2005

Feedback Classification of Multi-Input Nonlinear Control Systems

Issa Amadou Tall
Southern Illinois University Carbondale, itall@math.siu.edu

Published in Tall, I. A. (2005). Feedback classification of multi-input nonlinear control systems. *SIAM Journal on Control and Optimization, 43*(6), 2049-2070. doi: 10.1137/S0363012902410502.

SIAM J. CONTROL OPTIM.
Vol. 43, No. 6, pp. 2049–2070

FEEDBACK CLASSIFICATION OF MULTI-INPUT NONLINEAR CONTROL SYSTEMS*

ISSA AMADOU TALL†

Abstract. We study the feedback group action on multi-input nonlinear control systems with uncontrollable mode. We follow slightly an approach proposed in Kang and Krener [W. Kang and A. J. Krener, *SIAM J. Control. Optim.,* 30 (1992), pp. 1319–1337] which consists of analyzing the system and the feedback group step by step. We construct a normal form which generalizes, on one hand, the results obtained in the single-input case and, on the other hand, those recently obtained by the same author in the controllable case. We illustrate our results by studying the Caltech Multi-Vehicle Wireless Testbed (MVWT) and the prototype of Planar Vertical TakeOff and Landing aircraft (PVTOL). We also study the notion of *bifurcation of controllability* for systems with one nonzero uncontrollable mode. We first show that the equilibria for those systems is a p-dimensional submanifold (p equals number of inputs). Provided that one term in their normal form is nonzero, we show that these systems are linearly controllable, hence stabilizable, at any nearby equilibrium point of the origin.

Key words. feedback, multi-input, homogeneous, normal form, uncontrollable, control system

AMS subject classifications. 93B11, 93B17, 93B27

DOI. 10.1137/S0363012902410502

1. Introduction.

During the last twenty years the problem of transforming a nonlinear control system

$$\Pi \ : \ \dot{\zeta} = f(\zeta, u), \quad \zeta(\cdot) \in \mathbb{R}^n \quad u(\cdot) = (u_1(\cdot), \ldots, u_p(\cdot))^T \in \mathbb{R}^p$$

by a feedback transformation of the form

$$\Upsilon \ : \ \begin{array}{rcl} \bar{\zeta} & = & \varphi(\zeta), \\ u & = & \gamma(\zeta, \bar{u}) \end{array}$$

to a simpler form has been extensively studied by several authors. Necessary and sufficient geometric conditions for smooth linearizability, that is, smooth feedback equivalence to a linear system, have been obtained independently by Hunt and Su [16], Hunt, Su, and Meyer [17], and Jakubczyk and Respondek [21] among others. Those conditions turn out to be restrictive, except for the planar case, and a natural problem that arises is to find normal forms for nonlinearizable systems. Four basic methods have been proposed to study feedback equivalence problems. The first method is based on the theory of singularities of vector fields and distributions and their invariants, and using that method on a large variety of feedback classification problems have been solved; see, e.g., [4, 6, 10, 17, 18, 21, 22, 23, 31, 36, 39, 51]. The second approach, proposed by Gardner [10], uses Cartan's method of equivalence [5] and describes the geometry of feedback equivalence, [11, 12, 13, 35]. The third method, inspired by the Hamiltonian formalism for optimal control problems, has been developed by Bonnard [3, 4] and Jakubczyk [19, 20] and has led to a very nice

*Received by the editors July 1, 2002; accepted for publication (in revised form) June 3, 2004; published electronically April 14, 2005.
http://www.siam.org/journals/sicon/43-6/41050.html

†Department of Mathematics, University of California, One Shields Avenue, Davis, CA 95616 (tall@math.ucdavis.edu). This research was conducted while the author was on leave from the Department of Mathematics, Université Cheikh Anta Diop, Dakar, Sénégal.

description of feedback invariants in terms of singular extremals. Most recently, Kang and Krener [29] adapted Poincaré's technique for linearization of dynamical systems (see, e.g., [1]) to control systems. Their idea consists of analyzing the system Π and the feedback transformation Υ step by step in order to produce a simpler equivalent system, also step by step. They first obtained quadratic normal forms under quadratic changes of coordinates and feedback for single-input control systems with controllable linearization. Later, Kang [24] generalizes this result to all degrees for the same class of control systems. He also obtained [25] quadratic normal forms for systems with uncontrollable linearization. The method introduced by Kang and Krener finds its importance in replacing the solving of partial differential equations by that of algebraic equations.

Since then many results have followed. Tall [41, 43] and Tall and Respondek [49] obtained canonical forms and dual canonical forms for single-input nonlinear control systems with controllable linearization, then normal forms for single-input nonlinear control systems with uncontrollable linearization [44] (see also Krener, Kang, and [32]), as well as the corresponding homogeneous invariants. Hence, the feedback classification of single-input nonlinear control systems is almost complete, and the aim of this paper is to deal with the multi-input nonlinear control systems. Preliminary results for two-input control systems, with controllable mode, have been recently obtained by Tall and Respondek [47] and completed by Tall [42] for multi-input systems with controllable mode. This paper gives a generalization of those results to multi-input systems with uncontrollable mode.

Motivations for studying normal forms for multi-input systems are underlined by the huge varieties of applications derived for single-input systems. Indeed, in the single-input case, the theory of normal forms has proved to be very useful in analyzing structural properties of nonlinear control systems. It has been used to study bifurcations and stabilizations of nonlinear systems [7, 14, 26, 27, 28], has led to a complete description of symmetries around equilibrium [37, 38, 48], and allowed the characterization of systems equivalent to feedforward forms [45, 46, 50]. The same approach has been introduced to study observability of control systems [33, 34, 2], the problem of output regulation, and the model matching problem.

The study of linearly uncontrollable systems is also motivated by the numerous engineering applications and the fact that the qualitative properties like controllability and stabilizability are generic, that is, invariant under a small variation of parameters at a point where the system is linearly controllable. Furthermore, it is known that local bifurcations at a point where the system is linearly controllable can be removed or delayed by pole placement. For those systems that are not linearly controllable, nonlinear phenoma like bifurcations are expected around the critical points.

In this paper we construct a normal form for multi-input nonlinear control systems with uncontrollable linearization which generalizes the results obtained in the single-input case [24, 41, 43, 44, 49] and the two-input case [47].

The organization of the paper is as follows. Section 2 deals with basic notations. In section 3, we construct a normal form for multi-input nonlinear control systems with uncontrollable linearization. We illustrate our results by two physical examples. We also discuss the notion of *bifurcation of controllability* for systems with one nonzero uncontrollable mode. We first show that the set of equilibria of these systems is a p-dimensional surface, and at any nearby equilibrium point of the origin, these systems became linearly controllable. Section 4 deals with the proofs of our results.

2. Notations and preliminaries. All objects, that is, functions, maps, vector fields, control systems, etc., are considered in a neighborhood of $0 \in \mathbb{R}^n$ and assumed to be C^∞-smooth. Consider the system

$$\Pi \ : \ \dot{\zeta} = f(\zeta, u), \quad \zeta(\cdot) \in \mathbb{R}^n, \ u(\cdot) = (u_1(\cdot), \dots, u_p(\cdot))^T \in \mathbb{R}^p.$$

We will assume throughout the paper that the point $(0,0) \in \mathbb{R}^n \times \mathbb{R}^p$ is an equilibrium point, that is, $f(0,0) = 0$, and let

$$\Pi^{[1]} \ : \ \dot{\zeta} = F\zeta + Gu = F\zeta + G_1 u_1 + \cdots + G_p u_p$$

be its linear approximation around the equilibrium point $(0,0) \in \mathbb{R}^n \times \mathbb{R}^p$, where

$$F = \frac{\partial f}{\partial \zeta}(0,0), \qquad G_1 = \frac{\partial f}{\partial u_1}(0,0), \dots, G_p = \frac{\partial f}{\partial u_p}(0,0).$$

We assume that the linear approximation is uncontrollable which means that there exists a nonnegative integer $q \in \mathbb{N}^*$ such that

$$\operatorname{span}\left\{ F^i G_s : \ 0 \le i \le n-1, \ 1 \le s \le p \right\} = \mathbb{R}^{n-q}.$$

We will also assume that $G_1 \wedge \ldots \wedge G_p \neq 0$, that is, the $n \times p$ matrix whose columns are $G_1, \dots, G_p$ to be of constant rank p.

Let $(r_1, \dots, r_p)$, $1 \le r_1 \le \cdots \le r_p = r$, be the largest, in the lexicographic ordering, p-tuple of nonnegative integers, with $r_1 + \cdots + r_p = n$, such that

$$\operatorname{span}\left\{ F^i G_s : \ 0 \le i \le r_s - 1, \ 1 \le s \le p \right\} = \mathbb{R}^{n-q}.$$

For the simplicity of the presentation we will suppose that $r_1 = \cdots = r_p = r$, and we will show how the general case deduces.

By a smooth linear feedback transformation it is always possible to bring the pair (F, G) into the Brunovský–Jordan canonical pair $(\tilde{A}, \tilde{B})$, where

$$\tilde{A} = \operatorname{diag}(J, A_1, \dots, A_p), \ \tilde{B} = (0, B_1, \dots, B_p) = \operatorname{diag}(0, b_1, \dots, b_p)$$

with J the Jordan canonical form of dimension q, (A_s, b_s) the Brunovský single-input canonical form of dimension $r_s = r$ for any $1 \le s \le p$.

For simplicity we will set

$$A = \operatorname{diag}(A_1, \dots, A_p), \ B = (B_1, \dots, B_p) = \operatorname{diag}(b_1, \dots, b_p).$$

We will denote coordinates of $\mathbb{R}^q \times \mathbb{R}^{n-q}$ by $(z^T, x^T)^T$, where $z = (z_1, \dots, z_q)^T$ and $x = (x_1^T, \dots, x_p^T)^T$ with $x_s = (x_{s,1}, \dots, x_{s,r})^T$ for any $1 \le s \le p$. For the fixed value q, we will denote by $\mathcal{S}_q(\mathbb{R}, 0)$, the set of all functions, either smooth or formal, depending on the variables $z = (z_1, \dots, z_q)^T \in \mathbb{R}^q$.

Let $h = h(z, x, u)$ be a smooth $\mathbb{R}$-valued function defined in a neighborhood of the point $(0,0,0) \in \mathbb{R}^q \times \mathbb{R}^{n-q} \times \mathbb{R}^p$. By

$$h(z,x,u) = h^{[0]}(z,x,u) + h^{[1]}(z,x,u) + h^{[2]}(z,x,u) + \cdots = \sum_{m=0}^{\infty} h^{[m]}(z,x,u)$$

we denote its Taylor series expansion at $(0,0,0) \in \mathbb{R}^q \times \mathbb{R}^{n-q} \times \mathbb{R}^p$ with respect to the variables x and u, where $h^{[m]}(z,x,u)$ stands for a homogeneous polynomial of degree m

of the variables x and u whose coefficients are functions of the variable $z \in \mathbb{R}^q$, that is, in $\mathcal{S}_q(\mathbb{R}, 0)$.

To fix the ideas, the functions $x_{1,1}^3$, $z^2 x_{1,1}^3$, $\cos z\, x_{1,1}^3$, $(1-e^z)x_{1,1}x_{2,1}^2$, $\sin z\, x_{1,2}u_2^2$, and $z_1 z_2 u_1^2 u_2 + x_{1,1}x_{2,1}u_1$ are all polynomials of degree 3.

Choose $d \in \mathbb{N} \cup \{\infty\}$ large enough and consider the Taylor series expansion of order d of the system Π

$$\Pi^{\leq d} : \begin{cases} \dot{z} = Jz + \sum_{m=1}^{d} g^{[m-1]}(z, x, u) + O(z, x, u)^d, \\ \dot{x} = Ax + Bu + \sum_{m=0}^{d} f^{[m]}(z, x, u) + O(z, x, u)^{d+1}, \end{cases} \tag{2.1}$$

where we already assumed that the linear part is in Brunovský–Jordan form, and the Taylor series expansion of order d of the transformation Υ,

$$\Upsilon^{\leq d} : \begin{cases} \bar{z} = \psi(z, x) = z + \sum_{m=1}^{d} \psi^{[m-1]}(z, x) + O(z, x)^d, \\ \bar{x} = \phi(z, x) = x + \sum_{m=0}^{d} \phi^{[m]}(z, x) + O(z, x)^{d+1}, \\ u = \gamma(z, x, \bar{u}) = \bar{u} + \sum_{m=0}^{d} \gamma^{[m]}(z, x, \bar{u}) + O(z, x, \bar{u})^{d+1}. \end{cases} \tag{2.2}$$

The variables z and $\bar{z}$ (resp., (x, u) and $(\bar{x}, \bar{u})$) will be called the *uncontrollable variables* associated with the uncontrollable part (resp., *controllable variables* associated with the controllable part) of the system.

Above, and throughout the paper, we mean by $g^{[m-1]}(z, x, u)$ and $\psi^{[m-1]}(z, x)$ (resp., $f^{[m]}(z, x, u)$, $\phi^{[m]}(z, x)$ and $\gamma^{[m]}(z, x, \bar{u})$) that each of their components is a homogeneous polynomial of degree $m-1$ (resp., of degree m) of the *controllable variables*. Moreover, $O(\cdot)^k$ denotes terms of degree k and higher of the controllable variables.

Notice that although we bring the linear approximation (F, G) of the system into Brunovský–Jordan canonical form, that is, the uncontrollable part in Jordan form of dimension q, and the controllable part in Brunovský form of dimension $n-q$, we still have terms of degree 0 and degree 1. However, the first jets of these terms is zero at the origin.

3. Main results. In this section we will establish our main results. We will give, in subsection 3.1 below, the normal forms we obtain for general control systems. The results are given in the simplest case where the controllability indices are equal. We will show that the general case deduces by extended the system. In subsection 3.2, we will study two physical examples: The Caltech Multi-Vehicle Wireless Testbed (MVWT) and the prototype of a Planar Vertical TakeOff and Landing (PVTOL) aircraft. In subsection 3.3, we discuss the notion of *bifurcation of controllability* for systems with one nonzero uncontrollable mode. We first show that the set of equilibria of these systems is a p-dimensional surface, and at any nearby equilibrium point of the origin, these systems become linearly controllable.

3.1. Normal forms. Let $1 \leq s \leq p$. We denote

$$\bar{z} = (\bar{z}_1, \ldots, \bar{z}_q)^T, \qquad \bar{x}_s = (\bar{x}_{s,1}, \ldots, \bar{x}_{s,r})^T, \qquad \text{and} \qquad \bar{x}_{s,r+1} = \bar{u}_s$$

and we set $\hat{x}_{s,i} = (\bar{x}_{s,1}, \ldots, \bar{x}_{s,i})$ for any $1 \leq i \leq r+1$.

For any $1 \le s \le t \le p$ and any $1 \le i \le r+1$, we also denote

$$\pi^s_{t,i}(\bar{x}) = (\hat{x}_{1,i}, \ldots, \hat{x}_{s,i}, \hat{x}_{s+1,i-1}, \ldots, \hat{x}_{t-1,i-1}, \hat{x}_{t,i}, \hat{x}_{t+1,i-1}, \ldots, \hat{x}_{p,i-1})^T.$$

For $i = 1$ the expressions $\hat{x}_{t,i-1}$ will be taken to be empty. Few examples are given right after the theorem to make these notations comprehensible.

Our main result for general control systems, that is, for control systems with *uncontrollable linearization* is as follows.

THEOREM 3.1. *For any $d \in \mathbb{N} \cup \{\infty\}$, the system $\Pi^{\le d}$, defined by* (2.1)*, with* uncontrollable linearization *is feedback equivalent, by a feedback transformation $\Upsilon^{\le d}$ of the form* (2.2)*, to the following normal form:*

$$\Pi^{\le d}_{NF}: \begin{cases} \dot{\bar{z}} = J\bar{z} + \bar{g}^{[0]}(\bar{z}) + \sum\limits_{m=2}^{d} \bar{g}^{[m-1]}(\bar{z}, \bar{x}, \bar{u}) + O(\bar{z}, \bar{x}, \bar{u})^d, \\ \dot{\bar{x}} = A\bar{x} + B\bar{u} + \sum\limits_{m=2}^{d} \bar{f}^{[m]}(\bar{z}, \bar{x}, \bar{u}) + O(\bar{z}, \bar{x}, \bar{u})^{d+1}, \end{cases}$$

where for any m,

$$\begin{aligned} \bar{g}^{[m-1]}(\bar{z}, \bar{x}, \bar{u}) &= \sum_{j=1}^{q} \bar{g}^{[m-1]}_j(\bar{z}, \bar{x}, \bar{u}) \frac{\partial}{\partial \bar{z}_j} \\ \bar{f}^{[m]}(\bar{z}, \bar{x}, \bar{u}) &= \sum_{k=1}^{p} \sum_{j=1}^{r-1} \bar{f}^{k[m]}_j(\bar{z}, \bar{x}, \bar{u}) \frac{\partial}{\partial \bar{x}_{k,j}}, \end{aligned} \tag{3.1}$$

with

$$\begin{aligned} \bar{g}^{[1]}_j(\bar{z}, \bar{x}, \bar{u}) &= \sum_{1 \le s \le p} \bar{x}_{s,1} R_{j,s}(\bar{z}) \\ \bar{g}^{[m-1]}_j(\bar{z}, \bar{x}, \bar{u}) &= \sum_{1 \le s \le t \le p} \sum_{i=1}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i} P^{[m-3]}_{j,i,s,t}(\bar{z}, \pi^s_{t,i}(\bar{x})) \\ &\quad + \sum_{1 \le s < t \le p} \sum_{i=2}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i-1} Q^{[m-3]}_{j,i,s,t}(\bar{z}, \pi^t_{t,i-1}(\bar{x})) \end{aligned} \tag{3.2}$$

for any $1 \le j \le q$ and

$$\begin{aligned} \bar{f}^{k[m]}_j(\bar{z}, \bar{x}, \bar{u}) &= \sum_{1 \le s \le t \le p} \sum_{i=j+2}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i} P^{k[m-2]}_{j,i,s,t}(\bar{z}, \pi^s_{t,i}(\bar{x})) \\ &\quad + \sum_{1 \le s < t \le p} \sum_{i=j+2}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i-1} Q^{k[m-2]}_{j,i,s,t}(\bar{z}, \pi^t_{t,i-1}(\bar{x})) \end{aligned} \tag{3.3}$$

for any $1 \le k \le p$ and any $1 \le j \le r-1$.

Above, the functions $P^{[m-3]}_{j,i,s,t}$, $Q^{[m-3]}_{j,i,s,t}$, $P^{k[m-2]}_{j,i,s,t}$, and $Q^{k[m-2]}_{j,i,s,t}$ stand for homogeneous polynomials of the indicated controllable variables $\bar{x}$ and $\bar{u}$ whose coefficients depend on the uncontrollable variable $\bar{z}$.

To make the notations $\pi^s_{t,i}(\bar{x})$ somewhat understandable, suppose $p = 3$. Then we have

$$\pi^1_{2,2}(\bar{x}) = (\bar{x}_{1,1}, \bar{x}_{1,2}, \bar{x}_{2,1}, \bar{x}_{2,2}, \bar{x}_{3,1}) \quad \text{and} \quad \pi^2_{2,2}(\bar{x}) = (\bar{x}_{1,1}, \bar{x}_{1,2}, \bar{x}_{2,1}, \bar{x}_{2,2}, \bar{x}_{3,1}).$$

We also have

$$\pi^1_{2,1}(\bar{x}) = (\bar{x}_{1,1}, \bar{x}_{2,1}), \quad \pi^2_{2,1}(\bar{x}) = (\bar{x}_{1,1}, \bar{x}_{2,1}), \quad \text{and} \quad \pi^1_{3,1}(\bar{x}) = (\bar{x}_{1,1}, \bar{x}_{3,1}).$$

Notice that the above normal form is a natural combination of the two extreme cases: that of dynamical systems and that of systems with controllable linearization.

Indeed, if $q = n$, that is, we deal with a dynamical system, then the coordinates $(\bar{x}_1^T, \ldots, \bar{x}_p^T)^T$ are not present and the normal form $\Pi_{NF}^{\leq d}$ reduces to a dynamical system $\dot{\bar{z}} = J\bar{z} + \bar{g}^{[0]}(\bar{z})$ containing resonant terms only. This is, of course, Poincaré normal form of a vector field under a formal diffeomorphism. On the other hand, if $q = 0$, that is, if the linearization of the system is controllable, the coordinates $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_q)^T$ are not present and our normal form reduces to

$$\begin{aligned}\bar{f}_j^{k[m]}(\bar{x}, \bar{u}) = &\sum_{1\leq s\leq t\leq p} \sum_{i=j+2}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i} P_{j,i,s,t}^{k[m]}(\pi_{t,i}^s(\bar{x})) \\ &+ \sum_{1\leq s< t\leq p} \sum_{i=j+2}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i-1} Q_{j,i,s,t}^{k[m]}(\pi_{t,i-1}^t(\bar{x}))\end{aligned}$$

for any $1 \leq k \leq p$ and any $1 \leq j \leq r-1$, and $\bar{f}_j^{k[m]}(\bar{x}, \bar{u}) = 0$ otherwise. This latter case will be summarized in the following corollary. It gives the normal form for multi-input control systems with controllable linearization (see [42]).

COROLLARY 3.2. *The system* $\Pi^{\leq d}$*, defined by* (2.1)*, with* controllable linearization*, is feedback equivalent by a polynomial feedback transformation* $\Upsilon^{\leq d}$ *of the form* (2.2)*, to the following normal form:*

$$\Pi_{NF}^{\leq d} \; : \; \dot{\bar{x}} = A\bar{x} + B\bar{u} + \sum_{m=2}^{d} \bar{f}^{[m]}(\bar{x}, \bar{u}) + O(\bar{x}, \bar{u})^{d+1},$$

where

$$\bar{f}^{[m]}(\bar{x}, \bar{u}) = \sum_{k=1}^{p} \sum_{j=1}^{r-1} \bar{f}_j^{k[m]}(\bar{x}, \bar{u}) \frac{\partial}{\partial \bar{x}_{k,j}},$$

with

$$\begin{aligned}\bar{f}_j^{k[m]}(\bar{x}, \bar{u}) = &\sum_{1\leq s\leq t\leq p} \sum_{i=j+2}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i} P_{j,i,s,t}^{k[m-2]}(\pi_{t,i}^s(\bar{x})) \\ &+ \sum_{1\leq s< t\leq p} \sum_{i=j+2}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i-1} Q_{j,i,s,t}^{k[m-2]}(\pi_{t,i-1}^t(\bar{x}))\end{aligned}$$

for any $1 \leq k \leq p$ *and any* $1 \leq j \leq r-1$.

Particular case $p = 2$ and $r = 3$. In this particular case the normal form will be given by

$$\Pi^{\leq d} \; : \; \left\{ \begin{array}{lcl} \dot{\bar{x}}_{1,1} & = & \bar{x}_{1,2} + \bar{f}_1^{1[2]}(\bar{x}, \bar{u}) + \cdots + \bar{f}_1^{1[d]}(\bar{x}, \bar{u}) + O(\bar{x}, \bar{u})^{d+1}, \\ \dot{\bar{x}}_{1,2} & = & \bar{x}_{1,3} + \bar{f}_2^{1[2]}(\bar{x}, \bar{u}) + \cdots + \bar{f}_2^{1[d]}(\bar{x}, \bar{u}) + O(\bar{x}, \bar{u})^{d+1}, \\ \dot{\bar{x}}_{1,3} & = & \bar{u}_1, \\ \dot{\bar{x}}_{2,1} & = & \bar{x}_{2,2} + \bar{f}_1^{2[2]}(\bar{x}, \bar{u}) + \cdots + \bar{f}_1^{2[d]}(\bar{x}, \bar{u}) + O(\bar{x}, \bar{u})^{d+1}, \\ \dot{\bar{x}}_{2,2} & = & \bar{x}_{2,3} + \bar{f}_2^{2[2]}(\bar{x}, \bar{u}) + \cdots + \bar{f}_2^{2[d]}(\bar{x}, \bar{u}) + O(\bar{x}, \bar{u})^{d+1}, \\ \dot{\bar{x}}_{2,3} & = & \bar{u}_2, \end{array} \right.$$

where for any $2 \le m \le d$ and any $k = 1, 2$ we have

$$\begin{aligned}
\bar f_1^{k[m]}(\bar x, \bar u) &= \bar x_{1,3}^2 P_{1,3,1,1}^{k[m-2]}(\hat x_{1,3}, \hat x_{2,2}) + \bar x_{2,3}^2 P_{1,3,2,2}^{k[m-2]}(\hat x_{1,2}, \hat x_{2,3}) \\
&+ \bar x_{1,3}\bar x_{2,3} P_{1,3,1,2}^{k[m-2]}(\hat x_{1,3}, \hat x_{2,3}) + \bar x_{1,3}\bar x_{2,2} Q_{1,3,1,2}^{k[m-2]}(\hat x_{1,2}, \hat x_{2,2}) \\
&+ \bar u_1^2 P_{1,4,1,1}^{k[m-2]}(\hat u_1, \hat x_{2,3}) + \bar u_2^2 P_{1,4,2,2}^{k[m-2]}(\hat x_{1,3}, \hat u_2) + \bar u_1 \bar u_2 P_{1,4,1,2}^{k[m-2]}(\hat u_1, \hat u_2) \\
&+ \bar u_1 \bar x_{2,3} Q_{1,4,1,2}^{k[m-2]}(\hat x_{1,3}, \hat x_{2,3})
\end{aligned}$$

and

$$\begin{aligned}
\bar f_2^{k[m]}(\bar x, \bar u) &= \bar u_1^2 P_{2,4,1,1}^{k[m-2]}(\hat u_1, \hat x_{2,3}) + \bar u_2^2 P_{2,4,2,2}^{k[m-2]}(\hat x_{1,3}, \hat u_2) + \bar u_1 \bar u_2 P_{2,4,1,2}^{k[m-2]}(\hat u_1, \hat u_2) \\
&+ \bar u_1 \bar x_{2,3} Q_{2,4,1,2}^{k[m-2]}(\hat x_{1,3}, \hat x_{2,3}).
\end{aligned}$$

We recall that $\hat x_{1,i} = (\bar x_{1,1}, \ldots, \bar x_{1,i})$ and $\hat x_{2,i} = (\bar x_{2,1}, \ldots, \bar x_{2,i})$. Moreover,

$$\hat u_1 = (\bar x_{1,1}, \bar x_{1,2}, \bar x_{1,3}, \bar u_1) \text{ and } \hat u_2 = (\bar x_{2,1} \bar x_{2,2}, \bar x_{2,3}, \bar u_2).$$

When the initial system is affine in the control, then in the normal form, the homogeneous polynomials $P_{j,4,1,1}^{k[m-2]}(\hat u_1, \hat x_{2,3})$, $P_{j,4,2,2}^{k[m-2]}(\hat x_{1,3}, \hat u_2)$, and $P_{j,4,1,2}^{k[m-2]}(\hat u_1, \hat u_2)$ are all zero.

Generalization. Now let us assume that the controllability indices are not equal. Without loss of generality, we suppose that $1 \le r_1 \le \cdots \le r_p = r$. We then define the sequence of indices $d_1 \ge \cdots \ge d_p = 0$ so that $r_1 + d_1 = \cdots = r_p + d_p = r$.

It thus suffices to extend each subsystem, say the kth subsystem of (2.1) given by

$$\Pi_k^{\le d} : \begin{cases}
\dot x_{k,1} = x_{k,2} + \sum_{m=0}^{d} f_1^{k[m]}(z, x, u) + O(z, x, u)^{d+1}, \\
\quad \vdots \\
\dot x_{k,r-1} = x_{k,r} + \sum_{m=0}^{d} f_{r_k-1}^{k[m]}(z, x, u) + O(z, x, u)^{d+1}, \\
\dot x_{k,r} = u_k,
\end{cases}$$

as follows. We define $\tilde x_k = (\tilde x_{k,1}, \ldots, \tilde x_{k,r})$ so that

$$\tilde x_{k,d_k+1} = x_{k,1}, \ldots, \tilde x_{k,r} = x_{k,r_k} \quad \text{and} \quad \dot{\tilde x}_{k,1} = \tilde x_{k,2}, \ldots, \dot{\tilde x}_{k,d_k} = x_{k,1}.$$

This means that the kth subsystem is extended as

$$\tilde\Pi_k^{\le d} : \begin{cases}
\dot{\tilde x}_{k,1} = \tilde x_{k,2}, \\
\quad \vdots \\
\dot{\tilde x}_{k,d_k} = \tilde x_{k,d_k+1}, \\
\dot{\tilde x}_{k,d_k+1} = \tilde x_{k,d_k+2} + \sum_{m=0}^{d} \tilde f_{d_k+1}^{k[m]}(z, \tilde x, u) + O(z, \tilde x, u)^{d+1}, \\
\quad \vdots \\
\dot{\tilde x}_{k,r-1} = \tilde x_{k,r} + \sum_{m=0}^{d} \tilde f_{r-1}^{k[m]}(z, \tilde x, u) + O(z, \tilde x, u)^{d+1}, \\
\dot{\tilde x}_{k,r} = u_k,
\end{cases}$$

where

$$\tilde f_{d_k+1}^{k[m]}(z, \tilde x, u) = f_1^{k[m]}(z, x, u), \ldots, \tilde f_{r-1}^{k[m]}(z, \tilde x, u) = f_{r_k-1}^{k[m]}(z, x, u).$$

In this case all extended subsystems will have the same controllability index r, and by Theorem 3.1 the extended system will be equivalent to the normal form $\Pi_{NF}^{\leq d}$ given by (3.2)–(3.3) with $\bar{g}_j^{[m-1]}(\bar{z},\bar{x},\bar{u})$ and $\bar{f}_j^{k[m]}(\bar{z},\bar{x},\bar{u})$ *depending exclusively on the variables* $\bar{z},\bar{u}$ and $\bar{x}_{s,d_s+1},\ldots,\bar{x}_{s,r}$ (not on the added variables $\bar{x}_{s,1},\ldots,\bar{x}_{s,d_s}$) for all $1 \leq s \leq p$. Moreover, the first d_k components $\bar{f}_1^{k[m]}(\bar{z},\bar{x},\bar{u}),\ldots,\bar{f}_{d_k}^{k[m]}(\bar{z},\bar{x},\bar{u})$ remain zero (see Example 1 for illustration).

3.2. Examples. In this subsection we will illustrate our results by considering two physical examples: The MVWT and the prototype of a PVTOL.

Example 1. Multi-Vehicle Wireless Testbed. We consider the MVWT, presented in [8, 9] and we study the normal form of one vehicle. The equations of motion of an MVWT vehicle (see [8, 9]) are given by

$$\begin{cases} m\ddot{x} &= -\eta\dot{x} + (F_s+F_p)\cos\theta, \\ m\ddot{y} &= -\eta\dot{y} + (F_s+F_p)\sin\theta, \\ J\ddot{\theta} &= -\psi\dot{\theta} + (F_s-F_p)l, \end{cases}$$

where (x,y) denotes the position of the center mass of the vehicle, θ the angle of the axis of the vehicle with the horizontal (x-axis), m the mass of the vehicle, J the rotational inertia, F_s and F_p denote the starboard and port fan forces, respectively, and l (r in [8, 9]) the common moment arm of the forces. The center mass of the vehicle and the center of geometry are assumed to coincide. The constants η and ψ stand, respectively, for the coefficients of viscous friction and rotational friction.

Let us introduce the variables

$$\begin{array}{llllllll} z_1 = y, & x_{1,1} = x, & x_{2,1} = \theta, & u_1 = F_s+F_p, \\ z_2 = \dot{z}_1, & x_{1,2} = \dot{x}_{1,1}, & x_{2,2} = \dot{x}_{2,1}, & u_2 = F_s-F_p. \end{array}$$

The equations of motion of an MVWT vehicle are rewritten as

$$\begin{cases} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -\eta z_2 + u_1\sin x_{2,1}, \\ \dot{x}_{1,1} &= x_{1,2}, \\ \dot{x}_{1,2} &= -\eta x_{1,2} + u_1\cos x_{2,1}, \\ \dot{x}_{2,1} &= x_{2,2}, \\ \dot{x}_{2,2} &= -\phi x_{2,2} + u_2 l. \end{cases} \tag{3.4}$$

We can notice that the system is affine and its distribution $\mathcal{G} = \operatorname{span}\{g_1,g_2\}$, where

$$g_1 = \begin{pmatrix} 0 \\ \sin x_{2,1} \\ 0 \\ \cos x_{2,1} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

is involutive and of constant rank 2. An equilibrium point for the system (3.4) is given by any constant position and orientation

$$(z_1^e, z_2^e, x_{1,1}^e, x_{1,2}^e, x_{2,1}^e, x_{2,2}^e)^T = (z_1, 0, x_{1,1}, 0, x_{2,1}, 0)^T.$$

The linearization of the system (3.4) around an equilibrium (we assume $x_{2,1} = 0$) is given by

$$\left\{\begin{array}{lcl} \dot{z}_1 &=& z_2, \\ \dot{z}_2 &=& -\eta z_2, \\ \dot{x}_{1,1} &=& x_{1,2}, \\ \dot{x}_{1,2} &=& -\eta x_{1,2} + u_1, \\ \dot{x}_{2,1} &=& x_{2,2}, \\ \dot{x}_{2,2} &=& -\phi x_{2,2} + u_2 l. \end{array}\right.$$

It is easy to see that this linear system is not controllable because

$$\operatorname{span}\left\{ F^i G_k,\ 0 \le i \le 5,\ 1 \le k \le 2 \right\} = \mathbb{R}^4,$$

where

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -\phi \end{pmatrix}, \qquad G_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad G_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

It thus follows that $q = 2$, and the computation of the controllability matrix shows that $r_1 = r_2 = 2$.

The feedback transformation defined by

$$u_1 = \frac{1}{\cos x_{2,1}} \bar{u}_1 + \eta \frac{x_{1,2}}{\cos x_{2,1}} \qquad \text{and} \qquad \bar{u}_2 = \frac{u_2}{l} + \frac{\phi}{l} x_{2,2}$$

takes the system into the following form:

$$\left\{\begin{array}{lcl} \dot{z}_1 &=& z_2 \\ \dot{z}_2 &=& -\eta z_2 + \eta x_{1,2} \tan x_{2,1} + \bar{u}_1 \tan x_{2,1}, \\ \dot{x}_{1,1} &=& x_{1,2}, \\ \dot{x}_{1,2} &=& \bar{u}_1, \\ \dot{x}_{2,1} &=& x_{2,2}, \\ \dot{x}_{2,2} &=& \bar{u}_2. \end{array}\right.$$

The change of coordinates given by

$$\left\{\begin{array}{lcl} \bar{z}_1 &=& z_1, \\ \bar{z}_2 &=& z_2 - x_{1,2} \tan x_{2,1}, \\ \bar{x}_{1,1} &=& x_{1,1}, \\ \bar{x}_{1,2} &=& x_{1,2}, \\ \bar{x}_{2,1} &=& x_{2,1}, \\ \bar{x}_{2,2} &=& x_{2,2} \end{array}\right.$$

brings the system into the form

$$\left\{\begin{array}{lcl} \dot{\bar{z}}_1 &=& \bar{z}_2 + \bar{x}_{1,2} \tan \bar{x}_{2,1}, \\ \dot{\bar{z}}_2 &=& -\eta \bar{z}_2 - \bar{x}_{1,2} \bar{x}_{2,2} (1 + \tan^2 \bar{x}_{2,1}), \\ \dot{\bar{x}}_{1,1} &=& \bar{x}_{1,2}, \\ \dot{\bar{x}}_{1,2} &=& \bar{u}_1, \\ \dot{\bar{x}}_{2,1} &=& \bar{x}_{2,2}, \\ \dot{\bar{x}}_{2,2} &=& \bar{u}_2. \end{array}\right.$$

Since

$$\bar{x}_{1,2}\tan\bar{x}_{2,1}=\bar{x}_{1,2}\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\bar{x}_{2,1}^{2\nu+1}}{(2\nu+1)!}=\bar{x}_{1,2}\bar{x}_{2,1}\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\bar{x}_{2,1}^{2\nu}}{(2\nu+1)!},$$

$$\bar{x}_{1,2}\bar{x}_{2,2}(1+\tan^2\bar{x}_{2,1})=\bar{x}_{1,2}\bar{x}_{2,2}\left(1+\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\bar{x}_{2,1}^{2\nu+1}}{(2\nu+1)!}\right),$$

we conclude that the system is in normal form (compare with Theorem 3.1), with

$$\bar{g}_1^{[m-1]}(\bar{x})=\begin{cases}\bar{x}_{1,2}\bar{x}_{2,1}Q_{1,2}^{[m-3]}(\bar{x})=\pm\bar{x}_{1,2}\bar{x}_{2,1}\frac{\bar{x}_{2,1}^{m-3}}{(m-2)!} & \text{if } m \text{ is even,}\\ 0 & \text{if } m \text{ is odd}\end{cases}$$

and

$$\bar{g}_2^{[m-1]}(\bar{x})=\begin{cases}-\bar{x}_{1,2}\bar{x}_{2,2} & \text{if } m=3,\\ \bar{x}_{1,2}\bar{x}_{2,2}P_{2,2}^{[m-3]}(\bar{x})=\pm\bar{x}_{1,2}\bar{x}_{2,2}\frac{\bar{x}_{2,1}^{m-3}}{(m-3)!} & \text{if } m\geq 4 \text{ is odd,}\\ 0 & \text{if } m \text{ is even.}\end{cases}$$

Example 2. Planar Vertical TakeOff and Landing. In this example we study a simple toy aircraft of prototype PVTOL presented in [15, 40]. The equations of motion of the PVTOL (see [15, 40]) are given by

$$\begin{cases}\ddot{x} &= -\sin\theta u_1+\epsilon^2\cos\theta u_2,\\ \ddot{y} &= \cos\theta u_1+\epsilon^2\sin\theta u_2-1,\\ \ddot{\theta} &= u_2,\end{cases}$$

where (x,y) denotes the position of the center mass of the aircraft, θ the angle of the aircraft relative to the x-axis, "-1" the gravitational acceleration, and $\epsilon\neq 0$ the (small) coefficient giving the coupling between the rolling moment and the lateral acceleration of the aircraft. The control inputs u_1 and u_2 are the thrust (directed out the bottom of the aircraft) and the rolling moment.

We introduce the variables

$$\begin{array}{lllllllllllll} x_{1,1} &=& y, & x_{2,1} &=& x, & x_{2,3} &=& \theta, & w_1 &=& u_1-1,\\ x_{1,2} &=& \dot{x}_{1,1}, & x_{2,2} &=& \dot{x}_{2,1}, & x_{2,4} &=& \dot{x}_{2,3}, & w_2 &=& u_2.\end{array}$$

The equations of motion of the PVTOL are rewritten as

$$\begin{cases}\dot{x}_{1,1} &= x_{1,2},\\ \dot{x}_{1,2} &= \cos x_{2,3}w_1+\epsilon^2\sin x_{2,3}w_2+\cos x_{2,3}-1,\\ \dot{x}_{2,1} &= x_{2,2},\\ \dot{x}_{2,2} &= -\sin x_{2,3}w_1+\epsilon^2\cos x_{2,3}w_2-\sin x_{2,3},\\ \dot{x}_{2,3} &= x_{2,4},\\ \dot{x}_{2,4} &= w_2.\end{cases}\tag{3.5}$$

The system is affine and its distribution $\mathcal{G}=\operatorname{span}\{g_1,g_2\}$, given by

$$g_1=\begin{pmatrix}0\\ \cos x_{2,3}\\ 0\\ -\sin x_{2,3}\\ 0\\ 0\end{pmatrix}\qquad\text{and}\qquad g_2=\begin{pmatrix}0\\ \epsilon^2\sin x_{2,3}\\ 0\\ \epsilon^2\cos x_{2,3}\\ 0\\ 1\end{pmatrix},$$

is involutive and of constant rank 2. The equilibria is defined by

$$(x_{1,1}^e, x_{1,2}^e, x_{2,1}^e, x_{2,2}^e, x_{2,3}^e, x_{2,4}^e, w_1^e, w_2^e)^T = (c, 0, 0, 0, 0, 0, 0, 0)^T,$$

where c is any constant. The linearization of the system (3.5) around the equilibria is given by

$$\begin{cases} \dot{x}_{1,1} &=& x_{1,2}, \\ \dot{x}_{1,2} &=& w_1, \\ \dot{x}_{2,1} &=& x_{2,2}, \\ \dot{x}_{2,2} &=& -x_{2,3} + \epsilon^2 w_2, \\ \dot{x}_{2,3} &=& x_{2,4}, \\ \dot{x}_{2,4} &=& w_2. \end{cases}$$

It is easy to see that the linear system is controllable with controllability indices $r_1 = 2$ and $r_2 = 4$. Indeed,

$$\operatorname{span}\left\{ G_1, FG_1, G_2, FG_2, F^2G_2, F^3G_2 \right\} = \mathbb{R}^6,$$

where

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad G_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad G_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \epsilon^2 \\ 0 \\ 1 \end{pmatrix}.$$

Since $r_1 = 2 < r_2 = 4$ we have $d_1 = 2$ and $d_2 = 0$. Thus we extend the system as

$$\begin{cases} \dot{\tilde{x}}_{1,1} &=& \tilde{x}_{1,2}, \\ \dot{\tilde{x}}_{1,2} &=& \tilde{x}_{1,3}, \\ \dot{\tilde{x}}_{1,3} &=& \tilde{x}_{1,4}, \\ \dot{\tilde{x}}_{1,4} &=& \cos\tilde{x}_{2,3} w_1 + \epsilon^2 \sin\tilde{x}_{2,3} w_2 + \cos\tilde{x}_{2,3} - 1, \\ \dot{\tilde{x}}_{2,1} &=& \tilde{x}_{2,2}, \\ \dot{\tilde{x}}_{2,2} &=& -\sin\tilde{x}_{2,3} w_1 + \epsilon^2 \cos\tilde{x}_{2,3} w_2 - \sin\tilde{x}_{2,3}, \\ \dot{\tilde{x}}_{2,3} &=& \tilde{x}_{2,4}, \\ \dot{\tilde{x}}_{2,4} &=& w_2, \end{cases}$$

where

$$\tilde{x}_{1,3} = x_{1,1}, \quad \tilde{x}_{1,4} = x_{1,2}, \quad \tilde{x}_{2,1} = x_{2,1}, \quad \tilde{x}_{2,2} = x_{2,2}, \quad \tilde{x}_{2,3} = x_{2,3}, \quad \tilde{x}_{2,4} = x_{2,4}.$$

The feedback transformation defined by

$$w_1 = \frac{1}{\cos\tilde{x}_{2,1}} v_1 - \epsilon^2 \tan\tilde{x}_{2,1} v_2 + \frac{1}{\cos\tilde{x}_{2,1}} - 1 \qquad \text{and} \qquad w_2 = v_2$$

takes the system into the following form:

$$\begin{cases} \dot{\tilde{x}}_{1,1} &=& \tilde{x}_{1,2}, \\ \dot{\tilde{x}}_{1,2} &=& \tilde{x}_{1,3}, \\ \dot{\tilde{x}}_{1,3} &=& \tilde{x}_{1,4}, \\ \dot{x}_{1,4} &=& v_1, \\ \dot{\tilde{x}}_{2,1} &=& \tilde{x}_{2,2}, \\ \dot{\tilde{x}}_{2,2} &=& -\tan\tilde{x}_{2,3}v_1 + \frac{\epsilon^2}{\cos\tilde{x}_{2,3}}v_2 - \tan\tilde{x}_{2,3}, \\ \dot{\tilde{x}}_{2,3} &=& \tilde{x}_{2,4}, \\ \dot{\tilde{x}}_{2,4} &=& v_2. \end{cases}$$

The change of coordinates given by

$$\begin{cases} \bar{x}_{1,1} &=& \tilde{x}_{1,1}, \\ \bar{x}_{1,2} &=& \tilde{x}_{1,2}, \\ \bar{x}_{1,3} &=& \tilde{x}_{1,3} - \epsilon^2\int_0^{\tilde{x}_{2,3}}\frac{dt}{\cos t}, \\ \bar{x}_{1,4} &=& \tilde{x}_{1,4} + \tilde{x}_{1,4}\tan\tilde{x}_{2,3} - \frac{\epsilon^2}{\cos\tilde{x}_{2,3}}\tilde{x}_{2,4}, \\ \bar{x}_{2,1} &=& \tilde{x}_{2,1}, \\ \bar{x}_{2,2} &=& \tilde{x}_{2,2}, \\ \bar{x}_{2,3} &=& -\tan\tilde{x}_{2,3}, \\ \bar{x}_{2,4} &=& -\tilde{x}_{2,4}(1+\tan^2\tilde{x}_{2,3}) = \dot{\bar{x}}_{2,3} \end{cases}$$

followed by the feedback

$$\bar{u}_1 = v_1, \text{ and } \bar{u}_2 = \dot{\bar{x}}_{2,4} = -v_2(1+\tan^2\tilde{x}_{2,3}) - 2\tilde{x}_{2,4}^2\tan\tilde{x}_{2,3}(1+\tan^2\tilde{x}_{2,3})$$

brings the system into

$$\begin{cases} \dot{\bar{x}}_{1,1} &=& \bar{x}_{1,2}, \\ \dot{\bar{x}}_{1,2} &=& \bar{x}_{1,3}, \\ \dot{\bar{x}}_{1,3} &=& \bar{x}_{1,4}, \\ \dot{\bar{x}}_{1,4} &=& \bar{u}_1, \\ \dot{\bar{x}}_{2,1} &=& x_{2,2} + \bar{x}_{1,4}\bar{x}_{2,3}, \\ \dot{\bar{x}}_{2,2} &=& \bar{x}_{2,3} - \bar{x}_{1,4}\bar{x}_{2,4} + \epsilon^2(1-\bar{x}_{2,3}^2)\bar{x}_{2,4}^2, \\ \dot{\bar{x}}_{2,3} &=& \bar{x}_{2,4}, \\ \dot{\bar{x}}_{2,4} &=& \bar{u}_2. \end{cases}$$

Comparing with Corollary 3.2 we get

$$\begin{aligned} \bar{f}_1^{2[2]}(\bar{x}) &= \bar{x}_{1,4}\bar{x}_{2,3}Q_{1,4,1,2}^{2[0]}(\bar{x}), \\ \bar{f}_2^{2[2]}(\bar{x}) &= \bar{x}_{1,4}\bar{x}_{2,4}P_{2,4,1,2}^{2[0]}(\bar{x}) + \bar{x}_{2,4}\bar{x}_{2,4}P_{2,4,2,2}^{2[0]}(\bar{x}), \\ \bar{f}_2^{2[4]}(\bar{x}) &= \bar{x}_{2,4}\bar{x}_{2,4}P_{2,4,2,2}^{2[2]}(\pi_{2,4}^2(\bar{x})), \end{aligned}$$

where

$$Q_{1,4,1,2}^{2[0]}(\bar{x}) \equiv 1, \quad P_{2,4,1,2}^{2[0]}(\bar{x}) \equiv -1, \quad P_{2,4,2,2}^{2[0]}(\bar{x}) = \epsilon^2, \quad P_{2,4,2,2}^{2[2]}(\bar{x}) = -\epsilon^2\bar{x}_{2,3}^2.$$

This means that the system (3.5) is equivalent to the normal form

$$\begin{cases} \dot{\bar{x}}_{1,3} &=& \bar{x}_{1,4}, \\ \dot{\bar{x}}_{1,4} &=& \bar{u}_1, \\ \dot{\bar{x}}_{2,1} &=& x_{2,2} + \bar{x}_{1,4}\bar{x}_{2,3}, \\ \dot{\bar{x}}_{2,2} &=& \bar{x}_{2,3} - \bar{x}_{1,4}\bar{x}_{2,4} + \epsilon^2(1-\bar{x}_{2,3}^2)\bar{x}_{2,4}^2, \\ \dot{\bar{x}}_{2,3} &=& \bar{x}_{2,4}, \\ \dot{\bar{x}}_{2,4} &=& \bar{u}_2. \end{cases}$$

Notice that the *added variables* $\bar{x}_{1,1}$ and $\bar{x}_{1,2}$ are not present in the normal form. We may also notice that in both *Examples* 1 and 2 the transformations taking the corresponding systems into their normal forms are smooth, actually they are analytic. Though little is known about the convergence of the formal transformations taking a system into its normal form, this gives hope.

3.3. Nearby controllability. In this subsection we generalize a result obtained earlier in collaboration with Kang et al. [30]. We proved that for systems with one nonzero uncontrollable mode the set of equilibria is a smooth curve passing by the origin. Moreover, provided that some term in the normal form is nonzero, the system becomes linearly controllable at these equilibria (except at the origin). This is called a *bifurcation of controllability* and the conclusion drawn from this study is that we can stabilize the system at any nearby point of the origin.

A parallel analysis could be made for multi-input systems with one nonzero uncontrollable mode. Indeed, consider the system $\Pi^{\leq d}$ defined by (2.1) and assume that $q = 1$, i.e., $J = \lambda$. The equilibria set of this system is

$$E = \left\{ (z, x) \in \mathbb{R} \times \mathbb{R}^{n-1} \text{ such that } \exists u \in \mathbb{R}^p : H(z, x, u) = 0 \right\},$$

where $H(z, x, u) = (g(z, x, u), f(z, x, u))$ with

$$\begin{aligned} g(z, x, u) &= \lambda z + \sum_{m=1}^{d} g^{[m-1]}(z, x, u) + O(z, x, u)^d, \\ f(z, x, u) &= Ax + Bu + \sum_{m=0}^{d} f^{[m]}(z, x, u) + O(z, x, u)^{d+1}. \end{aligned}$$

We will show that E is a surface parameterized by $x^1 = (x_{1,1}, x_{2,1}, \ldots, x_{p,1})^T$. If we denote by $\mathbf{x}_s = (x_{s,2}, \ldots, x_{s,r})^T$ for all $1 \leq s \leq p$ and $\mathbf{x} = (\mathbf{x}_1^T, \ldots, \mathbf{x}_p^T)$, then we have

$$\left.\frac{\partial H(z, x, u)}{\partial(z, \mathbf{x}, u)}\right|_{(z,x,u)=0} = \text{diag } (\lambda, \mathrm{Id}_{\mathbb{R}^{n-1}}).$$

Since $\lambda \neq 0$, the matrix diag $(\lambda, \mathrm{Id}_{\mathbb{R}^{n-1}})$ is invertible. The *implicit function theorem* implies that the equation

$$H(z, x, u) = H(z, x^1, \mathbf{x}, u) = 0$$

has a solution in a neighborhood of the origin parameterized by the variables x^1, that is, there exist functions

$$z = z^e(x^1), \quad \mathbf{x} = \mathbf{x}^e(x^1), \quad u = u^e(x^1)$$

so that for some open neighborhood $\mathcal{V}$ of the origin in $\mathbb{R}^p$, we have

$$H(z^e(x^1), x^1, \mathbf{x}^e(x^1), u^e(x^1)) = 0 \text{ for all } x^1 \in \mathcal{V} \subset \mathbb{R}^p.$$

We thus deduce that the equilibria set is a surface parameterized by the variables x^1.

Let us denote by E_0 the subset of E defined by

$$E_0 = \{ (z, x) \in E \text{ such that } x_{s,1} \neq 0 \text{ for all } 1 \leq s \leq p \}.$$

Consider the normal form of $\Pi^{\leq d}$ given by (3.1)–(3.3), where (since $q=1$) we have

$$\begin{aligned} \bar{g}_1^{[m-1]}(\bar{z},\bar{x},\bar{u}) &= \sum_{1\leq s\leq t\leq p}\sum_{i=1}^{r+1}\bar{x}_{s,i}\bar{x}_{t,i}P_{1,i,s,t}^{[m-3]}(\bar{z},\pi_{t,i}^{s}(\bar{x})) \\ &+ \sum_{1\leq s<t\leq p}\sum_{i=2}^{r+1}\bar{x}_{s,i}\bar{x}_{t,i-1}Q_{1,i,s,t}^{[m-3]}(\bar{z},\pi_{t,i-1}^{t}(\bar{x})) \end{aligned}$$

for any $m\geq 3$.

An analogous result to that given in [30] could be formulated as follows.

THEOREM 3.3. *Consider the system $\Pi^{\leq d}$ defined by (2.1) for $d\in\mathbb{N}\cup\{\infty\}$ sufficiently large enough. If there are integers $3\leq m\leq d$ and $1\leq s\leq t\leq p$ so that*

$$P_{1,1,s,t}^{[m-3]}(\bar{z},\pi_{t,1}^{s}(\bar{x}))\Big|_{\bar{z}=0} = P_{1,1,s,t}^{[m-3]}(\bar{x}_{1,1},\ldots,\bar{x}_{s,1},\bar{x}_{t,1})\neq 0,$$

then the system $\Pi^{\leq d}$ is linearly controllable at any point of E_0.

The proof of this result is straightforward and follows the same steps as in [30]. The domain where the system is linearly controllable could be, of course, larger than the subset E_0, but this will depend on the normal form.

Example 3. Consider the system

$$\begin{cases} \dot{z} &= z+x_{1,1}x_{2,1}, \\ \dot{x}_{1,1} &= x_{1,2}, \\ \dot{x}_{1,2} &= x_{1,3}, \\ \dot{x}_{1,3} &= u_1, \\ \dot{x}_{2,1} &= x_{2,2}, \\ \dot{x}_{2,2} &= u_2, \end{cases}$$

which is already in normal form. Its equilibria set is given by

$$E=\left\{\,(-x_{1,1}x_{2,1},x_{1,1},0,0,x_{2,1},0)^T\in\mathbb{R}^6:\ (x_{1,1},x_{2,1})^T\in\mathbb{R}^2\,\right\}.$$

The system is not linearly controllable at the origin but it is at any other point of E. Indeed, put

$$f(z,x,u)=(z+x_{1,1}x_{2,1},x_{1,2},x_{1,3},0,x_{2,2},0)^T,$$

$$g_1(z,x,u)=(0,0,0,1,0,0)^T,\ \text{and}\ g_2(z,x,u)=(0,0,0,0,0,1)^T.$$

(i) If $x_{1,1}\neq 0$, we have

$$\operatorname{span}\left\{\,g_1,ad_fg_1,ad_f^2g_1,g_2,ad_fg_2,ad_f^2g_2\,\right\}(z,x)=\mathbb{R}^6$$

for all $(z,x)\neq(0,0)$ in E.

(ii) If $x_{2,1}\neq 0$, we have

$$\operatorname{span}\left\{\,g_1,ad_fg_1,ad_f^2g_1,ad_f^3g_1,g_2,ad_fg_2,\,\right\}(z,x)=\mathbb{R}^6$$

for all $(z,x)\neq(0,0)$ in E.

In this example the linear controllability occurs outside the subset E_0 for which we have $\bar{x}_{1,1}\neq 0$ and $\bar{x}_{2,1}\neq 0$.

However, if we replace $\dot{z}=z+x_{1,1}x_{2,1}$ by $\dot{z}=z+x_{1,1}^2x_{2,1}^2$, then the linear controllability will occur only inside E_0.

4. Proofs of main results. The aim of this section is to prove Theorem 3.1. The proof of Corollary 3.2 is given in [42].

Consider the system $\Pi^{\leq d}$ defined by (2.1). Following [44] it is possible to show that the terms $f^{[0]}(z)$ and $f^{[1]}(z,x)$ can be removed. Thus, without loss of generality we will assume that the system $\Pi^{\leq d}$ is of the form

$$\Pi^{\leq d} : \begin{cases} \dot z = Jz + g^{[0]}(z) + \sum\limits_{m=2}^{d} g^{[m-1]}(z,x,u) + O(z,x,u)^d, \\ \dot x = Ax + Bu + \sum\limits_{m=2}^{d} f^{[m]}(z,x,u) + O(z,x,u)^{d+1}. \end{cases} \tag{4.1}$$

We like to study the action of the feedback transformation

$$\Upsilon^m : \begin{cases} \bar z &= & z + \psi^{[m-1]}(z,x), \\ \bar x &= & x + \phi^{[m]}(z,x), \\ u &= & \bar u + \gamma^{[m]}(z,x,\bar u) \end{cases}$$

on the system $\Pi^{\leq d}$ up to some degree. First, remark that the inverse of this transformation is such that

$$\begin{cases} z &= & \bar z - \psi^{[m-1]}(\bar z,\bar x) + O(\bar z,\bar x)^m, \\ x &= & \bar x - \phi^{[m]}(\bar z,\bar x) + O(\bar z,\bar x)^{m+1}, \\ u &= & \bar u + \gamma^{[m]}(\bar z,\bar x,\bar u) + O(\bar z,\bar x,\bar u)^{m+1}. \end{cases}$$

Then the uncontrollable part is transformed as

$$\begin{aligned} \dot{\bar z} &= \dot z + \frac{\partial \psi^{[m-1]}}{\partial z}(z,x)\dot z + \frac{\partial \psi^{[m-1]}}{\partial x}(z,x)\dot x \\ &= Jz + g^{[0]}(z) + \cdots + g^{[m-1]}(z,x,u) + O(z,x,u)^m \\ &\quad + \frac{\partial \psi^{[m-1]}}{\partial z}(z,x)(Jz + g^{[0]}(z)) + \frac{\partial \psi^{[m-1]}}{\partial x}(z,x)(Ax+Bu) + O(z,x,u)^m \\ &= J\bar z + g^{[0]}(\bar z) + \cdots + g^{[m-1]}(\bar z,\bar x,\bar u) - \left(J + \frac{\partial g^{[0]}}{\partial z}(\bar z)\right)\psi^{[m-1]}(\bar z,\bar x) \\ &\quad + \frac{\partial \psi^{[m-1]}}{\partial z}(\bar z,\bar x)(J\bar z + g^{[0]}(\bar z)) + \frac{\partial \psi^{[m-1]}}{\partial x}(\bar z,\bar x)(A\bar x+B\bar u) + O(\bar z,\bar x,\bar u)^m. \end{aligned}$$

It clearly appears that the terms of degree $m-2$ or less of the uncontrollable part remain unmodified while the terms of degree $m-1$ or higher are modified.

Similarly, we can show that

$$\begin{aligned} \dot{\bar x} &= A\bar x + B\bar u + f^{[2]}(\bar z,\bar x,\bar u) + \cdots + f^{[m]}(\bar z,\bar x,\bar u) - A\phi^{[m]}(\bar z,\bar x) - B\gamma^{[m]}(\bar z,\bar x,\bar u) \\ &\quad + \frac{\partial \phi^{[m]}}{\partial z}(\bar z,\bar x)(J\bar z + g^{[0]}(\bar z)) + \frac{\partial \phi^{[m]}}{\partial x}(\bar z,\bar x)(A\bar x+B\bar u) + O(\bar z,\bar x,\bar u)^{m+1}, \end{aligned}$$

which means that the terms of degree $m-1$ or less of the controllable part are preserved while the terms of degree m or higher are modified.

To study the action of the feedback transformation Υ^m on the terms of degree m (terms of degree $m-1$ of the uncontrollable part and of degree m for the controllable part) of the system $\Pi^{\leq d}$, it is enough to study their action on a *homogeneous system* of the form

$$\Pi^m : \begin{cases} \dot z = Jz + g^{[0]}(z) + g^{[m-1]}(z,x,u), \\ \dot x = Ax + Bu + f^{[m]}(z,x,u). \end{cases} \tag{4.2}$$

The proof of Theorem 3.1 will follow if we show that, by a feedback transformation Υ^m, we can take the system (4.2) into the normal form

$$\Pi^m_{NF} : \begin{cases} \dot{\bar{z}} = J\bar{z} + g^{[0]}(\bar{z}) + \bar{g}^{[m-1]}(\bar{z}, \bar{x}, \bar{u}), \\ \dot{\bar{x}} = A\bar{x} + B\bar{u} + \bar{f}^{[m]}(\bar{z}, \bar{x}, \bar{u}), \end{cases}$$

where the components of $\bar{g}^{[m-1]}(\bar{z}, \bar{x}, \bar{u})$ and $\bar{f}^{[m]}(\bar{z}, \bar{x}, \bar{u})$ are given by (3.1)–(3.3).

Indeed, if this is true we then consider the system $\Pi^{\leq d}$ of the form (4.1) and we first apply a quadratic feedback transformation Υ^2 to take it to the form

$$\Pi^{\leq d} : \begin{cases} \dot{\bar{z}} = J\bar{z} + g^{[0]}(\bar{z}) + \bar{g}^{[1]}(\bar{z}, \bar{x}, \bar{u}) + \sum\limits_{m=3}^{d} g^{[m-1]}(\bar{z}, \bar{x}, \bar{u}) + O(\bar{z}, \bar{x}, \bar{u})^d, \\ \dot{\bar{x}} = A\bar{x} + B\bar{u} + \bar{f}^{[2]}(\bar{z}, \bar{x}, \bar{u}) + \sum\limits_{m=3}^{d} f^{[m]}(\bar{z}, \bar{x}, \bar{u}) + O(\bar{z}, \bar{x}, \bar{u})^{d+1}, \end{cases}$$

where the vector fields $\bar{g}^{[1]}(\bar{z}, \bar{x}, \bar{u})$, and $\bar{f}^{[2]}(\bar{z}, \bar{x}, \bar{u})$ are in their normal forms, and the vector fields $g^{[m-1]}(\bar{z}, \bar{x}, \bar{u})$ and $f^{[m]}(\bar{z}, \bar{x}, \bar{u})$ stand for the new transformed vector fields. We thus apply a cubic transformation Υ^3 to take the system above into the form

$$\Pi^{\leq d} : \begin{cases} \dot{\bar{z}} = J\bar{z} + g^{[0]}(\bar{z}) + \bar{g}^{[1]}(\bar{z}, \bar{x}, \bar{u}) + \bar{g}^{[2]}(\bar{z}, \bar{x}, \bar{u}) + \sum\limits_{m=4}^{d} g^{[m-1]}(\bar{z}, \bar{x}, \bar{u}) + O(\bar{z}, \bar{x}, \bar{u})^d, \\ \dot{\bar{x}} = A\bar{x} + B\bar{u} + \bar{f}^{[2]}(\bar{z}, \bar{x}, \bar{u}) + \bar{f}^{[3]}(\bar{z}, \bar{x}, \bar{u}) + \sum\limits_{m=4}^{d} f^{[m]}(\bar{z}, \bar{x}, \bar{u}) + O(\bar{z}, \bar{x}, \bar{u})^{d+1}, \end{cases}$$

where $\bar{g}^{[1]}(\bar{z}, \bar{x}, \bar{u})$, $\bar{g}^{[2]}(\bar{z}, \bar{x}, \bar{u})$, $\bar{f}^{[2]}(\bar{z}, \bar{x}, \bar{u})$, and $\bar{f}^{[3]}(\bar{z}, \bar{x}, \bar{u})$ are in their normal forms. The vectors $g^{[m-1]}(\bar{z}, \bar{x}, \bar{u})$ and $f^{[m]}(\bar{z}, \bar{x}, \bar{u})$, for $m \geq 4$ are the new transformed vector fields. The process continues until the original system is in the desired normal form.

Proof of Theorem 3.1. As stated above, we need to prove that the homogeneous system $\Pi^{[m]}$ could be transformed into the normal form $\Pi^{[m]}_{NF}$ by a homogeneous transformation Υ^m. The proof will be divided into two parts. In the first part we will deal with the controllable mode and in the second part we will consider the uncontrollable mode.

(i) Consider the kth subsystem

$$\Pi^{k[m]} : \begin{cases} \dot{x}_{k,1} &= x_{k,2} + f_1^{k[m]}(z, x, u), \\ &\vdots \\ \dot{x}_{k,r-1} &= x_{k,r} + f_{r-1}^{k[m]}(z, x, u), \\ \dot{x}_{k,r} &= u_k. \end{cases}$$

Let us denote by $\mathfrak{P}^m(\mathbb{R}^q \times \mathbb{R}^{n-q} \times \mathbb{R}^p)$ the set of all homogeneous polynomials of degree m in the variables $(x, u) \in \mathbb{R}^{n-q} \times \mathbb{R}^p$ whose coefficients are functions of the variable $z \in \mathbb{R}^q$.

For a fixed j, $1 \leq j \leq r - 1$ we define the set $\mathfrak{F}_j^m(\mathbb{R}^q \times \mathbb{R}^{n-q} \times \mathbb{R}^p)$ of all homogeneous polynomials $h^{[m]}(z,x,v) \in \mathfrak{P}^m(\mathbb{R}^q \times \mathbb{R}^{n-q} \times \mathbb{R}^p)$ such that

$$h^{[m]}(z,x,v) = \sum_{1\le s\le t\le p} \sum_{i=j+2}^{r+1} x_{s,i}x_{t,i}P^{k[m-2]}_{j,i,s,t}(z,\pi^s_{t,i}(x)) + \sum_{1\le s<t\le p} \sum_{i=j+2}^{r+1} x_{s,i}x_{t,i-1}Q^{k[m-2]}_{j,i,s,t}(z,\pi^t_{t,i-1}(x)).$$

For simplicity we will just refer to $\mathfrak{P}^m(\mathbb{R}^q \times \mathbb{R}^{n-q} \times \mathbb{R}^p)$ and $\mathfrak{F}^m_j(\mathbb{R}^q \times \mathbb{R}^{n-q} \times \mathbb{R}^p)$ as $\mathfrak{P}^m$ and $\mathfrak{F}^m_j$, respectively. Denote by $\mathfrak{C}^m_j$ the subspace of $\mathfrak{P}^m$ so that

$$\mathfrak{P}^m = \mathfrak{F}^m_j \oplus \mathfrak{C}^m_j.$$

We want to prove that the subsystem $\Pi^{k[m]}$ could be transformed into the normal form

$$\Pi^{k[m]}_{NF} : \begin{cases} \dot{\bar{x}}_{k,1} &= \bar{x}_{k,2} + \bar{f}^{k[m]}_1(\bar{z},\bar{x},\bar{u}), \\ &\vdots \\ \dot{\bar{x}}_{k,r-1} &= \bar{x}_{k,r} + \bar{f}^{k[m]}_{r-1}(\bar{z},\bar{x},\bar{u}), \\ \dot{\bar{x}}_{k,r} &= \bar{u}_k, \end{cases}$$

where for any $1 \le j \le r-1$, the homogeneous polynomial $\bar{f}^{k[m]}_j(\bar{z},\bar{x},\bar{u})$ is of the form (3.3). Assume that the first $j-1$ components $f^{k[m]}_1(z,x,u),\dots,f^{k[m]}_{j-1}(z,x,u)$ of $\Pi^{k[m]}$ are already in their normal forms and let us focus exclusively on the jth component $f^{k[m]}_j(z,x,u)$.

Since $f^{k[m]}_j(z,x,u) \in \mathfrak{P}^m$, it decomposes uniquely as

$$f^{k[m]}_j(z,x,u) = \bar{f}^{k[m]}_j(z,x,u) + \tilde{f}^{k[m]}_j(z,x,u),$$

where $\bar{f}^{k[m]}_j(z,x,u) \in \mathfrak{F}^m_j$ and $\tilde{f}^{k[m]}_j(z,x,u) \in \mathfrak{C}^m_j$. We may remark that $\tilde{f}^{k[m]}_j(z,x,u)$ is necessarily affine in u; otherwise its projection on $\mathfrak{F}^m_j$ will not be zero.

We may also suppose that $\tilde{f}^{k[m]}_j(z,x,u)$ doesn't depend on u. Indeed, if

$$\tilde{f}^{k[m]}_j(z,x,u) = \sum_{t=1}^{p} u_t R^{k[m-1]}_{j,t}(z,x) + \hat{f}^{k[m]}_j(z,x)$$

it suffices to take the change of variable

$$\bar{x}_{k,j} = x_{k,j} - \sum_{1\le t\le p} \int_0^{x_{t,r}} R^{k[m-1]}_{j,t}(z,x)d\epsilon,$$

to get rid of those terms that depend on u. Of course, in order for the integral to make sense, the variable $x_{t,r}$ of the polynomial $R^{k[m-1]}_{j,t}(z,x)$ will be replaced by the parameter of integration ϵ.

Now, if we assume that $\tilde{f}^{k[m]}_j(z,x,u)$ doesn't depend on u, it suffices to take the change of coordinates

$$\bar{x}_{k,j+1} = x_{k,j+1} + \tilde{f}^{k[m]}_j(z,x)$$

to get the jth component $f_j^{k[m]}(z,x,u)$ into its normal form. As we now see, the procedure didn't modify the previous $j-1$ components because the change of coordinates involves only the variables $x_{k,j}$ and $x_{k,j+1}$, which didn't appear *linearly* in these components. For the same reason, it doesn't modify the other subsystems. This ends the proof of this part.

(ii) (a) The proof of this part will be done by induction. Consider the subsystem

$$\Pi^{[m-1]} \;:\; \dot{z} = Jz + g^{[0]}(z) + g^{[m-1]}(z,x,u) \tag{4.3}$$

with $m \geq 3$ and assume that for some $2 \leq l \leq r$, this system has been transformed so that

$$g^{[m-1]}(z,x,u) = \tilde{g}^{[m-1]}(z,x,u) + \hat{g}^{[m-1]}(z,x,u),$$

where for any $1 \leq j \leq q$ we have

$$\begin{aligned}\tilde{g}_j^{[m-1]}(z,x,u) = &\sum_{1\leq s\leq t\leq p}\sum_{i=l+1}^{r+1} x_{s,i}x_{t,i}P_{j,i,s,t}^{[m-3]}(z,\pi_{t,i}^s(x))\\ &+\sum_{1\leq s<t\leq p}\sum_{i=l+1}^{r+1} x_{s,i}x_{t,i-1}Q_{j,i,s,t}^{[m-3]}(z,\pi_{t,i-1}^t(x))\end{aligned}$$

and $\hat{g}^{[m-1]}(z,x,u)$ depends only on the variables z and $x_{s,1},\ldots,x_{s,l}$ for $1 \leq s \leq p$. We emphasize here that $\hat{g}^{[m-1]}(z,x,u)$ doesn't depend on any variable $x_{s,k}$ for $k \geq l+1$.

Each component of $\hat{g}^{[m-1]}(z,x,u)$ could be uniquely decomposed as follows:

$$\begin{aligned}\hat{g}_j^{[m-1]}(z,x,u) = &\sum_{1\leq s\leq t\leq p} x_{s,l}x_{t,l}P_{j,l,s,t}^{[m-3]}(z,\pi_{t,l}^s(x))\\ &+\sum_{1\leq s<t\leq p} x_{s,l}x_{t,l-1}Q_{j,l,s,t}^{[m-3]}(z,\pi_{t,l-1}^t(x))\\ &+\sum_{1\leq t\leq p} x_{t,l}R_{j,l,t}^{[m-2]}(z,x) + S_{j,l}^{[m-1]}(z,x),\end{aligned}$$

where the polynomials $R_{j,l,t}^{[m-2]}(z,x)$ and $S_{j,l}^{[m-1]}(z,x)$ depend only on the variables z and $x_{s,1},\ldots,x_{s,l-1}$ for $1 \leq s \leq p$.

It then suffices to apply the change of variables given, for any $1 \leq j \leq q$, by

$$\bar{z}_j = z_j - \sum_{1\leq t\leq p}\int_0^{x_{t,l-1}} R_{j,l,t}^{[m-2]}(z,x)d\epsilon,$$

to get rid of the terms $\sum 1 \leq t \leq p x_{t,l}R_{j,l,t}^{[m-2]}(z,x)$. For the need of the integral we replace the variable $x_{t,l-1}$ in $R_{j,l,t}^{[m-2]}(z,x)$ by the parameter of integration ϵ.

This means that we transform the subsystem (4.3) into the form

$$\Pi^{[m-1]} \;:\; \dot{\bar{z}} = J\bar{z} + g^{[0]}(\bar{z}) + g^{[m-1]}(\bar{z},\bar{x},\bar{u})$$

with

$$g^{[m-1]}(\bar{z},\bar{x},\bar{u}) = \tilde{g}^{[m-1]}(\bar{z},\bar{x},\bar{u}) + \hat{g}^{[m-1]}(\bar{z},\bar{x},\bar{u}),$$

where for any $1 \leq j \leq q$ we have

$$\tilde{g}_j^{[m-1]}(\bar{z}, \bar{x}, \bar{u}) = \sum_{1 \leq s \leq t \leq p} \sum_{i=l}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i} P_{j,i,s,t}^{[m-3]}(\bar{z}, \pi_{t,i}^s(\bar{x})) + \sum_{1 \leq s < t \leq p} \sum_{i=l}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i-1} Q_{j,i,s,t}^{[m-3]}(\bar{z}, \pi_{t,i-1}^t(\bar{x}))$$

and $\hat{g}^{[m-1]}(\bar{z}, \bar{x}, \bar{u})$ depends only on the variables $\bar{z}$ and $\bar{x}_{s,1}, \ldots, \bar{x}_{s,l-1}$ for $1 \leq s \leq p$. This proves the induction argument. If we take $l = 2$, then $\hat{g}^{[m-1]}(\bar{z}, \bar{x}, \bar{u})$ will depend only on the variables $\bar{z}$ and $\bar{x}_{s,1}$ for $1 \leq s \leq p$, which means that

$$\hat{g}^{[m-1]}(\bar{z}, \bar{x}, \bar{u}) = \sum_{1 \leq s \leq t \leq p} \bar{x}_{s,1}\bar{x}_{t,1} P_{j,1,s,t}^{[m-3]}(z, \pi_{t,1}^s(x)).$$

We thus deduce that

$$g_j^{[m-1]}(\bar{z}, \bar{x}, \bar{u}) = \sum_{1 \leq s \leq t \leq p} \sum_{i=1}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i} P_{j,i,s,t}^{[m-3]}(\bar{z}, \pi_{t,i}^s(\bar{x})) + \sum_{1 \leq s < t \leq p} \sum_{i=2}^{r+1} \bar{x}_{s,i}\bar{x}_{t,i-1} Q_{j,i,s,t}^{[m-3]}(\bar{z}, \pi_{t,i-1}^t(\bar{x}))$$

and this achieves the proof of this part.

(ii)(b) When $m = 2$, the homogeneous vector field $g^{[m-1]}(z, x, u)$ is linear with respect to the variables x and u, that is,

$$g^{[m-1]}(z, x, u) = \sum_{1 \leq t \leq p} \sum_{i=1}^{r+1} x_{t,i} P_{i,t}^{[0]}(z),$$

where $P_{i,t}^{[0]}(z) = \left(P_{1,i,t}^{[0]}(z), \ldots, P_{q,i,t}^{[0]}(z)\right)^T$ is a vector field that depends exclusively on the variable z.

The method is to apply first a change of coordinates of the form

$$\tilde{z} = z - \sum_{1 \leq t \leq p} x_{t,r} P_{r+1,t}^{[0]}(z)$$

to annihilate the terms

$$\sum_{1 \leq t \leq p} x_{t,r+1} P_{r+1,t}^{[0]}(z) = \sum_{1 \leq t \leq p} u_t P_{r+1,t}^{[0]}(z).$$

Then, apply a change of coordinates of the form

$$\bar{z} = \tilde{z} - \sum_{1 \leq t \leq p} x_{t,r-1} \tilde{P}_{r,t}^{[0]}(\tilde{z})$$

to annihilate the terms

$$\sum_{1 \leq t \leq p} x_{t,r} \tilde{P}_{j,r,t}^{[0]}(\tilde{z}),$$

where $\tilde{P}^{[0]}_{j,r,t}(\tilde{z})$ denotes the new terms obtained after the first change of coordinates.

We keep applying this method until we get

$$g^{[1]}(z, x, u) = \sum_{1 \leq t \leq p} x_{t,1} P^{[0]}_{1,t}(z).$$

This completes the proof of item (ii) and that of the theorem. □

REFERENCES

[1] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, 2nd ed., Springer-Verlag, New York, 1988.

[2] J.-P. Barbot, I. Belmouhoub, L. Boutat-Baddas, *Observability normal forms*, in New Trends in Nonlinear Dynamics and Control, and Their Applications, W. Kang, M. Xiao, and C. Borges, eds., Springer-Verlag, Berlin, 2003, pp. 3–17.

[3] B. Bonnard, *Feedback equivalence for nonlinear systems and the time optimal control problem*, SIAM J. Control Optim., 29 (1991), pp. 1300–1321.

[4] B. Bonnard, *Quadratic control systems*, Math. Control Signals Systems, 4 (1991), pp. 139–160.

[5] E. Cartan, *La Théorie des Groupes Finis et Continus et la Géométrie Différentielle traitées par la Méthode du Repère Mobile*, Gauthier-Villars, Paris, 1937.

[6] S. Celikovsky and H. Nijmeijer, *Equivalence of nonlinear systems to triangular form: The singular case*, Systems Control Lett., 27 (1996), pp. 135–144.

[7] D. E. Chang, W. Kang, and A. J. Krener, *Normal forms and bifurcations of control systems*, in Proceedings of the 39th IEEE Conference on Decision and Control, Vol. 2, 2000, pp. 1602–1607.

[8] L. Cremean, W. B. Dunbar, D. V. Goh, J. Hickey, E. Kavins, J. Meltzer, and R. Murray, *The Caltech multi-vehicle wireless testbed*, in Proceedings of the 40th IEEE Conference on Decision and Control, Las Vegas, 2002, pp. 86–88.

[9] W. B. Dunbar and R. Murray, *Model predictive control of coordinated multi-vehicle formations*, in Proceedings of the 40th IEEE Conference on Decision and Control, Las Vegas, 2002, pp. 4631–4636.

[10] R. B. Gardner, *The Method of Equivalence and Its Applications*, CBMS-NSF Regional Conf. Ser. Appl. Math., 58, SIAM, Philadelphia, 1989.

[11] R. B. Gardner and W. Shadwick, *The GS algorithm for exact linearization to Brunovský normal form*, IEEE Trans. Automat. Control, 37 (1992), pp. 224–230.

[12] R. B. Gardner, W. F. Shadwick, and G. R. Wilkens, *A geometric isomorphism with applications to closed loop controls*, SIAM J. Control Optim., 27 (1989), pp. 1361–1368.

[13] R. B. Gardner, W. F. Shadwick, and G. R. Wilkens, *Feedback equivalence and symmetries of Brunovský normal forms*, in Dynamics and Control of Multibody Systems, Contemp. Math., 97, J. E. Marsden, P. S. Krishnaprasad, and J. C. Simo, eds., AMS, Providence, RI, 1989, pp. 115–130.

[14] B. Hamzi, *Analysis and stabilization of nonlinear systems with a zero-Hopf control bifurcation*, in Proceedings of the 41st IEEE Conference on Decision and Control, Vol. 4, 2002, pp. 3912–3917.

[15] J. Hauser, S. Sastry, and G. Meyer, *Nonlinear control design for slightly nonminimum phase systems: Application to v/stol aircraft*, Automatica J. IFAC, 28 (1992), pp. 665–679.

[16] L. R. Hunt and R. Su, *Linear equivalents of nonlinear time varying systems*, in Proceedings of the MTNS, Santa Monica, CA, 1981, pp. 119–123.

[17] L. R. Hunt, R. Su, and G. Meyer, *Design for multi-input nonlinear systems*, in Differential Geometric Control Theory, R. Brockett, R. Millman, and H. J. Sussmann, eds., Birkhäuser, Boston, 1983, pp. 268–298.

[18] B. Jakubczyk, *Equivalence and invariants of nonlinear control systems*, in Nonlinear Controllability and Optimal Control, H. J. Sussmann, ed., Marcel Dekker, New York, Basel, 1990, pp. 177–218.

[19] B. Jakubczyk, *Feedback Invariants, Critical Trajectories, and Hamiltonian Formalism*, in Nonlinear Control in the Year 2000, Vol. 1, A. Isidori, F. Lamnabhi-Lagarrigue, and W. Respondek, eds., Springer, London, 2001, pp. 545–568.

[20] B. Jakubczyk, *Critical Hamiltonians and feedback invariants*, in Geometry of Feedback and Optimal Control, B. Jakubczyk and W. Respondek, eds., Marcel Dekker, New York, Basel, 1998, pp. 219–256.

[21] B. Jakubczyk and W. Respondek, *On linearization of control systems*, Bull. Acad. Polon. Sci. Ser. Sci. Math., 28 (1980), pp. 517–522.
[22] B. Jakubczyk and W. Respondek, *Feedback classification of analytic control systems in the plane*, in Analysis of Controlled Dynamical Systems, B. Bonnard, B. Bride, J. P. Gauthier, and I. Kupka, eds., Birkhäuser, Boston, 1991, pp. 263–273.
[23] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.
[24] W. Kang, *Extended controller form and invariants of nonlinear control systems with a single input*, J. Math. System. Estim. Control, 4 (1994), pp. 253–256.
[25] W. Kang *Quadratic normal forms of nonlinear control systems with uncontrollable linearization*, in Proceedings of the 34th IEEE Conference on Decision and Control, New Orleans, Vol. 1, 1995, pp. 608–612.
[26] W. Kang, *Bifurcation and normal form of nonlinear control systems, Part* I, SIAM J. Control Optim., 36 (1998), pp. 193–212.
[27] W. Kang, *Bifurcation and normal form of nonlinear control systems, Part* II, SIAM J. Control Optim., 36 (1998), pp. 213–232.
[28] W. Kang, *Bifurcation control via state feedback for systems with a single uncontrollable mode*, SIAM J. Control Optim., 38 (2000), pp. 1428–1452.
[29] W. Kang and A. J. Krener, *Extended quadratic controller normal form and dynamic state feedback linearization of nonlinear systems*, SIAM J. Control Optim., 30 (1992), pp. 1319–1337.
[30] W. Kang, M. Xiao, J. Mao, and I. A. Tall, *Controllability and local accessibility—A normal form approach*, IEEE Trans. Automat. Control, 48 (2003), pp. 1724–1736.
[31] A. J. Krener, *Approximate linearization by state feedback and coordinate change*, Systems Control Lett., 5 (1984), pp. 181–185.
[32] A. J. Krener, W. Kang, and D. E. Chang, *Normal forms of linearly uncontrollable nonlinear control systems with single-input*, in Proceedings NOLCOS'01, St. Petersburg, Russia, 2001, pp. 134–139.
[33] A. J. Krener and M. Xiao, *Nonlinear observer design in the Siegel domain*, SIAM J. Control Optim., 41 (2002), pp. 932–953.
[34] A. J. Krener and M. Xiao, *Erratum: Nonlinear observer design*, SIAM J. Control Optim., 43 (2004), pp. 377–378.
[35] I. Kupka, *On feedback equivalence*, CMS Conf. Proc., 12 (1990), pp. 105–117.
[36] W. Respondek, *Feedback classification of nonlinear control systems in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$, in Geometry of Feedback and Optimal Control, B. Jakubczyk and W. Respondek, eds., Marcel Dekker, New York, 1998, pp. 347–381.
[37] W. Respondek, *Symmetries and minimal flat outputs of nonlinear control systems*, in New Trends in Nonlinear Dynamics and Control, and Their Applications, W. Kang, M. Xiao, and C. Borges, eds., Springer-Verlag, Berlin, 2003.
[38] W. Respondek and I. A. Tall, *How many symmetries does admit a nonlinear single-input control system around equilibrium*, in Proceedings of the 40th IEEE Conference on Decision and Control, FL, 2000, pp. 1795–1800.
[39] W. Respondek and M. Zhitomirskii, *Feedback classification of nonlinear control systems on 3-manifolds*, Math. Control Signals Systems, 8 (1995), pp. 299–333.
[40] S. Sastry, Mathematics of Control, Signals, and Systems, 8 (1995), pp. 299–333.
[41] I. A. Tall, *Classification par bouclage des systèmes de contrôles non linéaires mono-entrée: Formes normales, formes canoniques, invariants et symétries*, Ph.D. thesis, INSA de Rouen, Rouen, France, 2000.
[42] I. A. Tall, *Normal forms of multi-input nonlinear control systems with controllable linearization*, in New Trends in Nonlinear Dynamics and Control, and Their Applications, W. Kang, M. Xiao, and C. Borges, eds., Springer-Verlag, Berlin, 2003, pp. 87–100.
[43] I. A. Tall and W. Respondek, *Normal forms, canonical forms, and invariants of single-input control systems under feedback*, in Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, 2000, pp. 1625–1630.
[44] I. A. Tall and W. Respondek, *Normal forms and invariants of nonlinear single-input systems with noncontrollable linearization*, in Proceedings NOLCOS'01, St. Petersburg, Russia, 2001, pp. 122–127.
[45] I. A. Tall and W. Respondek, *Transforming a single-input nonlinear system to a strict feedforward form via feedback*, in Nonlinear Control in the Year 2000, Vol. 2, A. Isidori, F. Lamnabhi, and W. Respondek, eds., Springer-Verlag, London, 2001, pp. 527–542.
[46] I. A. Tall and W. Respondek, *Feedback equivalence to feedforward forms for nonlinear single-input systems*, in Dynamics, Bifurcations and Control, F. Colonius and L. Grune, eds., Springer-Verlag, Berlin, 2002, pp. 269–286.

[47] I. A. Tall and W. Respondek, *Normal forms of two-inputs nonlinear control systems*, in Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, 2002, pp. 2732–2737.

[48] I. A. Tall and W. Respondek, *Nonlinearizable single-input control systems do not admit stationary symmetries*, Systems Control Lett., 46 (2002), pp. 1–16.

[49] I. A. Tall and W. Respondek, *Feedback classification of nonlinear single-input control systems with controllable linearization: Normal forms, canonical forms, and invariants*, SIAM J. Control Optim., 41 (2002), pp. 1498–1531.

[50] I. A. Tall and W. Respondek, *Feedback equivalence to a strict feedforward form for nonlinear single-input systems*, Internat. J. Control, to appear.

[51] M. Zhitomirskii and W. Respondek, *Simple germs of corank one affine distributions*, in Singularities Symposium—Lojasiewicz 70, B. Jakubczyk, W. Pawlucki, and J. Stasica, eds., Banach Center Publ. 44, Polish Academy of Science, Warsaw, 1998, pp. 269–276.