bar{f}^{[m]}(x)$, for any m such that $m_0 \leq m \leq m_0+k-1$, is of the form (3.4) and (3.6) and

$$\tilde{f}^{[m_0+k]} = \bar{f}^{[m_0+k]} + [\bar{f}^{[m_0]}, \phi^{[k+1]}].$$

Since the transformation Γ^∞ leaves invariant all terms of degree smaller than m_0+l of the system (4.2), in particular it leaves invariant all terms of degree m_0+k, which is equivalent to

$$[\bar{f}^{[m_0]}, \phi^{[k+1]}] = 0.$$

Repeating the calculations done in the proof of Theorem 5 we deduce, by differentiating, the identity

$$\frac{\partial^{m_0+k} CA^{j^*-1}[\bar{f}^{[m_0]}, \phi^{[k+1]}]}{\partial x_1^{i_1+k} \partial x_2^{i_2} \cdots \partial x_{n-s}^{i_{n-s}}} = \gamma_k \frac{\partial^{m_0} \bar{f}_{j^*}^{[m_0]}}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_{n-s}^{i_{n-s}}} = 0.$$

Thus, due to the fact that

$$\frac{\partial^{m_0} \bar{f}_{j^*}^{[m_0]}}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_{n-s}^{i_{n-s}}} \neq 0,$$

we obtain $\gamma_k = 0$ and hence $(\phi^{[k+1]}, \alpha^{[k+1]}, \beta^{[k]}) = 0$, which contradicts the definition of $k+1$. As a conclusion, it follows that the transformation Γ^∞ is of the form (4.10) and (3.7). $\square$

Proof of Theorem 6. Let us consider two systems Σ_1^∞ and Σ_2^∞ and let

$$\Sigma_{1,CF}^\infty \ : \ \dot{x} = Ax + Bv + \sum_{m=m_0}^{\infty} \bar{f}^{[m]}(x)$$

and

$$\Sigma_{2,CF}^\infty \ : \ \dot{z} = Az + Bw + \sum_{m=m_1}^{\infty} \tilde{f}^{[m]}(z)$$

denote, respectively, their canonical forms, where m_0 and m_1 denote the degrees of the first nonlinearizable homogeneous parts. It is obvious that Σ_1^∞ and Σ_2^∞ are feedback equivalent if their canonical forms $\Sigma_{1,CF}^\infty$ and $\Sigma_{2,CF}^\infty$ coincide. To prove the converse, we assume that the systems Σ_1^∞ and Σ_2^∞ are feedback equivalent while their canonical forms fail to coincide. Since Σ_1^∞ and Σ_2^∞ are feedback equivalent, so are their canonical forms $\Sigma_{1,CF}^\infty$ and $\Sigma_{2,CF}^\infty$. It means that there exists a transformation Γ^∞ which brings $\Sigma_{1,CF}^\infty$ into $\Sigma_{2,CF}^\infty$. First remark that, from the definition of the integers m_0 and m_1, we necessarily have $m_0 = m_1$. Then, Theorem 2 and the fact that the components $\bar{f}_{j^*}^{[m_0]}$ and $\tilde{f}_{j^*}^{[m_0]}$ are normalized (see (3.5)) ensure that $\bar{f}^{[m_0]} = \tilde{f}^{[m_1]}$. Let l be the largest integer such that for any $i \le l$, we have $\bar{f}^{[m_0+i-1]} = \tilde{f}^{[m_0+i-1]}$. This means that the transformation Γ^∞ leaves invariant all terms of degree smaller than $m_0 + l$ of the system $\Sigma_{1,CF}^\infty$. Then Lemma 1 shows that the transformation Γ^∞ is of the form (4.10). Since the transformation Γ^∞ brings $\Sigma_{1,CF}^\infty$ into $\Sigma_{2,CF}^\infty$, we deduce that

$$\tilde{f}^{[m_0+l]} = \bar{f}^{[m_0+l]} + [\bar{f}^{[m_0]}, \phi^{[l+1]}]. \tag{4.12}$$

Following arguments in the proof of Theorem 5, we obtain

$$\frac{\partial^{m_0+l-2}\tilde{a}^{[m_0+l]j^*,s+2}}{\partial x_1^{i_1+l}\partial x_2^{i_2}\cdots\partial x_{n-s}^{i_{n-s}-2}} = \frac{\partial^{m_0+l-2}\bar{a}^{[m_0+l]j^*,s+2}}{\partial x_1^{i_1+l}\partial x_2^{i_2}\cdots\partial x_{n-s}^{i_{n-s}-2}} + \gamma_l \frac{\partial^{m_0}\bar{f}_{j^*}^{[m_0]}}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_{n-s}^{i_{n-s}}},$$

where

$$\left\{\bar{a}^{[m_0+l]j,i+2} \ : \ (j,i) \in \Delta\right\}$$

and

$$\left\{\tilde{a}^{[m_0+l]j,i+2} \ : \ (j,i) \in \Delta\right\}$$

denote, respectively, the set of $(m_0 + l)$-invariants associated with the homogeneous parts of degree $m_0 + l$ of the systems $\Sigma_{1,CF}^\infty$ and $\Sigma_{2,CF}^\infty$. Using Theorem 2, the last identity can be rewritten as

$$\frac{\partial^{m_0+l}\tilde{f}_{j^*}^{[m_0+l]}}{\partial x_1^{i_1+l}\partial x_2^{i_2}\cdots\partial x_{n-s}^{i_{n-s}}} = \frac{\partial^{m_0+l}\bar{f}_{j^*}^{[m_0+l]}}{\partial x_1^{i_1+l}\partial x_2^{i_2}\cdots\partial x_{n-s}^{i_{n-s}}} + \gamma_l \frac{\partial^{m_0}\bar{f}_{j^*}^{[m_0]}}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_{n-s}^{i_{n-s}}}. \tag{4.13}$$

Since

$$\frac{\partial^{m_0}\tilde{f}_{j^*}^{[m_0+l]}}{\partial x_1^{i_1}\cdots\partial x_{n-s}^{i_{n-s}}}(x_1, 0, \ldots, 0) = \frac{\partial^{m_0}\bar{f}_{j^*}^{[m_0+l]}}{\partial x_1^{i_1}\cdots\partial x_{n-s}^{i_{n-s}}}(x_1, 0, \ldots, 0) = 0,$$

the identity (4.13) gives

$$\gamma_l \frac{\partial^{m_0} \bar{f}_{j^*}^{[m_0]}}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_{n-s}^{i_{n-s}}} = 0,$$

which implies $\gamma_l = 0$, that is (recall the definition of γ_l), we have $a_{l+1} = 0$, and consequently $(\phi^{[l+1]}, \alpha^{[l+1]}, \beta^{[l]}) = 0$. Then the identity (4.12) reduces to

$$\tilde{f}^{[m_0+l]} = \bar{f}^{[m_0+l]},$$

which contradicts the definition of l. We conclude that the canonical forms $\Sigma_{1,CF}^{\infty}$ and $\Sigma_{2,CF}^{\infty}$ coincide. □

5. Dual normal form and dual m-invariants. In the normal form $\Sigma_{NF}^{[m]}$ given by (2.7), all the components of the control vector field $g^{[m-1]}$ are annihilated and all nonremovable nonlinearities are grouped in $f^{[m]}$. Kang and Krener in their pioneering paper [26] showed that it is possible to transform, via a transformation Γ^2 of degree 2, the homogeneous system

$$\Sigma^{[2]}: \ \dot{\xi} = A\xi + Bu + f^{[2]}(\xi) + g^{[1]}(\xi)u$$

into a dual normal form. In that form the components of the drift $f^{[2]}$ are annihilated, while this time all nonremovable nonlinearities are present in $g^{[1]}$. The aim of this section is to propose, for an arbitrary m, a dual normal form for the system $\Sigma^{[m]}$ and a dual normal form for the system Σ^{∞}. Our dual normal form on the one hand generalizes, for higher order terms, that given in [26] for second order terms, and on the other hand dualizes the normal form $\Sigma_{NF}^{[m]}$. The structure of this section will follow that of section 2: we will give the dual normal form, then define and study dual m-invariants; finally, we give an explicit construction of transformations bringing the system into its dual normal form.

Our first result asserts that we can always bring $\Sigma^{[m]}$ to a dual normal form.

THEOREM 7. *The homogeneous system $\Sigma^{[m]}$ is equivalent, via a homogeneous feedback transformation Γ^m, to the dual normal form $\Sigma_{DNF}^{[m]}$ given by*

$$\Sigma_{DNF}^{[m]} \ : \ \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3 + v x_n Q_{2,n}^{[m-2]}(x_1, \ldots, x_n), \\ \quad \vdots \\ \dot{x}_j = x_{j+1} + v \displaystyle\sum_{i=n-j+2}^{n} x_i Q_{j,i}^{[m-2]}(x_1, \ldots, x_i), \\ \quad \vdots \\ \dot{x}_{n-1} = x_n + v \displaystyle\sum_{i=3}^{n} x_i Q_{j,i}^{[m-2]}(x_1, \ldots, x_i), \\ \dot{x}_n = v, \end{cases} \tag{5.1}$$

where $Q_{j,i}^{[m-2]}(x_1, \ldots, x_i)$ are homogeneous polynomials of degree $m-2$ depending on the indicated variables.

Theorem 7 follows from Theorem 9, which gives explicit transformation bringing $\Sigma^{[m]}$ to its dual normal form $\Sigma_{DNF}^{[m]}$, and thus we omit its proof.

Now we will define dual m-invariants. To start with, recall that the homogeneous vector field $X_i^{[m-1]}$ is defined by taking the homogeneous part of degree $m-1$ of the vector field

$$X_i^{m-1} = (-1)^i ad^i_{A\xi+f^{[m]}}(B + g^{[m-1]}).$$

By $X_i^{[m-1]}(\pi_i(\xi))$ we will denote the vector field $X_i^{[m-1]}$ evaluated at the point $\pi_i(\xi) = (\xi_1, \ldots, \xi_i, 0, \ldots, 0)$ of the submanifold

$$W_i = \{\xi \in \mathbb{R}^n \ : \ \xi_{i+1} = \cdots = \xi_n = 0 \ \}.$$

Consider the system $\Sigma^{[m]}$ and, for any j such that $2 \leq j \leq n-1$, define the polynomial $b_j^{[m-1]}$ by setting

$$b_j^{[m-1]} = g_j^{[m-1]} + \sum_{k=1}^{j-1} L_B L_{A\xi}^{j-k-1} f_k^{[m]} - \sum_{i=1}^{n} L_B L_{A\xi}^{j-1} \int_0^{\xi_i} CX_{n-i}^{[m-1]}(\pi_i(\xi)) d\xi_i.$$

The homogeneous polynomials $b_j^{[m-1]}$ for $2 \leq j \leq n-1$ will be called the *dual m-invariants* of the homogeneous system $\Sigma^{[m]}$.

Consider two systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$ of the form (2.4) and (2.5). Let

$$\{ \, b_j^{[m-1]} \ : \ 2 \leq j \leq n-1 \, \}$$

and

$$\{ \, \tilde{b}_j^{[m-1]} \ : \ 2 \leq j \leq n-1 \, \}$$

denote, respectively, their dual m-invariants. The following result gives a dualization of Theorem 2.

THEOREM 8. *The dual m-invariants have the following properties:*

(i) *Two systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$ are equivalent via a homogeneous feedback transformation Γ^m if and only if*

$$b_j^{[m-1]} = \tilde{b}_j^{[m-1]}$$

for any $2 \leq j \leq n-1$.

(ii) *The dual m-invariants $\bar{b}_j^{[m-1]}$ of the dual normal form $\Sigma_{DNF}^{[m]}$, defined by* (5.1), *are given by*

$$\bar{b}_j^{[m-1]}(x) = \sum_{i=n-j+2}^{n} x_i Q_{j,i}^{[m-2]}(x_1, \ldots, x_i)$$

for any $2 \leq j \leq n-1$.

The above result asserts that the dual m-invariants, as do the m-invariants, form a set of complete invariants of the homogeneous feedback transformation. Notice, however, that the same information is encoded in both sets of invariants in different ways. We will give a proof of Theorem 8 in section 7.

Now, we define the following homogeneous polynomials:

$$
\begin{aligned}
\phi_1^{[m]} &= -\sum_{i=1}^{n} \int_0^{\xi_i} C X_{n-i}^{[m-1]}(\pi_i(\xi)) d\xi_i, \\
\phi_{j+1}^{[m]} &= f_j^{[m]} + L_{A\xi}\phi_j^{[m]}, \qquad 1 \le j \le n-1, \\
\alpha^{[m]} &= -\left(f_n^{[m]} + L_{A\xi}\phi_n^{[m]}\right), \\
\beta^{[m-1]} &= -\left(g_n^{[m-1]} + L_B \phi_n^{[m]}\right).
\end{aligned} \tag{5.2}
$$

The next result gives an explicit construction of feedback transformations bringing the system $\Sigma^{[m]}$ to its dual normal form $\Sigma_{DNF}^{[m]}$.

THEOREM 9. *The feedback transformation*

$$
\Gamma^m \ : \quad \begin{aligned} x &= \xi + \phi^{[m]}(\xi), \\ u &= v + \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v, \end{aligned}
$$

where $\alpha^{[m]}$, $\beta^{[m-1]}$, and the components $\phi_j^{[m]}$ of $\phi^{[m]}$ are defined by (5.2), brings the system $\Sigma^{[m]}$ into its dual normal form $\Sigma_{DNF}^{[m]}$ given by (5.1).

6. Dual canonical form. Consider the system Σ^∞ of the form (2.1) and assume that its linear part (F, G) is controllable. Apply successively to it a series of transformations Γ^m, $m = 1, 2, \dots$, such that each Γ^m brings $\Sigma^{[m]}$ to its dual normal form $\Sigma_{DNF}^{[m]}$; for instance we can take a series of transformations defined by (5.2). Successive repeating of Theorem 9 gives the following dual normal form.

THEOREM 10. *The system Σ^∞ can be transformed via a formal feedback transformation Γ^∞ into the dual normal form Σ_{DNF}^∞ given by*

$$
\Sigma_{DNF}^\infty \ : \ \left\{ \begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3 + v x_n Q_{2,n}(x_1, \dots, x_n), \\
&\ \ \vdots \\
\dot{x}_j &= x_{j+1} + v \sum_{i=n-j+2}^{n} x_i Q_{j,i}(x_1, \dots, x_i), \\
&\ \ \vdots \\
\dot{x}_{n-1} &= x_n + v \sum_{i=3}^{n} x_i Q_{j,i}(x_1, \dots, x_i), \\
\dot{x}_n &= v,
\end{aligned} \right. \tag{6.1}
$$

where $Q_{j,i}(x_1, \dots, x_i)$ are formal power series depending on the indicated variables.

Naturally, as with normal forms, a given system can admit different dual normal forms. We are thus interested in constructing a dual canonical form. Assuming that the linear part (F, G) of the system Σ^∞, of the form (2.1), is controllable, we denote by m_0 the degree of the first homogeneous term of the system Σ^∞ which cannot be annihilated by a feedback transformation. Thus, using Theorems 8 and 9, we can assume, after applying a suitable feedback, that Σ^∞ takes the form

$$
\Sigma^\infty \ : \ \dot{\xi} = A\xi + Bu + \bar{g}^{[m_0-1]}(\xi)u + \sum_{m=m_0+1}^{\infty} \left(f^{[m]}(\xi) + g^{[m-1]}(\xi)u\right),
$$

where (A, B) is in Brunovský canonical form and the first nonvanishing homogeneous vector field $\bar{g}^{[m_0-1]}$ is of the form

$$\bar{g}_j^{[m_0-1]}(\xi) = \begin{cases} \displaystyle\sum_{i=n-j+2}^{n} \xi_i Q_{j,i}^{[m_0-2]}(\xi_1, \dots, \xi_i), & 2 \le j \le n-1, \\ 0, & j = 1 \text{ and } j = n. \end{cases}$$

Define

$$j_* = \inf\left\{ j = 2, \dots, n-1 \ : \ \bar{g}_j^{[m_0-1]}(\xi) \neq 0 \right\}$$

and let $(i_1, \dots, i_n)$ such that $i_1 + \cdots + i_n = m_0 - 1$ be the largest, in the lexicographic ordering, n-tuple of nonnegative integers such that

$$\frac{\partial^{m_0-1} \bar{g}_{j_*}^{[m_0-1]}}{\partial \xi_1^{i_1} \cdots \partial \xi_n^{i_n}} \neq 0.$$

We get the following result.

THEOREM 11. *There exists a formal feedback transformation Γ^∞ which brings the system Σ^∞ into the following one:*

$$\Sigma_{DCF}^\infty : \dot{x} = Ax + Bv + \sum_{m=m_0}^{\infty} \bar{g}^{[m-1]}(x) v,$$

where for any $m \ge m_0$,

$$\bar{g}_j^{[m-1]} = \begin{cases} \displaystyle\sum_{i=n-j+2}^{n} x_i Q_{j,i}^{[m-2]}(x_1, \dots, x_i), & 2 \le j \le n-1, \\ 0, & j = 1 \text{ and } j = n. \end{cases} \tag{6.2}$$

Moreover,

$$\frac{\partial^{m_0-1} \bar{g}_{j_*}^{[m_0-1]}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = \pm 1, \tag{6.3}$$

and for any $m \ge m_0 + 1$

$$\frac{\partial^{m_0-1} \bar{g}_{j_*}^{[m-1]}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(x_1, 0, \dots, 0) = 0. \tag{6.4}$$

The form Σ_{DCF}^∞, which satisfies (6.2), (6.3), and (6.4), will be called the *dual canonical form* of Σ^∞. The name is justified by the following.

THEOREM 12. *The two systems Σ_1^∞ and Σ_2^∞ are formally feedback equivalent if and only if their dual canonical forms $\Sigma_{1,DCF}^\infty$ and $\Sigma_{2,DCF}^\infty$ coincide.*

Example 6. Let us consider the system

$$\Sigma \ : \ \dot{\xi} = f(\xi) + g(\xi)u, \quad \xi(\cdot) \in \mathbb{R}^3, \ u(\cdot) \in \mathbb{R},$$

whose linear part is assumed to be controllable. Theorem 10 ensures that the system Σ is formally feedback equivalent to the dual normal form Σ^{∞}_{DNF} given by

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_3 + v x_3 Q(x_1, x_2, x_3),\\ \dot{x}_3 &= v,\end{aligned}$$

where $Q(x_1, x_2, x_3)$ is a formal power series of variables x_1, x_2, x_3.

Assume for simplicity that $m_0 = 2$, which is equivalent to the condition that g, $ad_f g$, and $[g, ad_f g]$ are linearly independent at $0 \in \mathbb{R}^3$. This implies that we can represent $Q = Q(x_1, x_2, x_3)$ by

$$Q = c + x_1 Q_1(x_1) + x_2 Q_2(x_1, x_2) + x_3 Q_3(x_1, x_2, x_3),$$

where $c \in \mathbb{R}$, $c \neq 0$.

Observe that any Q of the above form gives a dual normal form Σ^{∞}_{DNF}. In order to get the dual canonical form we use Theorem 11, which ensures that the system Σ is formally feedback equivalent to its dual canonical form Σ^{∞}_{DCF} defined by

$$\begin{aligned}\dot{\tilde{x}}_1 &= \tilde{x}_2,\\ \dot{\tilde{x}}_2 &= \tilde{x}_3 + \tilde{v} \tilde{x}_3 \tilde{Q}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3),\\ \dot{\tilde{x}}_3 &= \tilde{v},\end{aligned}$$

where $\tilde{Q}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is a formal power series such that

$$\tilde{Q}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = 1 + \tilde{x}_2 \tilde{Q}_2(\tilde{x}_1, \tilde{x}_2) + \tilde{x}_3 \tilde{Q}_3(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3). \qquad \square$$

7. Proofs of dual results. In this section, we prove our dual results. The proof of Theorem 7 will be omitted because in the proof of Theorem 9 we give an explicit homogeneous feedback transformation bringing a given homogeneous system into its dual normal form. Theorem 10 follows from a successive application of Theorem 7. We will thus prove Theorems 8, 9, 11, and 12.

Proof of Theorem 8. (i) We will prove that if the system $\Sigma^{[m]}$ is equivalent to $\tilde{\Sigma}^{[m]}$ via a transformation Γ^m, then their dual m-invariants $b_j^{[m-1]}$ and $\tilde{b}_j^{[m-1]}$ coincide. The action of Γ^m can be decomposed into that of a pure feedback of the form

$$u = v + \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi) v$$

followed by that of a diffeomorphism

$$x = \xi + \phi^{[m]}(\xi)$$

of the state space. Since the first $n-1$ components of the vector fields $f^{[m]}$ and $g^{[m-1]}$, as well as those of $X^{[m-1]}_{n-i}$, are invariant under pure feedback, we can conclude that the functions $b_j^{[m-1]}$ for $2 \leq j \leq n-1$ are invariant under pure feedback. It remains to prove that they are also invariant under any diffeomorphism $x = \Phi(\xi)$ of the form $\Phi(\xi) = \xi + \phi^{[m]}(\xi)$.

The diffeomorphism Φ brings the system $\Sigma^{[m]}$ into the form

$$\tilde{\Sigma}^{[m]} \;:\; \dot{x} = Ax + Bu + \tilde{f}^{[m]}(x) + \tilde{g}^{[m-1]}(x) u,$$

where

$$\tilde{f}^{[m]} = f^{[m]} + [Ax, \phi^{[m]}],$$
$$\tilde{g}^{[m-1]} = g^{[m-1]} + L_B\phi^{[m]}.$$

Denoting by $b_j^{[m-1]}$ and $\tilde{b}_j^{[m-1]}$ for $2 \leq j \leq n-1$ the dual m-invariants associated, respectively, with the homogeneous systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$, we get

$$\tilde{b}_j^{[m-1]} = b_j^{[m-1]} + \hat{b}_j^{[m-1]},$$

where

$$\hat{b}_j^{[m-1]}(x) = L_B\phi_j^{[m]}(x) + \sum_{k=0}^{j-2} L_B L_{Ax}^{j-k-2} CA^k ad_{Ax}\phi^{[m]} - \sum_{i=1}^{n} L_B L_{Ax}^{j-1}\int_0^{x_i} C\hat{X}_{n-i}^{[m-1]}(\pi_i(x))dx_i$$

and

$$\hat{X}_{n-i}^{[m-1]}(x) = (-1)^{n-i} ad_{Ax+[Ax,\phi^{[m]}]}^{n-i}(B + L_B\phi^{[m]}) = (-1)^{n-i}(\Phi_* ad_{A\xi}^{n-i}(B))(x)$$
$$= A^{n-i}B + L_{A^{n-i}B}\phi^{[m]}(x).$$

We can deduce that

$$\begin{aligned}\hat{b}_j^{[m-1]}(x) &= L_B\phi_j^{[m]}(x) + \sum_{k=1}^{j-1} L_B L_{Ax}^{j-k}\phi_k^{[m]} - \sum_{k=2}^{j} L_B L_{Ax}^{j-k}\phi_k^{[m]} \\ &\quad - \sum_{i=1}^{n} L_B L_{Ax}^{j-1}\int_0^{x_i} L_{A^{n-i}B}\phi_1^{[m]}(\pi_i(x))dx_i \\ &= L_B\phi_j^{[m]}(x) + \sum_{k=1}^{j-1} L_B L_{Ax}^{j-k}\phi_k^{[m]} - \sum_{k=2}^{j} L_B L_{Ax}^{j-k}\phi_k^{[m]} - L_B L_{Ax}^{j-1}\phi_1^{[m]}(x) = 0,\end{aligned}$$

which gives

$$\tilde{b}_j^{[m-1]} = b_j^{[m-1]}.$$

Thus the functions $b_j^{[m-1]}$ are invariant under any diffeomorphism of the form $x = \Phi(\xi) = \xi + \phi^{[m]}(\xi)$. Therefore they remain invariant under the transformation Γ^m.

The fact that two homogeneous systems, whose dual m-invariants coincide, are feedback equivalent follows clearly from item (ii) of the theorem, which will be proved below. Indeed, by item (ii), both systems coincide when transformed to their canonical forms.

(ii) Denote by $\bar{b}_j^{[m-1]}$ for $2 \leq j \leq n-1$ the dual m-invariants associated with the dual normal form $\Sigma_{DNF}^{[m]}$. They are given by

$$\bar{b}_j^{[m-1]} = \bar{g}_j^{[m-1]} - \sum_{i=1}^{n} L_B L_{Ax}^{j-1}\int_0^{x_i} C\bar{X}_{n-i}^{[m-1]}(\pi_i(x))dx_i,$$

where the components $\bar{g}_j^{[m-1]}$ are given by (5.1) and

$$C\bar{X}_{n-i}^{[m-1]} = (-1)^{n-i} C ad_{Ax}^{n-i} \bar{g}^{[m-1]}.$$

It suffices to observe (see Lemma 2 below) that, on the one hand, $C\bar{X}_{n-i}^{[m-1]}$ is a linear combination of functions $L_{Ax}^s \bar{g}_j^{[m-1]}$ for $0 \le s \le n-i$ and $1 \le j \le n-i+1$ and, on the other hand, $\bar{g}_j^{[m-1]}(\pi_i(x)) = 0$ for all j such that $1 \le j \le n-i+1$. We thus conclude that $C\bar{X}_{n-i}^{[m-1]}(\pi_i(x)) = 0$, which implies

$$\bar{b}_j^{[m-1]} = \bar{g}_j^{[m-1]}$$

for any j such that $2 \le j \le n-1$. □

Proof of Theorem 9. Denote by

$$\tilde{\Sigma}^{[m]} \quad : \quad \dot{x} = Ax + Bv + \tilde{f}^{[m]}(x) + \tilde{g}^{[m-1]}(x)v$$

the system $\Sigma^{[m]}$ transformed via a homogeneous feedback transformation Γ^m defined by (5.2). From Proposition 1, it follows that for $\tilde{\Sigma}^{[m]}$ we have

$$\begin{aligned} \tilde{f}_j^{[m]} &= 0 \qquad \text{for} \quad 1 \le j \le n, \\ \tilde{g}_j^{[m-1]} &= 0 \qquad \text{for} \quad j = 1 \text{ and } j = n, \\ \tilde{g}_j^{[m-1]} &= g_j^{[m-1]} + L_B \phi_j^{[m]} \qquad \text{for} \quad 2 \le j \le n-1. \end{aligned} \tag{7.1}$$

It thus suffices to show that the components $\tilde{g}_j^{[m-1]}$ for $2 \le j \le n-1$ are in the dual normal form (5.1). We prove easily by an induction argument that

$$\begin{aligned} \phi_{j+1}^{[m]} &= \sum_{k=1}^{j} L_{A\xi}^{j-k} f_k^{[m]} + L_{A\xi}^{j} \phi_1^{[m]}, \\ L_B L_{A\xi}^{j} \phi_1^{[m]} &= \sum_{k=0}^{j} \tbinom{j}{k} L_{A\xi}^{j-k} L_{A^k B} \phi_1^{[m]}, \end{aligned}$$

which allows us to show that

$$\tilde{g}_{j+1}^{[m-1]} = g_{j+1}^{[m-1]} + \sum_{k=1}^{j} L_B L_{A\xi}^{k-1} f_{j-k+1}^{[m]} + \sum_{k=0}^{j} \tbinom{j}{k} L_{A\xi}^{j-k} L_{A^k B} \phi_1^{[m]}.$$

Now, from the identity

$$L_{A^k B} \phi_1^{[m]} = -CX_k^{[m-1]}(\pi_{n-k}(\xi)) - \sum_{i=n-k+1}^{n} \int_0^{\xi_i} \frac{\partial CX_{n-i}^{[m-1]}(\pi_i(\xi))}{\partial \xi_{n-k}} d\xi_i,$$

we can deduce that

$$\begin{aligned} \tilde{g}_{j+1}^{[m-1]} = g_{j+1}^{[m-1]} &+ \sum_{k=1}^{j} L_B L_{A\xi}^{k-1} f_{j-k+1}^{[m]} - \sum_{k=0}^{j} \tbinom{j}{k} L_{A\xi}^{j-k} CX_k^{[m]}(\pi_{n-k}(\xi)) \\ &- \sum_{k=1}^{j} \sum_{i=n-k+1}^{n} \tbinom{j}{k} L_{A\xi}^{j-k} \left(\int_0^{\xi_i} \frac{\partial CX_{n-i}^{[m-1]}(\pi_i(\xi))}{\partial \xi_{n-k}} d\xi_i \right). \end{aligned}$$

Taking into account that for any k such that $0 \leq k \leq j$ we have

$$L_{A\xi}^{j-k} CX_k^{[m]}(\pi_{n-k} \circ \pi_{n-j}(\xi)) = L_{A\xi}^{j-k} CX_k^{[m]}(\pi_{n-j}(\xi))$$

and that for any $i \geq n-j+1$ we have

$$\left(\int_0^{\xi_i} \frac{\partial CX_{n-i}^{[m-1]}(\pi_i(\xi))}{\partial \xi_{n-k}} d\xi_i \right) (\pi_{n-j}(\xi)) = 0,$$

we can conclude that

$$\tilde{g}_{j+1}^{[m-1]}(\pi_{n-j}(\xi)) = \left(g_{j+1}^{[m-1]} + \sum_{k=1}^{j} L_B L_{A\xi}^{j-k} f_k^{[m]} - \sum_{k=0}^{j} \binom{j}{k} L_{A\xi}^{j-k} CX_k^{[m]} \right) (\pi_{n-j}(\xi)).$$

Using Lemma 2 given below, we thus obtain

$$\tilde{g}_{j+1}^{[m-1]}(\pi_{n-j}(\xi)) = 0,$$

which proves that $\tilde{g}_j^{[m-1]}$ is in the dual normal form (5.1). □

Lemma 2. *Let $X_i^{[m-1]}$ be the homogeneous part of degree $m-1$ of*

$$X_i^{m-1} = (-1)^i ad_{A\xi + f^{[m]}}^i (B + g^{[m-1]}).$$

Then the following identities hold:

(i) *For any $j \geq 1$, we have*

$$CA^j X_1^{[m-1]} = \sum_{k=1}^{j} L_{AB} L_{A\xi}^{j-k} f_k^{[m]} - \sum_{k=0}^{j} \binom{j}{k} L_{A\xi}^{j-k} CX_{k+1}^{[m-1]}.$$

(ii) *For any j such that $0 \leq j \leq n-1$, we have*

$$\sum_{k=0}^{j} \binom{j}{k} L_{A\xi}^{j-k} CX_k^{[m-1]} = g_{j+1}^{[m-1]} + \sum_{k=1}^{j} L_B L_{A\xi}^{j-k} f_k^{[m]}.$$

Both identities can be proved by a direct calculation.

Proof of Theorem 11. In the first step we will normalize terms of degree at most m_0 while in the general step we will normalize terms of order $m_0 + l$.

First step. Consider the system Σ^∞ and recall that m_0 is the degree of the first nonlinearizable homogeneous part. We can assume (see Theorems 7 and 8) that after applying a suitable feedback transformation, the system Σ^∞ takes the form

$$\dot{\xi} = A\xi + Bu + \bar{g}^{[m_0-1]}(\xi)u + \sum_{m=m_0+1}^{\infty} \left(f^{[m]}(\xi) + g^{[m-1]}(\xi)u \right), \tag{7.2}$$

where the vector field $\bar{g}^{[m_0-1]}$ defined by

$$\bar{g}_j^{[m_0-1]}(\xi) = \begin{cases} \displaystyle\sum_{i=n-j+2}^{n} \xi_i Q_{j,i}^{[m_0-2]}(\xi_1, \ldots, \xi_i), & 2 \leq j \leq n-1, \\ 0, & j = 1 \text{ or } j = n, \end{cases}$$

is the first nonlinearizable homogeneous part. We can notice that the linear feedback transformation

$$\Gamma^1 \ : \quad \begin{aligned} x &= a_1\xi, \\ u &= \tfrac{1}{a_1}v, \end{aligned}$$

where $a_1 \in \mathbb{R}$ and $a_1 \neq 0$, brings the system (7.2) into the following one:

$$\dot{x} = Ax + Bv + \frac{1}{a_1^{m_0-1}}\bar{g}^{[m_0-1]}(x)v + \sum_{m=m_0+1}^{\infty}\left(\tilde{f}^{[m]}(x) + \tilde{g}^{[m-1]}(x)v\right).$$

Due to the definitions of $(i_1, \ldots, i_n)$ and j_*, we can suitably choose the parameter a_1 such that

$$\frac{\partial^{m_0-1}\bar{g}_{j_*}^{[m_0-1]}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}} = \pm 1.$$

General step. Now we assume that, for some $l \geq 1$, the system Σ^∞ takes the form

$$\dot{\xi} = A\xi + Bu + \sum_{m=m_0}^{m_0+l-1}\bar{g}^{[m-1]}(\xi)u + f^{[m_0+l]}(\xi) + g^{[m_0+l-1]}(\xi)u + r(\xi, u), \tag{7.3}$$

where $r(\xi, u) \in R^{\geq m_0+l+1}(\xi, u)$ and, for any m such that $m_0 \leq m \leq m_0 + l - 1$ and any $1 \leq j \leq n$, the components $\bar{g}_j^{[m-1]}$ satisfy the conditions (6.2), (6.3), and (6.4). Consider the transformation Γ^∞ given by (4.3), satisfying (3.7), and its decomposition $\Gamma^\infty = \Gamma^{\leq m_0+l} = \Gamma^{m_0+l} \circ \Gamma^{\leq m_0+l-1}$ given by (4.4)–(4.5). We can easily see that the transformation $\Gamma^{\leq m_0+l-1}$ brings the system (7.3) into the system

$$\dot{y} = Ay + Bw + \sum_{m=m_0}^{m_0+l-1}\bar{g}^{[m-1]}(y)w + \tilde{f}^{[m_0+l]}(y) + \tilde{g}^{[m_0+l-1]}(y)w + r(y, w), \tag{7.4}$$

where $r(y, w) \in R^{\geq m_0+l+1}(y, w)$ and

$$\begin{aligned} \tilde{f}^{[m_0+l]} &= f^{[m_0+l]} + \bar{g}^{[m_0-1]}\alpha^{[l+1]}, \\ \tilde{g}^{[m_0+l-1]} &= g^{[m_0+l-1]} + \left[\bar{g}^{[m_0-1]}, \phi^{[l+1]}\right] + \bar{g}^{[m_0-1]}\beta^{[l]}. \end{aligned}$$

Let $b_j^{[m_0+l-1]}$ and $\tilde{b}_j^{[m_0+l-1]}$ be the dual $(m_0 + l)$-invariants associated, respectively, to the homogeneous parts of degree $m_0 + l$ of the systems (7.3) and (7.4). We thus deduce that

$$\tilde{b}_j^{[m_0+l-1]} = b_j^{[m_0+l-1]} + \hat{b}_j^{[m_0+l-1]}, \tag{7.5}$$

where

$$\hat{b}_j^{[m_0+l-1]} = CA^{j-1}\left[\bar{g}^{[m_0-1]}, \phi^{[l+1]}\right] + \beta^{[l]}\bar{g}_j^{[m_0-1]} - \sum_{i=1}^{n} L_B L_{Ay}^{j-1}\int_0^{y_i} C\hat{X}_{n-i}^{[m-1]}(\pi_i(y))dy_i$$

and

$$\begin{aligned} \hat{X}_{n-i}^{[m-1]} = (-1)^{n-i}ad_{Ay}^{n-i}&\left(\left[\bar{g}^{[m_0-1]}, \phi^{[l+1]}\right] + \bar{g}^{[m_0-1]}\beta^{[l]}\right) \\ &+ \sum_{k=0}^{n-i-1}(-1)^k ad_{Ay}^k ad_{A^{n-i-k-1}B}\left(\bar{g}^{[m_0-1]}\alpha^{[l+1]}\right). \end{aligned}$$

First notice (see Lemma 2) that $C\hat{X}_{n-i}^{[m-1]}$ is a linear combination, over the ring of polynomials, of the components $CA^{j-1}\bar{g}^{[m_0-1]}$ and $CA^{j-1}\left[\bar{g}^{[m_0-1]},\phi^{[l+1]}\right]$, $1 \le j \le n-i+1$, and their derivatives. Since

$$CA^{j-1}\left[\bar{g}^{[m_0-1]},\phi^{[l+1]}\right] = \sum_{k=1}^{j}\frac{\partial\phi_j^{[l+1]}}{\partial y_k}\bar{g}_k^{[m_0-1]} - \frac{\partial\bar{g}_j^{[m_0-1]}}{\partial y}\phi^{[l+1]},$$

it follows that $C\hat{X}_{n-i}^{[m-1]}$ is a linear combination, over the ring of polynomials, of the components $\bar{g}_j^{[m_0-1]}$, $1 \le j \le n-i+1$, and their derivatives. Taking into account the fact that $\bar{g}^{[m_0-1]}$ satisfies (6.2), we obtain, for any $1 \le j \le n-i+1$,

$$\bar{g}_j^{[m_0-1]}(\pi_i(y)) = 0.$$

Thus, we deduce that

$$\hat{X}_{n-i}^{[m-1]}(\pi_i(y)) = 0,$$

which leads to the identity

$$\hat{b}_j^{[m_0+l-1]} = CA^{j-1}\left[\bar{g}^{[m_0-1]},\phi^{[l+1]}\right] + \beta^{[l]}\bar{g}_j^{[m_0-1]}.$$

Putting $j = j_*$ and due to the fact that $\bar{g}_1^{[m_0-1]} = \cdots = \bar{g}_{j_*-1}^{[m_0-1]} = 0$, we get

$$\hat{b}_{j_*}^{[m_0+l-1]} = -\frac{\partial\bar{g}_{j_*}^{[m_0-1]}}{\partial y}\phi^{[l+1]} + L_{A^{n-j_*}B}(\phi_{j_*}^{[l+1]})\bar{g}_{j_*}^{[m_0-1]} + \beta^{[l]}\bar{g}_{j_*}^{[m_0-1]}.$$

Since the triplet $(\phi^{[l+1]},\alpha^{[l+1]},\beta^{[l]})$ satisfies the condition (3.7), it is easy to see that for any $1 \le j \le n-1$, we have

$$L_{A^{n-j}B}\phi_j^{[l+1]} + \beta^{[l]} = 0$$

and then conclude that

$$\hat{b}_{j_*}^{[m_0+l-1]} = -\frac{\partial\bar{g}_{j_*}^{[m_0-1]}}{\partial y}\phi^{[l+1]}.$$

Now, let us differentiate this last expression, taking into account that $(i_1,\ldots,i_n)$ is the largest n-tuple of nonnegative integers such that

$$\frac{\partial^{m_0-1}\bar{g}_{j_*}^{[m_0-1]}}{\partial y_1^{i_1}\cdots\partial y_n^{i_n}} \neq 0.$$

We obtain

$$\frac{\partial^{i_1+l}\hat{b}_{j_*}^{[m_0+l-1]}}{\partial y_1^{i_1+l}} = -\left(d_1\frac{\partial^{i_1+1}\bar{g}_{j_*}^{[m_0-1]}}{\partial y\partial y_1^{i_1}}\frac{\partial^l\phi^{[l+1]}}{\partial y_1^l} + d_2\frac{\partial^{i_1}\bar{g}_{j_*}^{[m_0-1]}}{\partial y\partial y_1^{i_1-1}}\frac{\partial^{l+1}\phi^{[l+1]}}{\partial y_1^{l+1}}\right), \tag{7.6}$$

where d_1 and d_2 are strictly positive integers. Since

$$\frac{\partial^l\phi^{[l+1]}}{\partial y_1^l} = a_{l+1}(l+1)!(y_1,y_2,\ldots,y_n)^T$$

and

$$\frac{\partial^{l+1}\phi^{[l+1]}}{\partial y_1^{l+1}} = a_{l+1}(l+1)!(1,0,\dots,0)^T,$$

and due to the fact that

$$\sum_{k=2}^{n} \frac{\partial \bar{g}_{j_*}^{[m_0-1]}}{\partial y_k} y_k = (m_0 - i_1 - 1)\bar{g}_{j_*}^{[m_0-1]},$$

the identity (7.6) gives

$$\frac{\partial^{i_1+l}\bar{b}_{j_*}^{[m_0+l-1]}}{\partial y_1^{i_1+l}} = \theta_l \frac{\partial^{i_1}\bar{g}_{j_*}^{[m_0-1]}}{\partial y_1^{i_1}},$$

where $\theta_l = -a_{l+1}(l+1)!\,(d_1(m_0 - i_1 - 1) + d_2)$. Plugging this last expression into (7.5), where we put $j = j_*$, we obtain, after differentiating, the following relation:

$$\frac{\partial^{m_0+l-1}\tilde{b}_{j_*}^{[m_0+l-1]}}{\partial y_1^{i_1+l}\partial y_2^{i_2}\cdots\partial y_n^{i_n}} = \frac{\partial^{m_0+l-1}\bar{b}_{j_*}^{[m_0+l-1]}}{\partial y_1^{i_1+l}\partial y_2^{i_2}\cdots\partial y_n^{i_n}} + \theta_l \frac{\partial^{m_0-1}\bar{g}_{j_*}^{[m_0-1]}}{\partial y_1^{i_1}\partial y_2^{i_2}\cdots\partial y_n^{i_n}}.$$

Because of the definition of θ_l, we can choose suitably the parameter a_{l+1} such that

$$\frac{\partial^{m_0+l-1}\tilde{b}_{j_*}^{[m_0+l-1]}}{\partial y_1^{i_1+l}\partial y_2^{i_2}\cdots\partial y_n^{i_n}} = 0,$$

which is equivalent to

$$\frac{\partial^{m_0-1}\tilde{b}_{j_*}^{[m_0+l-1]}}{\partial y_1^{i_1}\partial y_2^{i_2}\dots\partial y_n^{i_n}}(y_1,0,\dots,0) = 0.$$

Now, transforming the homogeneous part of degree $m_0 + l$ of the system (7.4) to its normal form via a homogeneous transformation Γ^{m_0+l} and taking into account Theorem 8, we bring the system (7.4) into the form

$$\dot{x} = Ax + Bv + \sum_{m=m_0}^{m_0+l} \bar{g}^{[m-1]}(x)v + r(x,v), \tag{7.7}$$

where $r(x,v) \in R^{\geq m_0+l+1}(x,v)$, and for any m such that $m_0 \leq m \leq m_0 + l$, the components $\bar{g}_j^{[m-1]}$ for $2 \leq j \leq n-1$ satisfy the conditions (6.2), (6.3), and (6.4). This ends the proof of Theorem 11. ∎

Proof of Theorem 12. The proof of this theorem follows the same line as that of Theorem 6. We notice only that the transformation Γ^∞ leaves invariant all terms of degree smaller than $m_0 + l$ of the system (7.3) if and only if it is of the form (4.10), given by Lemma 1. ∎

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