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Michael C. Sullivan
Southern Illinois University Carbondale, msulliva@math.siu.edu

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COMPOSITE KNOTS IN THE FIGURE-8 KNOT COMPLEMENT CAN HAVE ANY NUMBER OF PRIME FACTORS

Michael C. SULLIVAN

Department of Mathematics, University of Texas, Austin, TX 78712, USA,

mike@math.utexas.edu

We study an Anosov flow ϕ_t in $S^3 - \{\text{figure-8 knot}\}$. Birman and Williams conjectured in [2] that the knot types of the periodic orbits of this flow could have at most two prime factors. Below, we give a geometric method for constructing knots in this flow with any number of prime factors.

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flows knot complements periodic orbits templates

The periodic orbits of a smooth flow on S^3 with positive topological entropy form infinitely many knots [3]. These knots and how they are linked have been studied with the aid of *templates* or *knots holders*, which are compact, 2-dimensional branched manifolds with smooth semi-flows [2].

The complement of the figure-8 knot in S^3 can be fibered with fiber a punched torus which has the figure-8 knot as its boundary. Let

$$\pi : S^3 - \{\text{figure-8 knot}\} \to S^1$$

be a projection map for the fibration. By integrating along the gradient of π we obtain a flow. Any such flow induces a diffeomorphism on the fibers, which is called the monodromy of the flow. The monodromy of this flow was shown by Thurston [6] to be isotopic to the map

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

where we view the punched torus as the unit square in $(R^2 - Z^2)/Z^2$. This map is Anosov. By the work of Asimov and Franks [1] we know that any flow with monodromy in the same isotopy class as an Anosov map, will have at least the knots given by the suspension of the Anosov map.

Birman and Williams [2] showed that the knots in the suspension of A, and how they are linked, are mimicked by the periodic orbits of the template, T_1, shown in FIGURE 1. Be to more precise, for any finite link in this template there is an ambient isotopy to a link in the suspension flow (with one or two exceptions that won't concern us). They conjectured that these knots could have at most two prime factors and that for any template there would be a bound on the number of prime factors of the periodic orbits in the semi-flow. However, we can now show the opposite.

Theorem. *For any positive integer N, there is a knot with N prime factors in the suspension flow A_t of A in the figure 8 knot complement.*

Given N we will show how to construct a knot with N prime factors. Our construction will show that all of the factors can be Lorenz knots, that is knots which occur on the Lorenz template (see FIGURE 3b), which are known to be prime [7], or mirror images of Lorenz knots.

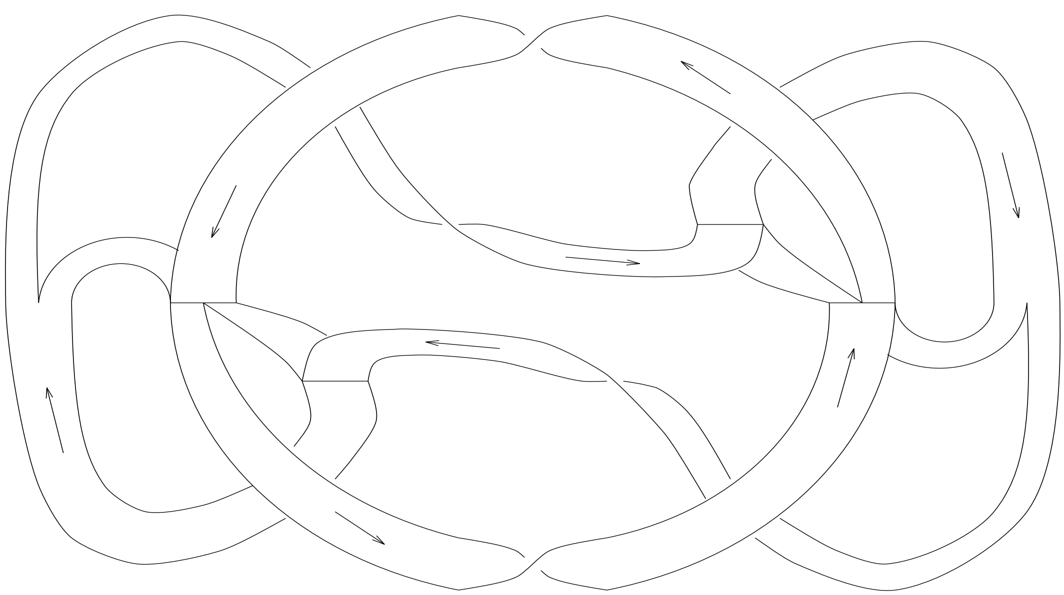


FIGURE 1: The Birman-Williams Template, T_1.

We will arrive at our result by studying a subtemplate of T_1. To this end we delete certain branches and cut along certain orbits. In cutting the template we will add a one or two closed orbits, but they are not important. This is demonstrated in FIGURES 2a & 2b. We will call the new template T_2.

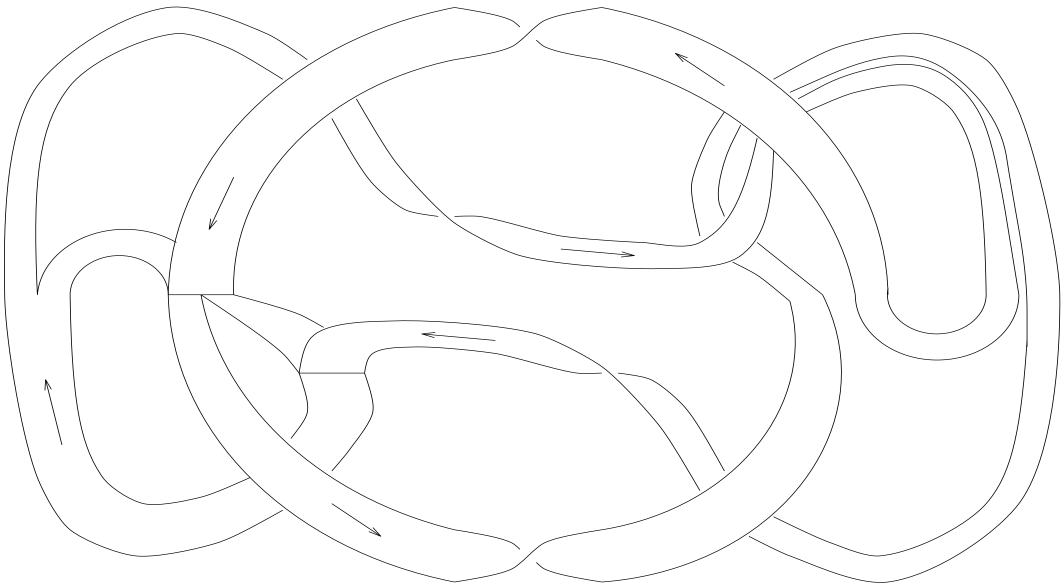


FIGURE 2a: A subtemplate of T_1.

The branch two lines on the right hand side of FIGURE 1 have been eliminated in FIGURE 2a by discarding many orbits. Next, the long branches are isotoped towards the left and the two pairs of twists cancel out. This gives us the figure in the upper left hand corner of FIGURE 2b. This is then straightened out before we delete more orbits to get the left hand figure in the middle of FIGURE 2b. We then push the branches around, "dragging" the branch lines along. Finally we arrive at the template T_2 in the lower right hand corner. Every orbit in T_2 is in T_1, save one that was created when we cut the branch in the upper right hand corner of T_1. But this with not effect our results.

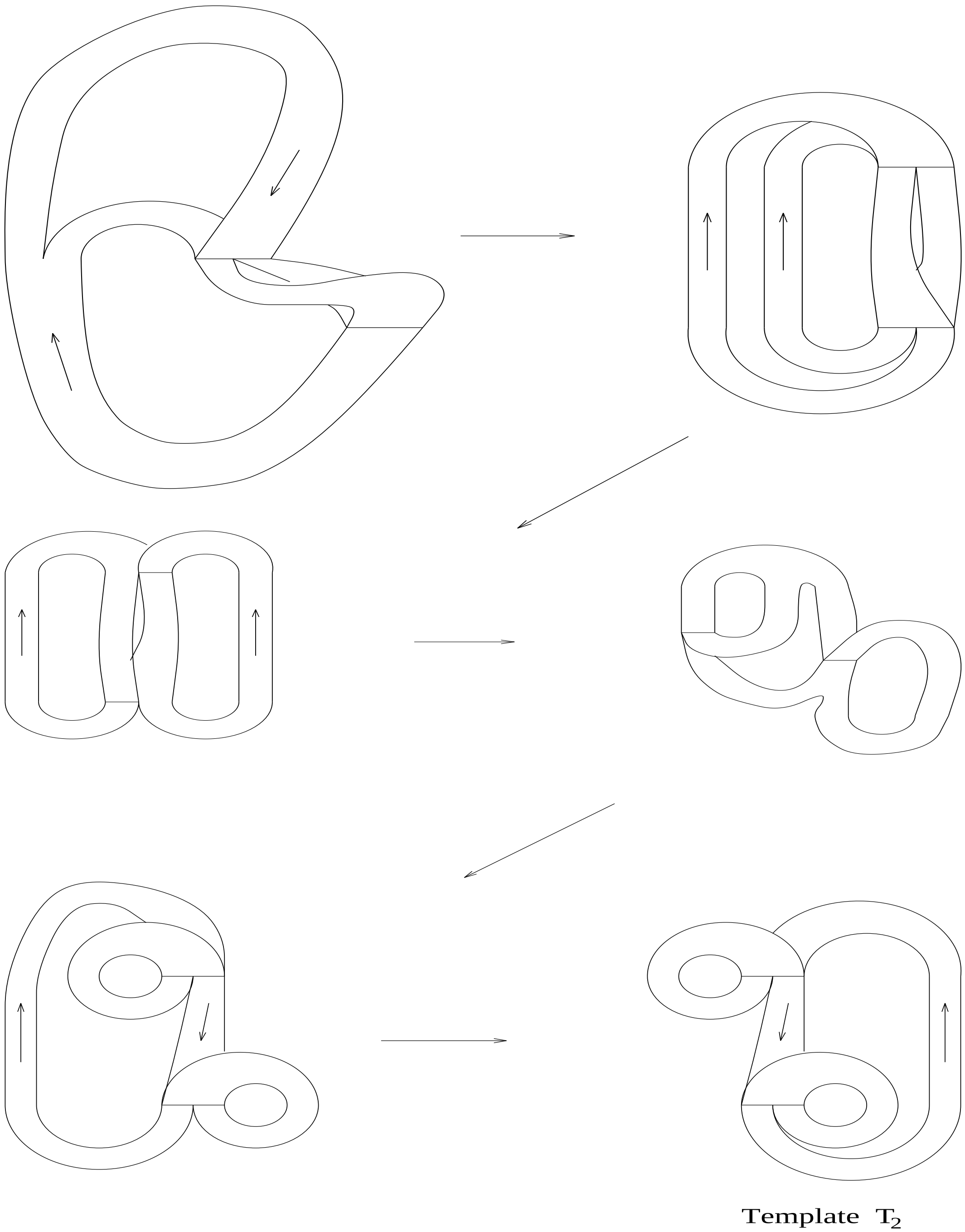
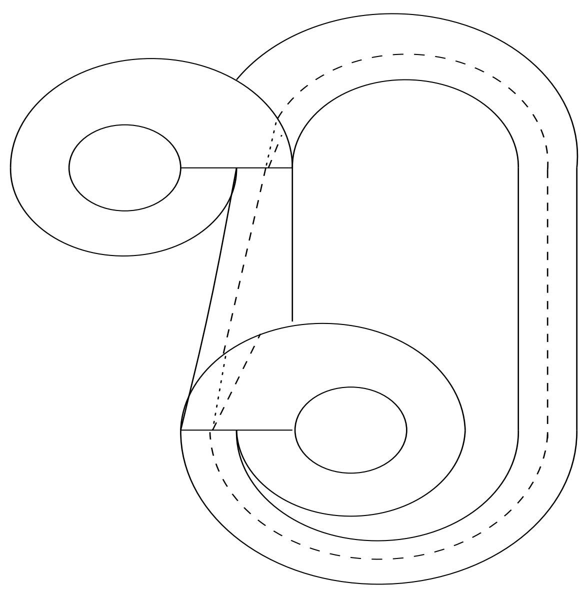


FIGURE 2b: We finally arrive at a subtemplate, T_2, of T_1.

Template T_2 contains composite knots each of which factor into a Lorenz knot and the mirror image of a Lorenz knot. This factoring is accomplished with the cutting sphere shown in FIGURE 3. The portion of the template outside the cutting sphere contains all Lorenz knots. The inside portion contains a mirror image of the Lorenz template. A knot on T_2 which has only one strand passing in and out of the sphere is the connected sum of a Lorenz knot and the mirror image of a Lorenz knot. It was this observation that led Birman and Williams to their conjecture, as they could find no other composite knots in their template.

a)



b)

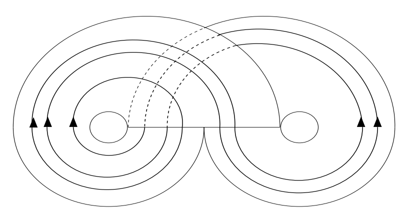


FIGURE 3: a) The dashed lines show the intersection of a sphere and T_2. b) The Lorenz template with a trefoil knot.

FIGURE 4a shows a new subtemplate within T_2. In FIGURE 4b we deform it, cutting away some additional parts and disregarding many non-periodic flow lines. We call this template T_3. Notice that its is very similar in appearance to T_2 except that the lower looped branch is attached on the opposite side.

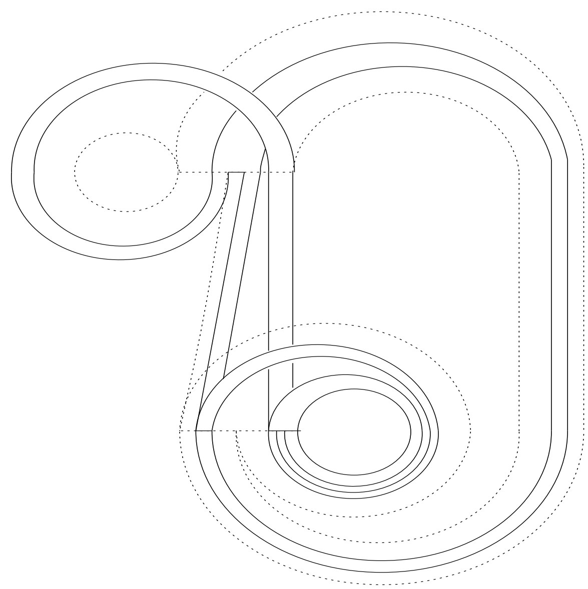


FIGURE 4a: A subtemplate inside T_2.

The proof is carried out as follows. Template T_3 is surgered along certain orbits in the semi-flow (FIGURES 5 & 6). We then see that it contains itself as a subtemplate in a nontrivial manner (FIGURES 7 & 8) and a copy of the Lorenz template (FIGURES 9 & 10). By studying FIGURE 11 the reader should be able to see the these two subtemplates are disjoint and unlinked. Let k_1 be any knot on T_3 (which we could choose to be Lorenz) and k_2 be any Lorenz knot. We will show that their connected sum is on T_3. The proof will then be complete by induction.

FIGURE 12 is a close up of the region in the dashed circle of FIGURE 11. The "parent" version of T_3 is depicted with dots. We show the strands of k_1 and k_2 that are closest to the center, or split point, of the branch line. Next we form their connected sum on the branched manifold. While this is not an orbit of the semi-flow, in can be homotopied to a periodic orbit of the same knot type.

FIGURE 13 shows that the mirror image of a Lorenz template is in T_3. The construction is similar to the one before. Thus the factors can include any ratio of Lorenz knots to mirror images of Lorenz knots.

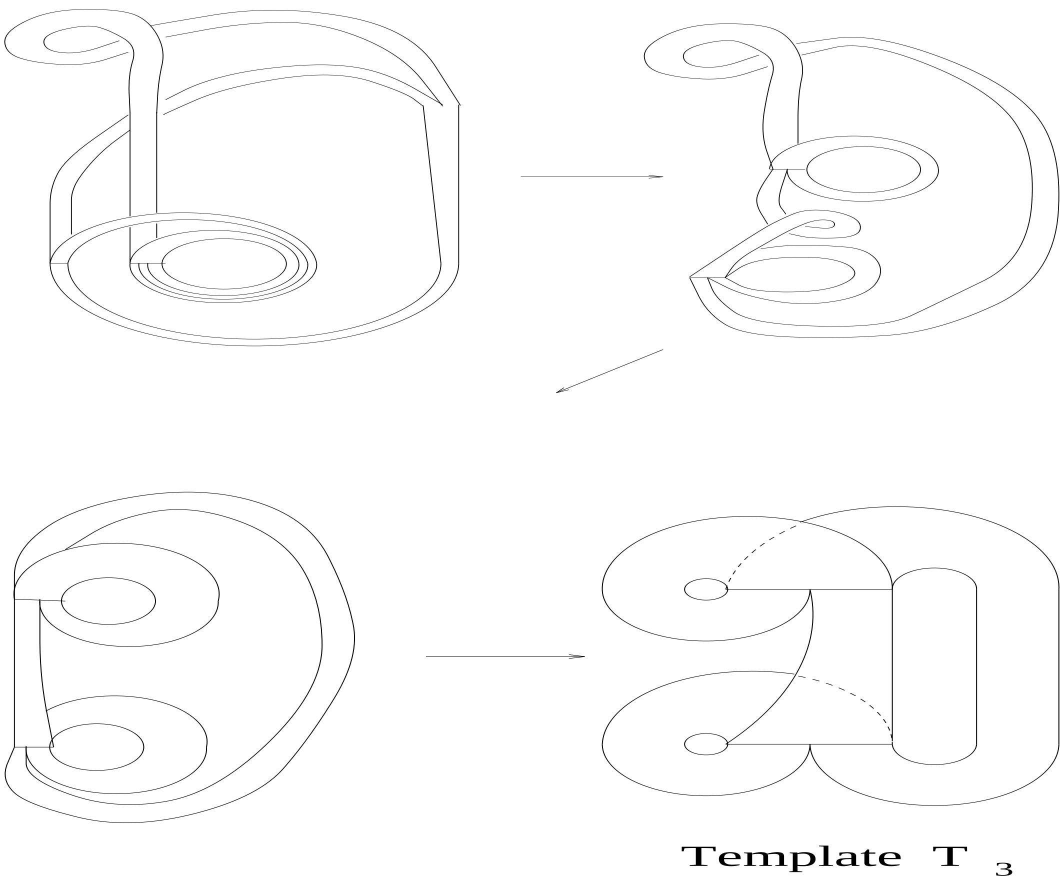


FIGURE 4b: A sub-subtemplate, T_3.

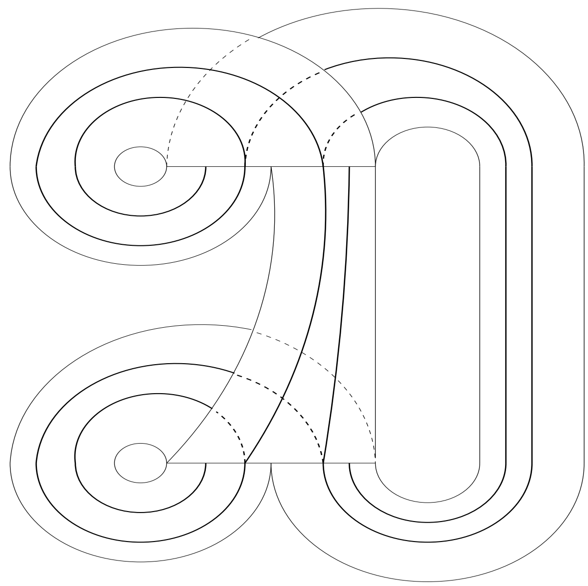


Figure 5: Template T_3 and orbits to be cut along.

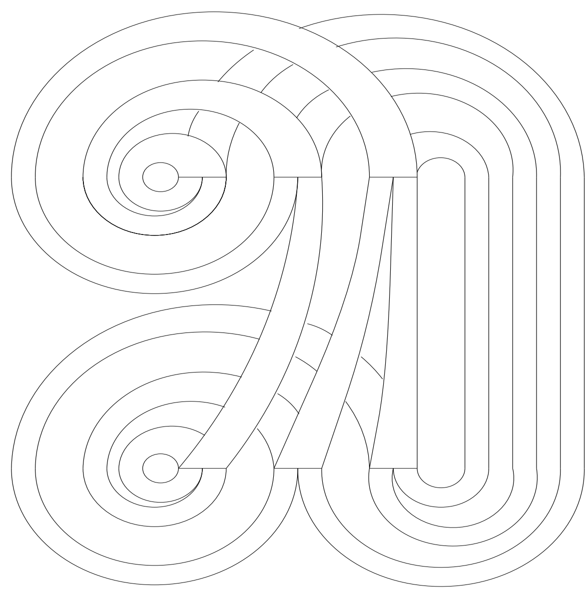


Figure 6: Template T_3 cut open along orbits.

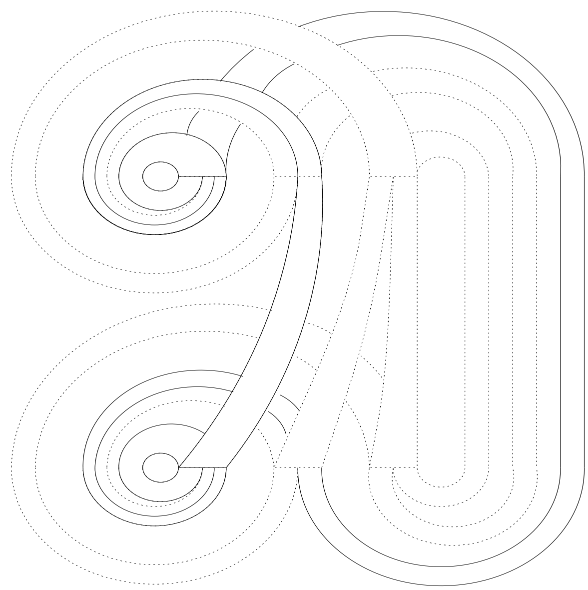


FIGURE 7: A subtemplate of T_3.

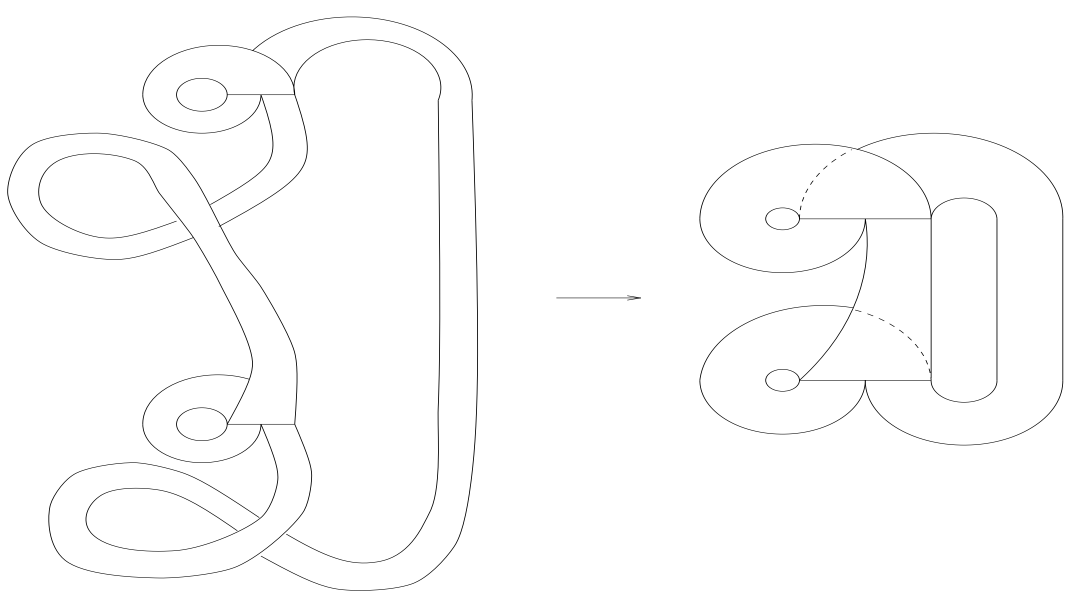


FIGURE 8: The subtemplate is T_3.

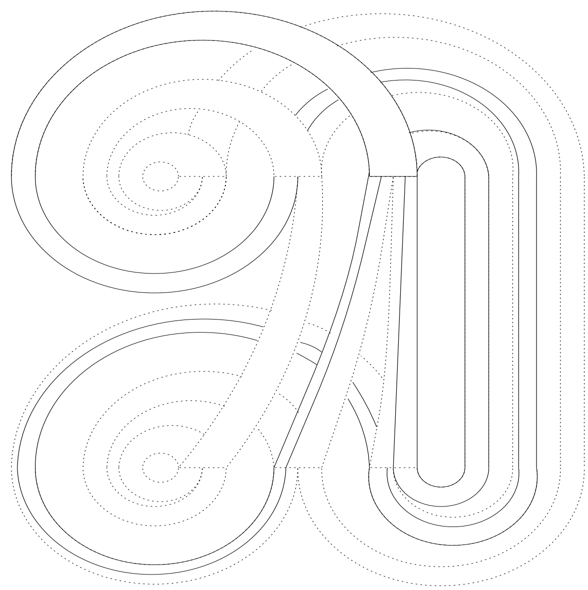


FIGURE 9: Another subtemplate of T_3.

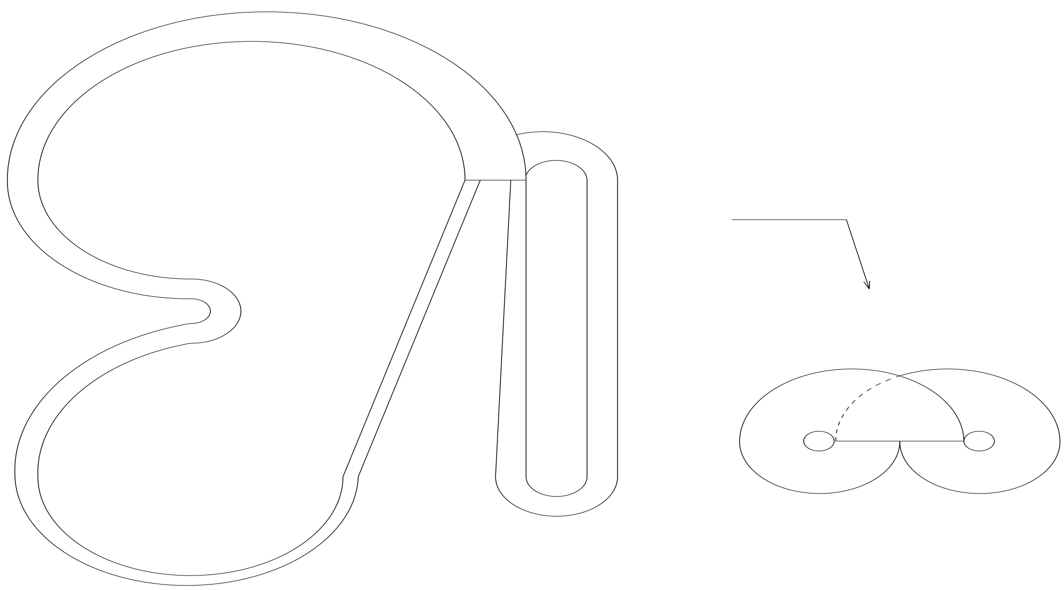


FIGURE 10: This time the subtemplate is the Lorenz template.

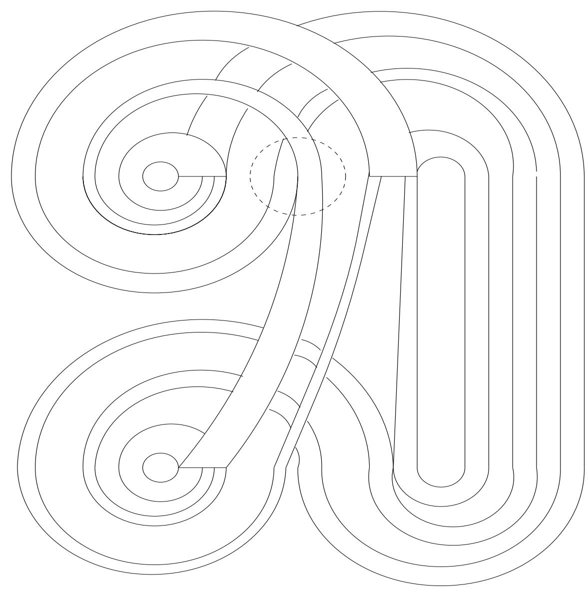


FIGURE 11: The two subtemplates are unlinked.

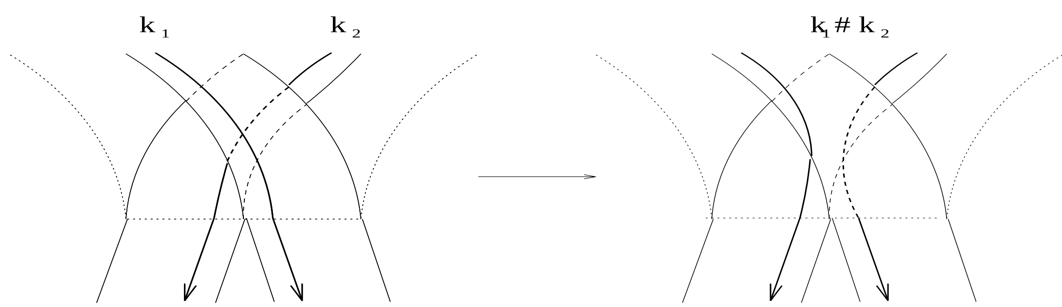


FIGURE 12: Form connected sum.

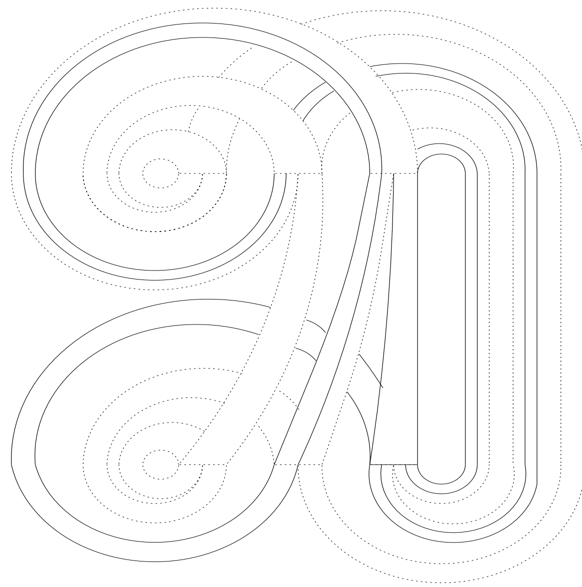


FIGURE 13: The mirror image of a Lorenz template in T_3.

The two templates in FIGURE 14 have only left handed crossings. Thus, the knots must all be positive braids (in the sign convention of [2]). The one on the right contains only prime knots and one of the left has composite knots, but with no more then two prime factors. In fact both factors must be Lorenz knots. The proof of these statements will be published in the author's forthcoming dissertation [5]. This points to the following weakened form of the Birman-Williams conjecture.

Conjecture. *There is an upper bound on the number of prime factors of the periodic orbits on a template with only one type of crossing.*

As further evidence for the conjecture, when Williams proved that all Lorenz knots are prime [7] he also showed that if one adds any number of left handed half twists to one branch of the Lorenz template, the new template will have only prime knots. Williams also observed that if one adds a single right handed half twist to one branch, then prime knots do exist. More recently it has been shown that prime knots exist when any number of right handed half twists are added to one branch of the Lorenz template [4] and that in fact there is no bound to the number of prime factors the knots can have [5].

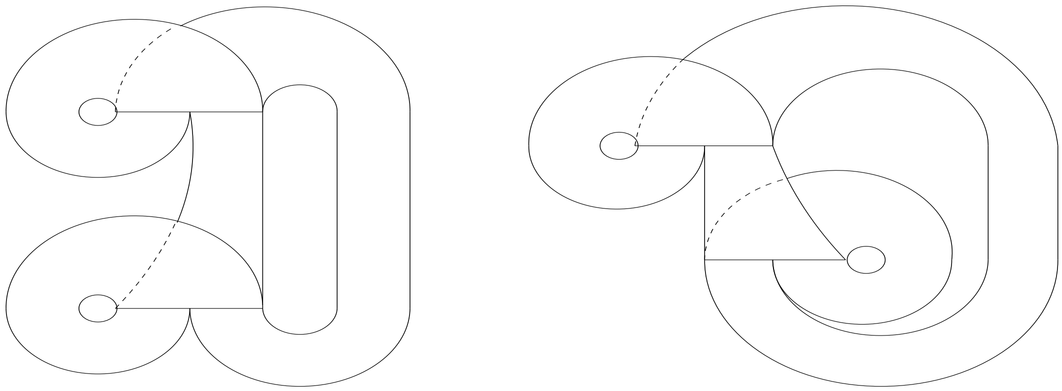


FIGURE 14: The template on the right has only prime knots while the other one has composites but with at most two factors.

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