12-1993

On SNR as a Measure of Performance for Narrowband Interference Rejection in Direct Sequence Spread Spectrum Systems

Arif Ansari
Southern Illinois University Carbondale

R. Viswanathan
Southern Illinois University Carbondale, viswa@engr.siu.edu

Published in Ansari, A., & Viswanathan, R. (1993). On SNR as a measure of performance for narrowband interference rejection in direct sequence spread spectrum systems. IEEE Global Telecommunications Conference, 1993, including a Communications Theory Mini-Conference. Technical Program Conference Record, IEEE in Houston (GLOBECOM '93), v. 4, 55-59. doi: 10.1109/GLOCOM.1993.318427

On SNR as a Measure of Performance for Narrowband Interference Rejection in Direct Sequence Spread Spectrum Systems

Arif Ansari and R. Viswanathan
Department of Electrical Engineering
Southern Illinois University
Carbondale, IL 62901

Abstract We simulate a nonlinearized Kalman [5], Kalman and a modified Kalman (linear) filter for suppressing a narrowband Gaussian interference in direct sequence spread spectrum receiver and examine the suitability of Signal-to-Noise Ratio (SNR) of the test statistic as a measure of performance of the receiver. We consider Gaussian autoregressive interference with a peaked spectrum and the three cases: small processing gain (PG) and short pseudonoise (PN) sequence, small PG and long PN sequence, and moderate PG and PN sequence. Based on the simulations, we conclude that for the two cases corresponding to small processing gain, if the thermal noise variance is small and the interference is strong, the Gaussian approximation to the test statistic does not yield the correct *BER* for any of the receivers. For small PG and short PN sequence, even though the *SNR* corresponding to nonlinear filter is significantly higher than the *SNR*'s of the two linear filters, the *BER* of the non-linear is higher than that of the linear receivers. *SNR* is not a useful measure in these situations.

I Introduction

Processing the received signal prior to correlating with the PN sequence has been employed to improve the suppression of narrowband interference[1]. Linear least squares estimation techniques to estimate and subtract the narrowband interference have been studied in [2]-[4]. Nonlinear techniques for interference suppression in spread spectrum have been investigated in [5]. Signal-to-noise ratio (*SNR*) at the output of the interference rejection filter has been used for evaluating the performance of spread spectrum systems. Theoretical and simulated *SNR* improvements are given in [2]. *SNR* improvement factor resulting from narrowband interference suppression has been calculated in [3]. *BER* performance has been evaluated by applying a Gaussian assumption to the calculated *SNR* and also through simulations. Derivations of *BER* expressions for transversal filters and maximum-likelihood receivers are given in [4,6]. Performances of Kalman filter and a nonlinear modification have been examined in [5] using simulated *SNR* improvement as a measure of performance.

In this paper, we study the suitability of *SNR* as a measure of *BER* performances of direct sequence spread spectrum systems employing Kalman filter, or a linear modification, or the non-linear modification of [5] for narrowband interference rejection. Simulations of these filters are carried out for estimating the SNR's at the filters' output ,which are sequences at the chip rate, *SNR*'s of the test statistics, and *BER*'s of the receivers.

1.1 Model for the received signal

A direct-sequence modulation waveform is given by

$$m(t) = \sum_{k=0}^{N-1} c_k \, q(t-kt_c) \tag{1}$$

where N is the length of PN sequence, τ_c is the chip interval, c_k is the kth chip of the PN sequence, and $q(t)$ is a rectangular pulse of duration τ_c starting at $t=0$. Let the message bit duration be $T_b=L\tau_c$. In the sequel, we assume two cases, a whole PN sequence embedded in each bit ($L=N$) and several bits with a small processing gain covering a long PN sequence ($L \ll N$).

The total transmitted signal may be expressed as

$$s(t) = \sum_l d_l \, m(t-lT_b) \tag{2}$$

where $\{d_l\}$ is the i.i.d. binary message sequence, $d_l \in \{-1,1\}$. The received signal is of the form

$$z(t) = \alpha \, s(t-\tau) + n(t) + i(t) \tag{3}$$

where α is the signal amplitude, τ is a delay, n(t) is white Gaussian noise and $i(t)$ is narrowband interference. Assuming $\alpha=1$ and $\tau=0$, and that the received signal has been chip matched filtered and sampled at the chip rate of the PN sequence, the following samples are obtained

$$z_k = s_k + n_k + i_k \tag{4}$$

Here $s_k=dc_k$, where d denotes the message sequence. Even though the message bit can change every T_b seconds, with a little abuse of notation, we denote the message sequence as d. This is especially convenient later on when we consider the decision statistic for a given bit. $\{n_k\}$ is i.i.d. zero mean, variance σ_n^2, Gaussian random sequence. The narrowband interference, $\{i_k\}$, is modeled as a Gaussian autoregressive process of order p and variance σ_i^2.

$$i_k = \sum_{i=1}^{p} \phi_i \, i_{k-i} + e_k \tag{5}$$

where $\{e_k\}$ is white excitation noise and the autoregressive parameters, φ's, are known to the receiver. The sequences $\{s_k\}$, $\{n_k\}$ and $\{i_k\}$ are mutually independent.

II Filtering for Narrowband Interference Rejection

The filtering structure for narrowband interference rejection and the bit decision procedure for the direct sequence receiver are shown in Fig. 1. The output of the filter, ε_k, is the input to the PN correlator. The output of the correlator gives the test statistic *TS*. The bit decision is obtained as follows.

$$TS = \sum_{k=0}^{L-1} \varepsilon_k \, c_k \underset{-1}{\overset{+1}{\gtrless}} 0 \tag{6}$$

where, without any loss of generality, $k=0$ is assumed to correspond to the first chip of the data bit under investigation. The per chip *SNR* at the output of the filter is defined as[5]

0-7803-0917-0/93$03.00

$$SNR_O = \frac{E(s_k^2)}{E(|\varepsilon_k - s_k|^2)} \tag{7}$$

The test statistic SNR is defined as

$$SNR_{TS} = \frac{E^2(TS)}{Var(TS)} \tag{8}$$

Since interference rejection filter cannot eliminate the narrowband interference completely, its output ε_k has some residual correlation from chip to chip, specially when the interference is strong. Hence SNR_{TS} cannot be estimated from SNR_O unless the residual correlations and any signal distortion are accounted for.

When the filter used for stationary narrowband interference rejection is a Kalman filter, which is asymptotically (as time increases without bound) a Wiener filter, due to a large filter memory, the test statistic for the current bit is affected by a number of previous bits. Since the filter is linear, this effect maybe studied by applying superposition. Let $f_K(.)$ be the function of the present and all past observations defining the Kalman filter operation. The test statistic corresponding to this filter maybe written as

$$\begin{aligned} TS &= \sum_{k=0}^{L-1}\left[z_k - f_K(z_k)\right]c_k \\ &= \sum_{k=0}^{L-1}\left[\{dc_k - f_K(dc_k)\} + \{n_k - f_K(n_k)\} + \{i_k - f_K(i_k)\}\right]c_k \end{aligned} \tag{9}$$

Define $t_s = \sum_{k=0}^{L-1}\left[dc_k - f_K(dc_k)\right]c_k$ as the contribution of the signal component to the receiver test statistic.

The test statistic of a receiver employing Kalman filter is conditionally Gaussian given all the bits in the filter memory that affect the current test statistic. In the case of a two-sided transversal filter with tap length less than the processing gain, the current test statistic is affected by the previous and the next bit. The true error rate is the average of the four conditional error rates given the four possible combinations of the two neighboring bits [6]. Hence the variable t_S for this transversal filter, given the true bit value, has a density consisting of four impulses. In a similar way, the density of t_S for the Kalman filter provides a measure of the ISI effect on the test statistic . A large variance for t_S indicates that the interference from previous bits in affecting the current test statistic is significant. Hence if the averaged test statistic *SNR* is used to estimate the *BER* instead of averaging the conditional error rates, the estimate of *BER* can be significantly different from the true *BER* for certain interference and noise variances.

If we assume the PN sequence to be truly random, then s_k can be regarded as a random variable taking values +1 and -1 with equal probability. In this case z_k can be regarded as i_k received in Gaussian mixture, s_k+n_k. Hence, a nonlinear filter is optimal for estimating the narrowband interference in this setting [5]. The test statistic corresponding to a nonlinear filter is, in general, non-Gaussian and its distribution is required in order to estimate the *BER*. Moreover, any inference on the comparative performance of linear and non-linear filters based on test statistic *SNR* could be misleading. This maybe specially true for low *BER* , since the nature of the tail of the test statistics' densities of linear and nonlinear filters maybe different.

In order to verify the observations in the preceding two paragraphs, the following comparisons are made of various quantities obtained through simulations at different noise and interference variances.

(i) For the two linear filters, the *BER* estimate obtained from the test statistic *SNR* ,by assuming the test statistic to be Gaussian, is compared to the *BER* of the system from simulations.

(ii) The test statistic *SNR* estimate of the nonlinear filter and its BER estimate are compared to the respective estimates for the linear filters to study the adequacy of *SNR* in the comparisons of linear and nonlinear filters.

(iii) The test statistic *SNR* estimate is compared to the *SNR* of the test statistic obtained from the estimate of the *SNR* at the output of the filter, neglecting any residual correlation.

The densities of t_S for the Kalman filter and TS for all the three filters are also estimated in order to draw inferences regarding the use of test statistic *SNR*. Simulations are carried out for the three cases of small processing gain / short PN sequence, moderate processing gain / moderate length PN sequence, and small processing gain / long PN sequence. The first two cases correspond to a whole PN sequence embedded in each bit.

III Kalman Filter, Nonlinear Filter, and a Modified Linear Filter.

The received signal (4) with i_k as in (5) can be represented in the state space with the following model

$$x_k = \Phi x_{k-1} + w_k \tag{10}$$

where

$$x_k = \left[i_k \quad i_{k-1} \quad \cdots \quad i_{k-p+1}\right]^T, \Phi = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$w_k = \left[e_k \quad 0 \quad \cdots \quad 0\right]^T, H = \left[1 \quad 0 \quad \cdots \quad 0\right], \quad v_k = s_k + n_k, \quad z_k = Hx_k + v_k$$

The first component of the state vector is the interference. The first component of the estimated state, therefore, gives an estimate of the interference which can be subtracted from the received signal. Minimum mean squared error estimates are the conditional expectations which give the filtered and predicted estimates , respectively,

$$\hat{x}_k = E\{x_k | Z^k\}, \tilde{x}_k = E\{x_k | Z^{k-1}\} \tag{11}$$

where Z^k denotes all past observations $\{z_k\}$. Linear minimum mean square estimates are same as optimal estimates if the observations are Gaussian. Linear minimum mean square estimates are obtained recursively using the Kalman Bucy filtering equations [5]. Viewing v_k as a variate from a mixture of two Gaussian densities with means +1 and -1, an approximate conditional mean (ACM) filter is derived in [5]. The ACM filter update equations are identical to those in the Kalman filter while the measurement equations involve correcting the predicted value by a nonlinear function of the prediction residual.

3.1 A Modified Linear Filter

A different linear filter is obtained by modifying the state space model of the received spread spectrum signal as follows.

$$x_k = \begin{bmatrix} i_k & i_{k-1} & \cdots & i_{k-p+1} & d \end{bmatrix}^T,$$

$$\Phi_k = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & j_k \end{bmatrix} \quad (12)$$

with $j_k = \begin{cases} 1, & k \neq 0 \\ 0, & k = 0 \end{cases}$ $\quad w_k = \begin{bmatrix} e_k & 0 & \cdots & 0 & u_k \end{bmatrix}^T,$

$$u_k = \begin{cases} 0, & k \neq 0 \\ \pm 1 \text{ with equal probability}, & k = 0 \end{cases}$$

$$H_k = \begin{bmatrix} 1 & 0 & \cdots & 0 & c_k \end{bmatrix}, x_k = \Phi_k x_{k-1} + w_k, z_k = H_k x_k + n_k$$

Above, $j_k = 1$ for $k \neq 0$ is used to represent the fact that bit contribution to each chip within a bit is the same . The first chip in a new bit corresponds to k=0 and in this case j_k =0 is assumed, since the true contribution(which is either +1 or -1) is unknown. Equations (12)-(15) with H, Q and Φ replaced by H_k, Q_k and Φ_k respectively are the filtering equations for this modified state space representation,

$$Q_k = \begin{bmatrix} E\{e_k^2\} & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \vdots & \vdots \quad \vdots \\ 0 & 0 \cdots u_k^2 \end{bmatrix}, u_k^2 = \begin{cases} 0, & k \neq 0 \\ 1, & k = 0 \end{cases}$$

IV Simulation

Computer simulations of the Kalman filter and its linear and nonlinear modifications are carried out. The interference is modeled as

$$i_k = 1.98 i_{k-1} - 0.9801 i_{k-2} + e_k \quad (13)$$

where $\{e_k\}$ is a zero mean white Gaussian noise. The spectrum of this interfering signal is sharply peaked.

Simulations are carried out to estimate the two *SNR*'s and the *BER* as follows. For a given interference power and thermal noise power, the received signal samples are obtained according to (4). The bit value is set at +1 or -1 with equal probability. $\{c_k\}$ is a maximal length PN sequence. The following three combinations of processing gain (L) and PN sequence length (N) are considered:
(i) L=N=7 (ii) L=7, N=1023 (iii)L=N=63

The signal (4) thus generated is the input to the three filters described in the previous section. The IMSL routine DKALMN is used for simulating the filters. The regular Kalman filter and its linear modification are directly available from this routine. The nonlinear filter uses only the time update of the routine while the measurement update is done by a separate subroutine. For estimating SNR_O and SNR_{TS}, 1000 samples of ε_k, and sample mean and variances of 1000 test statistic samples are averaged over 10 trials. At least 10^5 and upto 10^6 bit decisions are simulated to estimate BER.

The simulation results and some related calculations are shown in Table I (L=N=7), Table II (L=7,N=1023) and Table III (L=N=63) for various values of the thermal noise and interference variances, σ_n^2 and σ_i^2 respectively. For all the three filters, the estimate of SNR_{TS} and the *BER* estimate obtained through simulations are shown as T and P_b^* respectively. Also, if the test statistic is assumed to be Gaussian, another estimate of the BER is obtained as

$$P_T^* = Q(\sqrt{T}) \quad (14)$$

where $Q(.)$ is one minus the standard normal cdf. In addition, for the case L=N=7 (Table I), the test statistic *SNR* obtained as L times the estimate of SNR_O (this ignores residual chip correlation and signal distortion) is shown as γ_B and the *BER* estimate obtained from this SNR as

$$P_S^* = Q(\sqrt{\gamma_B}) \quad (15)$$

The estimates of the densities of t_S (9) for three sets of parameters
(L=N=7, σ_n^2 = 0.01 and σ_i^2 =10,000),
(L=N=7, σ_n^2 = 1.0 and σ_i^2 =1000), and
(L=7, N=1023, σ_n^2 = 0.01 and σ_i^2 =10,000)
were obtained by providing the IMSL routine DDESPL with 5000 samples of t_S.

V Discussion

A. *Small Processing Gain and Short PN Sequence (L=N=7)*

For the two linear filters, comparing γ_B and T (Table I), it is seen that when the thermal noise variance is small and the interference is strong ($\sigma_n^2 = 0.01$ and $\sigma_i^2 = 10{,}000$), the two estimates differ . This is because for these parameters the residual correlation at the filter output is significant. Also, for the two linear filters, the estimate of *BER* from the test statistic *SNR*, P_T^*, agrees with the *BER* estimate P_b^* for weak interference and relatively large thermal noise variance due to low variance of t_S and hence low ISI (Fig. 2). When the interference is strong and thermal noise variance is small (Figs. 3) the variance of t_S is large and the density estimate clearly shows the ISI effect (Fig. 3). The contribution of the signal component to the test statistic, t_S, is strongly dependent on the previous bit and this ISI causes the two *BER* estimates to be different. If we estimate the conditional SNR_{TS} , conditioned on the previous bit l, and obtain an error estimate

$\sum_{l\in\{-1,1\}} \frac{1}{2} Q\left(\sqrt{T} \mid l, d = +1\right)$, this estimate agrees with the *BER* estimate P_b^*.

When the filter is nonlinear, the test statistic is in general non-Gaussian and its *SNR* cannot be used to estimate the *BER* of the receiver. However, when the interference is not strong and thermal noise variance is relatively high, (σ_i^2=1000, σ_n^2= 0.1 or 1.0), the Gaussian approximation to the test statistic of the nonlinear filter also yields the correct *BER* (Table I). In general, it may not be reasonable to infer *BER*s of linear and non-linear filters based on test statistic *SNR*. A heavier tail in the density of the non-linear filter test statistic may lead to a higher *BER* even when its test statistic *SNR* is higher compared to that of a linear filter. Although the test statistic *SNR* of the nonlinear filter is much larger than that of the modified linear filter (Table I, σ_n^2=0.01 and σ_i^2=10000), the *BER* of the nonlinear filter is also higher. For another set of noise parameters (Table I, σ_n^2=0.1 and σ_i^2=10000), the test statistic *SNR*s of all three filters are comparable but the *BER* for the nonlinear filter is higher than the *BER*s of the two linear filters.

B. Small Processing Gain and Long PN Sequence (L=7, N=1023)

For the Kalman filter, the *BER* estimates P_T^* and P_b^* (Table II) disagree for weak thermal noise and strong interference ($\sigma_n^2 = 0.01$ and $\sigma_i^2 = 10{,}000$). This is due to the non-Gaussian nature of t_S as shown by its density estimate (Fig. 4). For the modified linear filter, these estimates agree for all the parameters considered.

The nonlinear filter performs equally well or better than both the linear filters. For weak thermal noise and strong interference, the nonlinear filter shows an error rate about two orders lower than those of the linear filters. The PN sequence being long, it can be regarded as truly random and the density of v_k is approximated well by a weighted sum of two Gaussian densities, warranting the use of nonlinear filter [6]. However, the disagreement between P_T^* and P_b^* indicates that SNR is not a reliable measure even when long PN sequences are used. For (σ_n^2=0.01 , σ_i^2=10000), the test statistic SNR estimate (*T*) for the nonlinear filter is almost 10 dB higher than that of the nonlinear predictor, but the *BER* estimates (P_b^*) are almost the same. The *BER* estimate of the Kalman filter is two orders higher than that of the nonlinear filter, while the estimate *T* is almost 20dB higher for the nonlinear filter. In Gaussian curve, 20dB SNR would translate to much faster decrease of the error rate than two orders.

C. Moderate Processing Gain and PN Sequence (L=N=63)

For moderate processing gain, the simulations had to be restricted to not too small thermal noise variances in order to observe enough errors and obtain an estimate of the *BER*. For all the parameters considered (Table III), P_T^* and P_b^* estimates agree. Also, it is observed that all the three types of filters exhibit nearly the same bit error rate.

In conclusion, the estimate of *SNR* should be used with caution as a measure of performance of a direct sequence spread spectrum system employing narrowband interference rejection filters. In particular, the following remarks are made.

(i) The test statistic *SNR* corresponding to a linear filter is only conditionally Gaussian for small processing gains, short PN sequences, low thermal noise and strong interference, given all the bits in the filter memory that significantly affect the current bit test statistic. This ISI should be accounted for by averaging the conditional *BER*s. For a long PN sequence and small processing gain, the test statistic corresponding to the Kalman filter is non-Gaussian. Therefore, test statistic *SNR* does not yield the correct *BER* through application of Gaussian assumption.

(ii) Any conclusions regarding comparative performance of linear and nonlinear filters based on *SNR* of the test statistics can be misleading, as shown by a situation (low thermal noise, strong interference) where the *SNR* of the nonlinear filter is much higher than those of the linear filters, but its error rate is also higher than the other two error rates. Even the comparison of two nonlinear filters based on *SNR* could be misleading as in the case of nonlinear predictor and filter showing almost same *BER*s but differing in *SNR*s by almost 10 dB.

(iii) The *SNR* of the chip rate sequence at the output of the filter does not lead to a good estimate of the test statistic *SNR* when the narrowband interference is strong, because of significant residual chip correlation.

(iv) For moderate processing gain and PN sequence, *SNR* provides reasonably accurate error estimates for all the filters and parameters that were tested. These error estimates are close for all three filters.

References

[1] L.B. Milstein, "Interference rejection techniques in spread spectrum communications," Proc. IEEE, vol. 76, No.6, June 1988, pp 657-671.

[2] F.M. Hsu and A. A. Giordano, "Digital whitening techniques for improving spread spectrum communications performance in the presence of narrowband jamming and interference," IEEE Trans. on Commun., Vol. COM-26, No. 2, Feb. 1978, pp 209-216.

[3] J. W. Ketchum and J. G. Proakis, "Adaptive algorithms for estimating and suppressing narrowband interference in PN spread spectrum systems," IEEE Trans. on Commun.,Vol COM-30, No. 5, May 1982, pp 913-924.

[4] R. A. Iltis and L. B. Milstein, "Performance analysis of narrowband interference rejection techniques in DS spread spectrum systems," IEEE Trans. on Commun.,Vol COM-32, No. 11, Nov. 1984, pp 1169-1177.

[5] R. Vijayan and H. V. Poor, "Nonlinear techniques for interference suppression in spread spectrum systems," IEEE Trans. on Commun, Vol. 38, No. 7, July 1990, pp 1060-1065.

[6] Arif Ansari and R. Viswanathan, "Performance comparison of Maximum-Likelihood and transversal filters for the rejection of narrowband interference in direct-sequence spread spectrum systems," IEEE Trans. Com. accepted for publication.

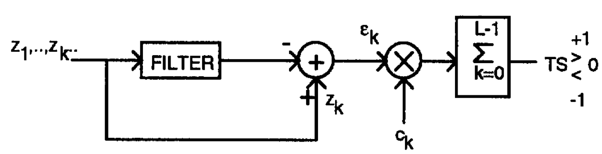

Fig. 1. Filter Structure for Narrowband Interference Rejection and Bit Decision of Spread Spectrum Receiver

Table I
SNR and BER Estimates

L=N=7			KALMAN		MODIFIED		NON-LIN.	
σ_n^2	σ_i^2		FILT.	PRED.	FILT.	PRED.	FILT.	PRED.
1.0	1000	T	5.4290	5.4290	5.1442	5.1442	5.3215	5.4254
		P_T^*	9.9E-3	9.9E-3	1.2E-2	1.2E-2	1.0E-2	9.9E-3
		γ_B	7.3093	4.3601	7.7073	4.23	7.7429	4.411
		P_S^*	3.4E-3	1.8E-2	2.7E-3	1.9E-2	3.2E-3	1.8E-2
		P_b^*	9.8E-3	9.8E-3	1.2E-2	1.2E-2	9.5E-3	9.5E-3
0.01	10000	T	8.6684	8.6684	41.725	41.725	843.36	87.309
		P_T^*	1.6E-3	1.6E-3	5.E-11	5.E-11	1E-185	4.E-21
		γ_B	13.614	7.2408	8.6836	2.7871	2073.4	85.0
		P_S^*	1.1E-4	3.5E-3	1.6E-3	4.7E-2	0**	1.E-20
		P_b^*	2.7E-4	2.7E-4	#	#	4.0E-5	4.0E-5
0.1	10000	T	6.8427	6.8427	6.6322	6.6322	7.9695	8.9847
		P_T^*	4.4E-3	4.4E-3	5.0E-3	5.0E-3	2.3E-3	1.3E-3
		γ_B	12.68	6.4178	10.51	4.503	23.627	9.91
		P_S^*	1.8E-4	5.6E-3	5.9E-4	1.7E-2	5.8E-7	8.2E-4
		P_b^*	1.7E-3	1.7E-3	8.2E-4	8.2E-4	8.9E-3	6.4E-3
1.0	10000	T	3.3297	3.3297	2.8303	2.8303	3.2994	3.4016
		P_T^*	3.4E-2	3.4E-2	4.6E-2	4.6E-2	3.4E-2	3.2E-2
		γ_B	7.007	2.954	7.481	2.831	7.1305	2.9890
		P_S^*	4.0E-3	4.3E-2	3.1E-3	4.6E-2	3.8E-3	4.2E-2
		P_b^*	3.9E-2	3.9E-2	5.0E-2	5.0E-2	3.8E-2	3.8E-2

#1 error observed in 10^6 trials **Zero up to machine precision

Table II
SNR and BER Estimates

L=7, N=1023			KALMAN		MODIFIED		NONLINEAR	
σ_n^2	σ_i^2		FILTER	PRED.	FILTER	PRED.	FILTER	PRED.
1.0	1000	T	4.2585	4.2585	4.18026	4.18026	4.2388	4.3052
		P_T^*	1.95E-2	1.95E-2	2.04E-2	2.04E-2	1.97E-2	1.89E-2
		P_b^*	2.32E-2	2.32E-2	2.33E-2	2.33E-2	2.33E-2	2.30E-2
0.01	10000	T	8.083	8.083	2.699	2.699	708.548	78.598
		P_T^*	2.23E-3	2.23E-3	5.02E-2	5.02E-2	0**	3.8E-19
		P_b^*	1.63E-2	1.63E-2	4.82E-2	4.82E-2	2.2E-4	1.9E-4
0.1	10000	T	7.0672	7.0672	4.7866	4.7866	8.951	10.692
		P_T^*	3.92E-3	3.92E-3	1.43E-2	1.43E-2	1.38E-2	5.38E-4
		P_b^*	1.80E-2	1.80E-2	2.45E-2	2.45E-2	1.35E-2	1.24E-2
1.0	10000	T	2.9706	2.9706	2.8912	2.8912	2.909	2.988
		P_T^*	4.24E-2	4.24E-2	4.45E-2	4.45E-2	4.40E-2	4.19E-2
		P_b^*	4.72E-2	4.72E-2	4.82E-2	4.82E-2	4.79E-2	4.74E-2

**Zero up to machine precision

Table III
BER Estimates

L=N=63			KALMAN		MODIFIED		NONLINEAR	
σ_n^2	σ_i^2		FILT.	PRED.	FILT.	PRED.	FILT.	PRED.
10.0	10,000	P_T^*	1.84E-2	1.84E-2	1.83E-2	1.83E-2	1.85E-2	1.85E-2
		P_b^*	1.98E-2	1.98E-2	1.97E-2	1.97E-2	1.98E-2	1.98E-2
10.0	100,000	P_T^*	3.79E-2	3.79E-2	3.75E-2	3.75E-2	3.79E-2	3.79E-2
		P_b^*	3.38E-2	3.38E-2	3.35E-2	3.33E-2	3.38E-2	3.38E-2
3.0	100,000	P_T^*	1.2E-3	1.11E-3	1.09E-3	1.09E-3	1.20E-3	1.11E-3
		P_b^*	1.4E-3	1.4E-3	1.43E-3	1.43E-3	1.38E-3	1.4E-3
5.0	100,000	P_T^*	7.13E-3	7.13E-3	6.90E-3	6.90E-3	7.16E-3	7.14E-3
		P_b^*	7.57E-3	7.57E-3	7.22E-3	7.22E-3	7.59E-3	7.58E-3

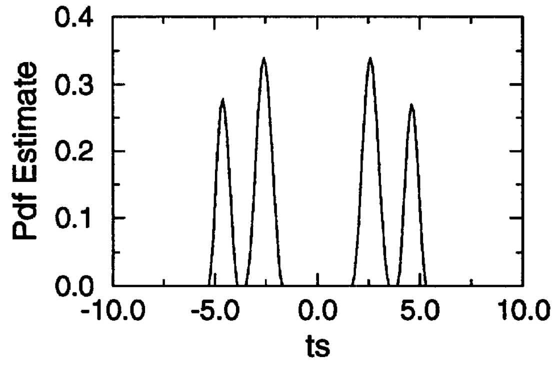

Fig. 2 Estimate of the density of t_S for Kalman (L=N=7) $\sigma_n^2 = 0.01, \sigma_i^2 = 10000$

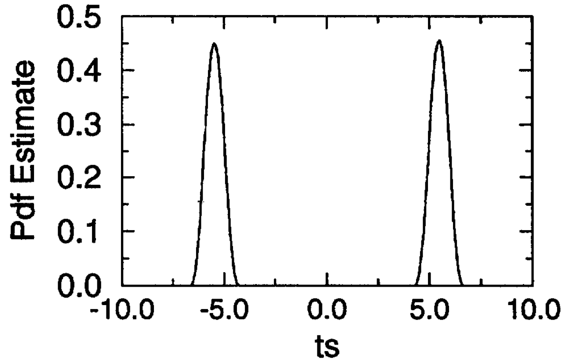

Fig. 3 Estimate of the density of t_S for Kalman (L=N=7) $\sigma_n^2 = 1.0, \sigma_i^2 = 1000$

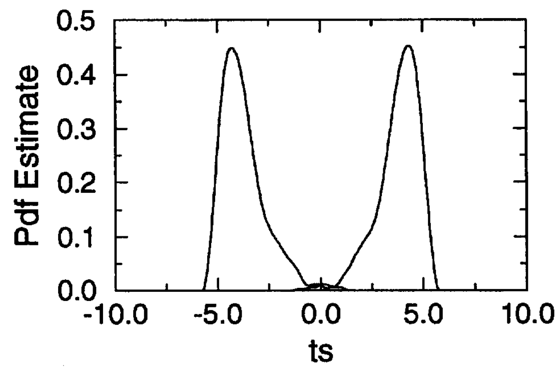

Fig. 4 Estimate of the density of t_S for Kalman(L=7,N=1023) $\sigma_n^2 = 0.01, \sigma_i^2 = 10000$