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The Stable Manifold Theorem for Nonlinear Stochastic Systems with Memory: II. The Local Stable Manifold Theorem

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THE STABLE MANIFOLD THEOREM FOR NONLINEAR STOCHASTIC SYSTEMS WITH MEMORY

II: THE LOCAL STABLE MANIFOLD THEOREM.*

Salah-Eldin A. Mohammed[‡] and Michael K. R. Scheutzow[†]

Abstract. We state and prove a *Local Stable Manifold Theorem* (Theorem 4.1) for non-linear stochastic differential systems with finite memory (viz. stochastic functional differential equations (sfde's)). We introduce the notion of hyperbolicity for stationary trajectories of sfde's. We then establish the existence of smooth stable and unstable manifolds in a neighborhood of a hyperbolic stationary trajectory. The stable and unstable manifolds are stationary and asymptotically invariant under the stochastic semiflow. The proof uses infinite-dimensional multiplicative ergodic theory techniques developed by D. Ruelle, together with interpolation arguments.

1. Preliminaries.

This paper is a sequel to [M-S.3]. In [M-S.3], we constructed a smooth locally compact stochastic semiflow for a large class of non-linear stochastic functional differential equations (sfde's) exemplified by (I) below. In this paper, we will use the stochastic semiflow constructed in [M-S.3] in order to develop a non-linear multiplicative ergodic theory for sfde's. The theory is used to characterize local stability of trajectories of the sfde in the neighborhood of a stationary trajectory. In order to describe this characterization

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more precisely, and for the rest of the article, we will recall some of the formulation and notation in [M-S.3].

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote by $\bar{\mathcal{F}}$ the P-completion of $\mathcal{F}$, and let $(\Omega, \bar{\mathcal{F}}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a complete filtered probability space satisfying the usual conditions ([Pr]).

Denote by $W : \mathbf{R}\times\Omega \to \mathbf{R}^p$, p-dimensional Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}}, P)$. Throughout the paper, we will adopt the following set-up:

(i) Let $\theta : \mathbf{R} \times \Omega \to \Omega$ be a P-preserving flow on Ω, viz.

  (a) θ is $(\mathcal{B}(\mathbf{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable,

  (b) $\theta(t + s, \cdot) = \theta(t, \cdot) \circ \theta(s, \cdot), \quad s, t \in \mathbf{R}$,

  (c) $\theta(0, \cdot) = I_\Omega$, the identity map on Ω,

  (d) $P \circ \theta(t, \cdot)^{-1} = P, \quad t \in \mathbf{R}$.

(ii) θ is ergodic.

(iii) Let $\{\mathcal{F}_t^s : -\infty < s \leq t < \infty\}$ be a family of sub-σ-algebras of $\bar{\mathcal{F}}$ satisfying the following conditions:

  (a) $\theta(-r, \cdot)(\mathcal{F}_t^s) = \mathcal{F}_{t+r}^{s+r}$ for all $r \in \mathbf{R}, -\infty < s \leq t < \infty$.

  (b) For each $s \in \mathbf{R}$, $(\Omega, \bar{\mathcal{F}}, (\mathcal{F}_{s+u}^s)_{u\geq 0}, P)$ is a filtered probability space satisfying the usual conditions, and $\mathcal{F}_t^0 = \mathcal{F}_t$, $t \geq 0$ ([Pr]).

(iv) The Brownian motion is a *helix* with respect to θ: For every $s \in \mathbf{R}$, there exists a sure event $\Omega_s \in \mathcal{F}$ such that

$$W(t + s, \omega) = W(t, \theta(s, \omega)) + W(s, \omega)$$

for all $t \in \mathbf{R}$, all $\omega \in \Omega_s$.

Consider the autonomous sfde:

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t)\,dt + G(x(t))\,dW(t), \quad t > 0 \\ x(0) &= v \in \mathbf{R}^d, \quad x_0 = \eta \in L^2([-r,0], \mathbf{R}^d), \end{aligned}\right\} \qquad (I)$$

driven by the Brownian motion $W : \mathbf{R} \times \Omega \to \mathbf{R}^p$. Let $r > 0$. The solution $x : [-r, \infty) \times \Omega \to \mathbf{R}^d$ is $(\mathcal{B}([-r,\infty)) \otimes \mathcal{F}, \mathcal{B}(\mathbf{R}^d))$-measurable and $(\mathcal{F}_t)_{t\geq 0}$-adapted. For each $t \geq 0$, $x_t \in L^2([-r,0], \mathbf{R}^d)$ is the segment

$$x_t(\cdot, \omega)(s) := x(t+s, \omega), \quad s \in [-r, 0], \quad \omega \in \Omega.$$

The coefficients H and G in (I) are continuous non-linear functionals $H : M_2 \to \mathbf{R}^d$, $G : \mathbf{R}^d \to L(\mathbf{R}^p, \mathbf{R}^d)$, satisfying the regularity hypotheses $(SMW)_{k,\delta}$ stated below. Recall that the space $M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d)$ carries the natural Hilbert norm

$$\|(v,\eta)\|_{M_2}^2 := |v|^2 + \|\eta\|_{L^2}^2, \qquad v \in \mathbf{R}^d,\ \eta \in L^2([-r,0], \mathbf{R}^d).$$

For a general theory of sfde's of type (I) the reader may refer to [Mo.1] and [Mo.4].

In order to specify our regularity hypotheses on the coefficients of (I), we shall recall some notation from [M-S.3] which will be used throughout this article.

Let E, N, K, L be real Banach spaces. Denote by $L^k(E,N)$ the Banach space of all continuous k-multilinear maps $A : E^k \to N$ with the uniform norm $\|A\| := \sup\{|A(v_1, v_2, \cdots, v_k)| : v_i \in E, |v_i| \leq 1, i = 1, \cdots, k\}$. Suppose $U \subseteq E$ is an open set. A map $f : U \to N$ is said to be of class $C^{k,\delta}$ $(k \geq 1, \delta \in (0,1])$ if it is C^k and if $D^k f : U \to L^k(E,N)$ is δ-Hölder continuous on bounded sets in U. A $C^{k,\delta}$ map $f : U \to N$ $(k \geq 1, \delta \in (0,1])$ is said to be of class $C_b^{k,\delta}$ if all its derivatives $D^j f : U \to L^k(E,N)$, $1 \leq j \leq k$, are globally bounded on U, and $D^k f : U \to L^k(E,N)$ is δ-Hölder continuous on U. When U is bounded, we denote by $C^{k,\delta}(U,N)$ the Banach space of all $C^{k,\delta}$ maps $f : U \to N$ given the norm

$$\|f\|_{k,\delta} := \sum_{0\leq j\leq k} \sup_{v\in U} \|D^j f(v)\| + \sup_{(v,v')\in (U\times U)\cap \triangle^c} \frac{\|D^k f(v) - D^k f(v')\|}{|v - v'|^\delta}$$

where $\triangle := \{(v,v) : v \in E\}$. Let

$$Y : \mathbf{R} \times E \times N \times K \times \Omega \to L$$

$$(t,z,v,\eta,\omega) \mapsto Y(t,z,v,\eta,\omega)$$

be a random field that is a.s. Fréchet differentiable in (z,v,η). We will denote its partial Fréchet derivatives with respect to z, v, η by $D_2Y(t,z,v,\eta,\omega) : E \to L, D_3Y(t,z,v,\eta,\omega) : N \to L$ and $D_4Y(t,z,v,\eta,\omega) : K \to L$ respectively.

The following hypotheses will be imposed on (I) throughout this paper.

Hypotheses $(SMW)_{k,\delta}$**.**

(1) $H : M_2 \to \mathbf{R}^d$ is of class $C_b^{k,\delta}$ and is globally bounded.

(2) $G : \mathbf{R}^d \to L(\mathbf{R}^p, \mathbf{R}^d)$ is of class $C_b^{k+1,\delta}$.

Assume Hypotheses $(SMW)_{k,\delta}$ for some $k \geq 1, \delta \in (0,1]$. Then by Theorem 4.1 ([M-S.3]), the sfde (I) has a stochastic semiflow which we will denote by $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$, where $X(t,(v,\eta),\cdot) := (x^{(v,\eta)}(t), x_t^{(v,\eta)})$ a.s. for all $(t,(v,\eta)) \in \mathbf{R}^+ \times M_2$, and $x^{(v,\eta)}$ is the unique solution of (I) through $(v,\eta) \in M_2$. The stochastic semiflow of (I) has a version, also denoted by X, such that the pair (X,θ) is a perfect cocycle on M_2, viz.

$$X(t_1+t_2,(v,\eta),\omega) = X(t_2, X(t_1,(v,\eta),\omega), \theta(t_1,\omega))$$

for all $\omega \in \Omega$, $t_1, t_2 \geq 0$, $(v,\eta) \in M_2$. Furthermore, each $X(t,\cdot,\omega)$ is locally compact for $t \geq r$, of class $C^{k,\epsilon}$ for any $\epsilon \in (0,\delta)$, and $DX(t,(v,\eta),\omega)$ is compact linear for every $(v,\eta) \in M_2$ ([M-S.3], Theorem 4.1).

Our main objective in this article is to prove a random non-linear saddle-point property for the sfde (I) under the regularity Hypotheses $(SMW)_{k,\delta}$ on the coefficients (Theorem 4.1). Theorem 4.1 is a *local stable manifold theorem* for the sfde (I). Like its deterministic counterpart, this theorem gives a local non-linear random set of coordinates in a neighborhood of a hyperbolic stationary trajectory. Such a set of coordinates consists of random stationary families of infinite-dimensional stable manifolds and a corresponding stationary family of finite-dimensional unstable manifolds for the stochastic semiflow. The stable and unstable manifolds intersect transversally at the stationary trajectory and are asymptotically invariant under the stochastic semiflow.

We next give a broad outline of the key ideas that go into the proof of the above result.

- By definition, a *stationary* random point $Y(\omega) \in M_2$ is invariant under the semiflow X; viz $X(t, Y) = Y(\theta(t, \cdot))$ for all times t.
- We linearize the semiflow X along the stationary point $Y(\omega)$ in M_2. In view of the stationarity of Y and the cocycle property of X, this gives a linear perfect cocycle $(D_2X(t, Y), \theta(t, \cdot))$ in $L(M_2)$, where D_2 denotes the first spatial (Fréchet) derivative in the M_2-variable.
- In view of the ergodicity of θ, we can introduce the notion of *hyperbolicity* for a stationary trajectory of (I) as follows. Use local compactness of the semiflow for times greater than the delay r (Part I, Theorem 4.1 (iii)), and apply Ruelle-Oseledec's multiplicative ergodic theorem in order to yield a discrete non-random Lyapunov spectrum $\{\lambda_i : i \geq 1\}$ for the linearized cocycle. Say that Y is *hyperbolic* if $\lambda_i \neq 0$ for every $i \geq 1$.
- Assuming that $\|Y\|^\epsilon$ is integrable (for small ϵ) and using the method of construction of the semiflow in Part I, we show that the linearized cocycle satisfies the hypotheses for "perfect versions" of the ergodic theorem and Kingman's subadditive ergodic theorem (Lemmas 5.1, 5.2). These refined versions yield invariance of the Oseledec spaces under the continuous-time linearized cocycle. In particular, the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear semiflow X.

- We establish continuous-time integrability estimates on the spatial derivatives of the non-linear cocycle X in a neighborhood of the stationary point Y. These estimates follow from the construction of the stochastic semiflow in Part I coupled with known global spatial estimates for finite-dimensional stochastic flows.
- We introduce the auxiliary perfect cocycle

$$Z(t,\cdot,\omega) := X(t,(\cdot)+Y(\omega),\omega) - Y(\theta(t,\omega)), \quad t \in \mathbf{R}^+, \omega \in \Omega.$$

  By refining the arguments in proofs by Ruelle ([Ru.2], Theorems 5.1 and 6.1), we construct local stable/unstable manifolds for the discrete cocycle $(Z(nr,\cdot,\omega), \theta(nr,\omega))$ near 0 and hence (by translation) for $X(nr,\cdot,\omega)$ near $Y(\omega)$ for all ω sampled from a $\theta(t,\cdot)$-invariant sure event in Ω. This is possible because of the continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem (Lemmas 3.2, 5.1, 5.2). By interpolating within delay periods of length r and further refining the arguments in the proofs of Ruelle's theorems (Theorems 5.1, 6.1, [Ru.2]), we then show that the above manifolds also serve as local stable/unstable manifolds for the *continuous-time* semiflow X near Y.
- The final key step is to establish the asymptotic invariance of the local stable manifolds under the stochastic semiflow X. This is achieved by appealing to the arguments underlying the proofs of Theorems 4.1 and 5.1 in Ruelle [Ru.2] and some additional estimates using the continuous-time integrability properties, and the perfect subadditive ergodic theorem. The asymptotic invariance of the local unstable manifolds follows by employing the concept of a *history process* for X (Theorem 4.1 (d)) coupled with similar arguments to the above. The existence of the history process compensates for the lack of invertibility of the semiflow.

Remark.

The results in this paper can be extended to cover the following class of sfde's driven by Kunita-type spatial semimartingales ([M-S.3]):

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t)\,\mu(dt) + G(dt, x(t), g(x_t)), \quad t > 0 \\ x(0) &= v \in \mathbf{R}^d, \quad x_0 = \eta \in L^2([-r, 0], \mathbf{R}^d). \end{aligned}\right\} \tag{I'}$$

In (I'), H, G, g, μ satisfy the hypotheses in Section 5(i), (GE)(i), and (C$'$) of [M-S.3]. In addition, assume that for every finite $T > 0$, the random variable $\sup_{0 \le t \le T} \mu(t, \cdot)$ has moments of all orders. We further assume that H and g are $C_b^{k,\delta}$, $C_b^{k+1,\delta}$ (resp.) and are globally bounded. Furthermore, G is a helix with respect to a P-preserving ergodic shift $\theta : \mathbf{R} \times \Omega \to \Omega$ and μ is an adapted non-decreasing continuous helix.

2. Stationary Trajectories. Hyperbolicity.

In this section, we will introduce the notion of a stationary hyperbolic trajectory for the sfde (I). This is an essential ingredient of the local stable manifold theorem for (I) (Theorem 4.1).

Definition 2.1.

Say that the sfde (I) has a *stationary point* if there exists an $(\mathcal{F}, \mathcal{B}(M_2))$-measurable random variable $Y : \Omega \to M_2$ such that

$$X(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{1}$$

for all $t \in \mathbf{R}^+$ and every $\omega \in \Omega$. We will refer to $X(t, Y)$ as a *stationary trajectory* of (I).

Note that, in general, a stationary trajectory is anticipating. On the other hand, the distribution of a non-anticipating stationary trajectory is an invariant measure for the Markov trajectory $\{(x^{(v,\eta)}(t), x_t^{(v,\eta)}) : (t, (v, \eta)) \in \mathbf{R}^+ \times M_2\}$ of (I). More precisely, suppose $Y : \Omega \to M_2$ is an $\mathcal{F}$-measurable stationary random point for the sfde (I) satisfying the identity (1) and independent of the Brownian motion $W(t), t \ge 0$. Let $\rho := P \circ Y^{-1}$ be the distribution of Y. Using the independence of Y and and $W(t), t \geq 0$, the reader may check directly that ρ is an invariant probability measure on M_2 for the Markov trajectory $\{(x^{(v,\eta)}(t), x_t^{(v,\eta)}) : (t,(v,\eta)) \in \mathbf{R}^+ \times M_2\}$ of (I). (Cf. [A], [Ba], [Cr], [Le], [L-Y].)

Example.

Consider the affine linear sfde

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t)\,dt + G\,dW(t), \quad t > 0, \\ x(0) &= v \in \mathbf{R}^d, \quad x_0 = \eta \in L^2([-r,0], \mathbf{R}^d), \end{aligned}\right\} \tag{I''}$$

where $H : M_2 \to \mathbf{R}^d$ is a continuous linear map, $G : \mathbf{R}^p \to \mathbf{R}^d$ is linear, and W is p-dimensional Brownian motion. Assume that the linear deterministic fde

$$dy(t) = H(y(t), y_t)\,dt, \quad t \geq 0,$$

has a semiflow $T_t \in L(M_2), t \geq 0$, which is uniformly asymptotically stable. Set

$$Y := \int_{-\infty}^{0} T_{-u}(G\,dW(u), 0). \tag{2}$$

Using integration by parts and the fact that

$$W(t, \theta(t_1, \omega)) = W(t + t_1, \omega) - W(t_1, \omega), \quad t, t_1 \in \mathbf{R}, \tag{3}$$

the reader may check that Y has an $(\mathcal{F}, \mathcal{B}(M_2))$-measurable version satisfying (1). Note also that Y is Gaussian and thus has finite moments of all orders. See ([Mo.1], Theorem 4.2, Corollary 4.2.1, pp. 208-217.) More generally, when H is hyperbolic, one can show that a stationary point of (I'') exists ([Mo.1]).

Sufficient conditions for the existence (and uniqueness) of stationary points for the sfde (I) are given in [I-N] and the appendix to this paper.

Remarks.

(i) If (1) holds for each $t \in \mathbf{R}^+$ on a sure event Ω_t that may depend on t, then there is a version of Y such that (1) holds identically for all $\omega \in \Omega$ and all $t \in \mathbf{R}^+$ ([Sc]).

(ii) The stationary trajectory extends to a meaningful trajectory for *negative* times; that is

$$X(t, Y(\theta(s,\omega)), \theta(s,\omega)) = Y(\theta(t+s,\omega)) \tag{4}$$

for all $s \in \mathbf{R}$, $t \in \mathbf{R}^+$ and every $\omega \in \Omega$. To see this, we let the sfde start at negative initial instants t_0 and then solve forward in time:

$$\left.\begin{aligned} x(t) &= v + \int_{t_0}^{t} H(x(u), x_u)\, du + \int_{t_0}^{t} G(x(u))\, dW(u), \quad t \geq t_0 \\ x(t) &= \eta(t - t_0), \quad t_0 - r < t < t_0 \end{aligned}\right\} \tag{I'''}$$

where $(v,\eta) \in L^2(\Omega, M_2; \mathcal{F}_{t_0})$. Denote by $X_t^{t_0}((v,\eta),\omega)$ the trajectory $\{(x(t), x_t) : t \geq t_0,\ (x(t_0), x_{t_0}) = (v,\eta)\}$ of (I'''). Then by the remark following the proof of Theorem 4.1 ([M-S.3]), one has $X_t^{t_0}((v,\eta),\omega) = X(t - t_0, (v,\eta), \theta(t_0,\omega)),\ t \geq t_0,\ \omega \in \Omega, (v,\eta) \in M_2$. In particular, (1) implies that $X_t^{t_0}(Y(\theta(t_0,\omega)),\omega) = Y(\theta(t,\omega)),\ t \geq t_0,\ \omega \in \Omega$.

We now describe a procedure for generating stationary points when the sfde (I) admits stationary solutions in the sense of [I-N].

Without loss of generality, assume that the sfde (I) and its driving Brownian motion W are defined on the canonical filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}}, P)$; viz. $\Omega := C(\mathbf{R}, \mathbf{R}^p; 0), \mathcal{F} := \mathcal{B}(C(\mathbf{R}, \mathbf{R}^p; 0))$, P is Wiener measure on Ω, $\mathcal{F}_t :=$ the P-completion of the σ-algebra $\sigma\{e_u - e_v : v \leq u \leq t\}, t \in \mathbf{R}$, and $e_u : C(\mathbf{R}, \mathbf{R}^p; 0) \ni \omega \mapsto \omega(u) \in \mathbf{R}^p, u \in \mathbf{R}$, are evaluation maps.

Define $\tilde{\Omega} := C(\mathbf{R}, \mathbf{R}^d) \times C(\mathbf{R}, \mathbf{R}^p; 0)$. Furnish $\tilde{\Omega}$ with the σ-algebra $\tilde{\mathcal{F}} := \mathcal{B}(C(\mathbf{R}, \mathbf{R}^d)) \otimes \mathcal{B}(C(\mathbf{R}, \mathbf{R}^p; 0))$. In the following computations, sample points from $\tilde{\Omega}$ will be denoted by

$\tilde{\omega} := (f, \omega) \in C(\mathbf{R}, \mathbf{R}^d) \times C(\mathbf{R}, \mathbf{R}^p; 0)$. Define the processes $x^\infty : \mathbf{R} \times \tilde{\Omega} \to \mathbf{R}^d$ and $W^\infty : \mathbf{R} \times \tilde{\Omega} \to \mathbf{R}^p$ by

$$x^\infty(t, \tilde{\omega}) := f(t), \ \ W^\infty(t, \tilde{\omega}) := W(t, \omega) = \omega(t)$$

for all $t \in \mathbf{R}, \ \tilde{\omega} := (f, \omega) \in \tilde{\Omega}$.

Assume that x^∞ is a *stationary solution* of the sfde (I) (cf. [I-N], pp. 2-3). That is, there exists a probability measure P^∞ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ such that the following is true:

(i) W^∞ is p-dimensional standard Brownian motion on $(\tilde{\Omega}, \tilde{\mathcal{F}}, P^\infty)$.

(ii) (x^∞, dW^∞) are *strictly stationarily correlated* in the sense that the law of the process

$$(x^\infty(t, \cdot), W^\infty(u, \cdot) - W^\infty(v, \cdot), t \in \mathbf{R}, v \leq u)$$

is invariant under time-shifts.

(iii) The σ-algebra $\sigma\{x^\infty(u) : u \leq t\} \vee \sigma\{W^\infty(u, \cdot) - W^\infty(v, \cdot), v \leq u \leq t\}$ is independent of $\sigma\{W^\infty(u, \cdot) - W^\infty(v, \cdot), t \leq v \leq u\}$ under P^∞ for each $t \in \mathbf{R}$.

(iv) x^∞ is a two-sided solution of (I) when W is replaced by W^∞:

$$dx^\infty(t) = H(x^\infty(t), x_t^\infty)\, dt + G(x^\infty(t))\, dW^\infty(t), \quad t > s > -\infty. \qquad (I^\infty)$$

See ([I-N]) and the appendix to this article for a method of constructing stationary solutions of (I).

We will show below that the stationary solution x^∞ gives rise to a stationary point in the sense of Definition 2.1.

Let $\tilde{\theta} : \mathbf{R} \times \tilde{\Omega} \to \tilde{\Omega}$ denote the two-sided shift

$$\tilde{\theta}(t, \tilde{\omega}) := (f(t + \cdot), \theta(t, \omega)), \quad t \in \mathbf{R}, \ \tilde{\omega} := (f, \omega) \in \tilde{\Omega},$$

where $\theta : \mathbf{R} \times C(\mathbf{R}, \mathbf{R}^p; 0) \to C(\mathbf{R}, \mathbf{R}^p; 0)$ is the canonical Brownian shift

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in C(\mathbf{R}, \mathbf{R}^p; 0).$$

It is easy to check that $(W^\infty, \tilde{\theta})$ is a helix. Next observe that P^∞ is invariant under the two-sided shift $\tilde{\theta}(t,\cdot) : \tilde{\Omega} \to \tilde{\Omega}$, $t \in \mathbf{R}$, viz. $P^\infty \circ \tilde{\theta}(t,\cdot)^{-1} = P^\infty$ for all $t \in \mathbf{R}$. This is a consequence of the definition of $\tilde{\theta}$ and the fact that (x^∞, dW^∞) are strictly stationarily correlated.

Let $(X(t,\cdot,\omega), \theta(t,\omega))$, $t \geq 0$, be the perfect cocycle on M_2 associated with the sfde (I). Define the random field $\tilde{X} : \mathbf{R}^+ \times M_2 \times \tilde{\Omega} \to M_2$ by

$$\tilde{X}(t,(v,\eta),\tilde{\omega}) := X(t,(v,\eta),\omega), \quad t \geq 0, \tilde{\omega} := (f,\omega) \in \tilde{\Omega}, (v,\eta) \in M_2.$$

It is easy to see that $(\tilde{X}(t,\cdot,\tilde{\omega}), \tilde{\theta}(t,\tilde{\omega})), t \geq 0$, is the perfect cocycle on M_2 generated by trajectories of the sfde (I) on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, P^\infty)$, and where W is replaced by W^∞.

Define the $(\tilde{\mathcal{F}}, \mathcal{B}(C([-r,0],\mathbf{R}^d)))$-measurable random variable $Z : \tilde{\Omega} \to C([-r,0],\mathbf{R}^d)$ by $Z(\tilde{\omega}) := x_0^\infty(\cdot,\tilde{\omega})$ for all $\tilde{\omega} \in \tilde{\Omega}$. It follows directly from the definitions of x^∞ and $\tilde{\theta}$ that

$$Z(\tilde{\theta}(t,\tilde{\omega})) = x_t^\infty(\cdot,\tilde{\omega}), \quad t \in \mathbf{R},\ \tilde{\omega} \in \tilde{\Omega}.$$

Now define the random mapping $Y : \tilde{\Omega} \to M_2$ by

$$Y(\tilde{\omega}) := (Z(\tilde{\omega})(0), Z(\tilde{\omega})), \quad \tilde{\omega} \in \tilde{\Omega}.$$

Clearly Y is $(\tilde{\mathcal{F}}, \mathcal{B}(M_2))$-measurable. Furthermore, for P^∞-a.a. $\tilde{\omega} \in \tilde{\Omega}$, we have

$$\begin{aligned} Y(\tilde{\theta}(t,\tilde{\omega})) &= (x^\infty(t,\tilde{\omega}), x_t^\infty(\cdot,\tilde{\omega})) \\ &= \tilde{X}(t,(x^\infty(0,\tilde{\omega}), x_0^\infty(\cdot,\tilde{\omega})),\tilde{\omega}) \\ &= X(t,(Z(\tilde{\omega})(0), Z(\tilde{\omega})),\omega) \\ &= \tilde{X}(t, Y(\tilde{\omega}), \tilde{\omega}) \end{aligned}$$

for all $t \geq 0$. Hence there is an $\tilde{\mathcal{F}}$-measurable version of Y (also denoted by the same symbol) such that the equality

$$Y(\tilde{\theta}(t,\tilde{\omega})) = \tilde{X}(t, Y(\tilde{\omega}), \tilde{\omega})$$

holds for *all* $\tilde{\omega} \in \tilde{\Omega}$ and all $t \geq 0$ (Remark (i) above, [Sc]). This shows that Y is a stationary point for the cocycle $(\tilde{X}, \tilde{\theta})$ in the sense of Definition 2.1. Furthermore, and in order to satisfy the set-up in Section 1, we stipulate that the stationary measure P^∞ is ergodic with respect to the two-sided shift $\tilde{\theta}$.

Note that if we pick a stationary solution of (I) in the sense of [I-N] (Appendix, Theorem 6.1), then Y will be independent of the forward increments $\{W^\infty(u,\cdot)-W^\infty(v,\cdot), 0 \leq v \leq u\}$ under P^∞, because in this case x_0^∞ will have the same property.

Lemma 2.1.

Assume Hypotheses $(SMW)_{k,\delta}$ $(k \geq 1, \delta \in (0,1])$. *Let* Y *be a stationary point of (I) such that* $E(\|Y\|^{\epsilon_0}) < \infty$ *for some* $\epsilon_0 > 0$. *Then the semiflow* X *of (I) satisfies*

$$\int_\Omega \log^+ \sup_{0 \leq t_1, t_2 \leq T} \|D_2 X(t_2, Y(\theta(t_1,\omega)), \theta(t_1,\omega))\|_{L(M_2)} \, dP(\omega) < \infty \tag{5}$$

for any fixed $0 < T < \infty$.

In particular, the linearized semiflow $(D_2 X(t, Y(\omega), \omega), \theta(t,\omega))$ *is an* $L(M_2)$*-valued perfect cocycle with a discrete fixed Lyapunov spectrum* $\{-\infty < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. *If the Lyapunov spectrum is infinite, then* $\lambda_{i+1} < \lambda_i$ *for all* $i \geq 1$*; otherwise there is a fixed (non-random) integer* $N > 1$ *such that* $\{\lambda_N = -\infty < \lambda_{N-1} < \cdots < \lambda_2 < \lambda_1\}$. *Furthermore, each finite* $\lambda_i (\in \mathbf{R})$ *has finite non-random multiplicity.*

Proof.

The proof of the lemma is based on linearizing the random variational integral equation underlying (I), which was established in [M-S.3]. More specifically, the sfde (I) is equivalent to the following random integral equation:

$$\zeta(t, x(t,\omega), \omega) = v + \int_0^t F(u, \zeta(u, x(u,\omega), \omega), x(u,\omega), x_u(\cdot,\omega), \omega) \, du, \tag{6}$$

where $0 \leq t \leq T, (v,\eta) \in M_2$, and $F : [0,\infty) \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ is given by

$$F(t, z, v, \eta, \omega) := \{D\psi(t, z, \omega)\}^{-1} H(v, \eta) \tag{7}$$

for all $t \geq 0, z, v \in \mathbf{R}^d, \eta \in L^2([-r,0],\mathbf{R}^d), \omega \in \Omega$. In (6), the random field $\zeta : [0,\infty) \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ is defined by

$$\zeta(t,x,\omega) := \psi(t,\cdot,\omega)^{-1}(x), \quad t \geq 0,\, x \in \mathbf{R}^d,\, \omega \in \Omega.$$

In (6) and (7), ψ is the $C^{k+1,\epsilon}$ $(0 < \epsilon < \delta)$ stochastic flow of the stochastic ordinary differential equation (without delay)(sode):

$$\left.\begin{aligned} d\psi(t) &= G(\psi(t))\, dW(t), \quad t \geq 0 \\ \psi(0) &= x \in \mathbf{R}^d. \end{aligned}\right\} \tag{8}$$

The sode (8) generates a perfect cocycle (ψ, θ):

$$\psi(t_1 + t_2, \cdot, \omega) = \psi(t_2, \cdot, \theta(t_1,\omega)) \circ \psi(t_1, \cdot, \omega), \quad t_1, t_2 \geq 0, \omega \in \Omega.$$

We quote the following estimates on ψ from [M-S.2] and [Ku]:

$$\sup_{0 \leq t \leq T} |\psi(t,x,\omega)| \leq K(\omega)[1 + |x|(\log^+ |x|)^\epsilon] \tag{9}$$
$$\sup_{0 \leq t \leq T} |\zeta(t,x,\omega)| \leq K(\omega)[1 + |x|(\log^+ |x|)^\epsilon] \tag{10}$$
$$\sup_{0 \leq t \leq T} \|D^j\psi(t,x,\omega)\| \leq K(\omega)(1 + |x|^\epsilon) \tag{11}$$
$$\sup_{0 \leq t \leq T} \|[D\psi(t,x,\omega)]^{-1}\| \leq K(\omega)(1 + |x|^\epsilon) \tag{12}$$

for each $\epsilon > 0$, $1 \leq j \leq k+1$, some $K = K(\epsilon, \omega, T) > 0$ and all $x \in \mathbf{R}^d$. The $\mathcal{F}$-measurable random variable $K(\epsilon, \cdot, T)$ has moments of all orders.

Write $x(t,(v,\eta),\omega) := x^{0,(v,\eta)}(t,\omega) = \psi(t, \zeta(t, x(t), \omega), \omega)$, and take Fréchet derivatives in (v,η) to obtain

$$D_2x(t,(v,\eta),\theta(t_1,\omega))(v_1,\eta_1)$$
$$= D_2\psi(t, \zeta(t, x(t,(v,\eta),\theta(t_1\omega)), \theta(t_1\omega))\Big[v_1 +$$
$$\int_0^t \{D_2F(u, \zeta(u, x(u,(v,\eta),\theta(t_1,\omega)),\theta(t_1,\omega)), x(u,(v,\eta),\theta(t_1,\omega)), x_u(\cdot,(v,\eta),\theta(t_1,\omega)), \theta(t_1,\omega))\cdot$$

$$
\begin{aligned}
&\cdot D_2\zeta(u, x(u,(v,\eta),\theta(t_1,\omega)),\theta(t_1,\omega)) D_2 x(u,(v,\eta),\theta(t_1,\omega))(v_1,\eta_1)\\
&+ D_3F(u,\zeta(u,x(u,(v,\eta),\theta(t_1,\omega)),\theta(t_1,\omega)), x(u,(v,\eta),\theta(t_1,\omega)), x_u(\cdot,(v,\eta),\omega),\theta(t_1,\omega))\cdot\\
&\qquad \cdot D_2 x(u,(v,\eta),\theta(t_1,\omega))(v_1,\eta_1)\\
&+ D_4F(u,\zeta(u,x(u,(v,\eta),\theta(t_1\omega)),\theta(t_1,\omega)), x(u,(v,\eta),\theta(t_1,\omega)), x_u(\cdot,(v,\eta),\omega),\theta(t_1,\omega))\cdot\\
&\qquad \cdot D_2 x_u(\cdot,(v,\eta),\theta(t_1,\omega))(v_1,\eta_1)\} \, du\Big]
\end{aligned}
\tag{13}
$$

for any fixed $(v,\eta),(v_1,\eta_1) \in M_2, \omega \in \Omega$ and $0 < t < T$.

In the estimates below, we will denote by $\epsilon > 0$ an arbitrarily small number, T a positive real number and $K_i := K_i(\epsilon,\cdot,T), i = 1,2,3,\cdots$, positive $\mathcal{F}$-measurable random constants that have moments of all orders. *For the rest of this proof, the choice of $\epsilon > 0$ may vary from line to line.* For brevity of notation, set $y(t) := \zeta(t, x(t,(v,\eta),\theta(t_1,\omega)),\theta(t_1,\omega))$.

We claim that there is a random positive constant K_1 such that

$$
\left.
\begin{aligned}
|F(t,z,v,\eta,\theta(t_1,\omega))| &\le K_1(\omega)(1+|z|^\epsilon)\\
\|D_iF(t,z,v,\eta,\theta(t_1,\omega))\| &\le K_1(\omega)(1+|z|^\epsilon)
\end{aligned}
\right\}
\tag{14}
$$

for $0 \le t, t_1 \le T, \omega \in \Omega, z, v \in \mathbf{R}^d, \eta \in L^2([-r,0],\mathbf{R}^d)$, $i = 2,3,4$. We will prove the first inequality in (14), and leave the proof of the second inequality to the reader. The following inequalities follow directly from (7), the global boundedness of H, the cocycle property for ψ, the chain rule, and (9)-(12):

$$
\begin{aligned}
|F(t,z,v,\eta,&\theta(t_1,\omega))|\\
&\le C_1 \|[D\psi(t,z,\theta(t_1,\omega))]^{-1}\|\\
&\le C_2(\omega)\|D\psi(t_1,\psi(t_1,\cdot,\omega)^{-1}(z),\omega)\| \cdot \|[D\psi(t+t_1,\psi(t_1,\cdot,\omega)^{-1}(z),\omega)]^{-1}\|\\
&\le C_3(\omega)[1+|\psi(t_1,\cdot,\omega)^{-1}(z)|^\epsilon]^2\\
&\le C_4(\omega)[1+|1+|z|(\log^+|z|)^\epsilon|^\epsilon]^2\\
&\le K_1(\omega)[1+|z|^\epsilon]
\end{aligned}
$$

for $0 \le t, t_1 \le T, \omega \in \Omega, z, v \in \mathbf{R}^d, \eta \in L^2([-r,0],\mathbf{R}^d)$. In the above inequalities, $C_i, i = 1, 2, \cdots, 4$, are (possibly) random positive constants with moments of all orders. This completes the proof of the first inequality in (14).

From (13), (14) and (12), it follows that

$$\begin{aligned}
&\|D_2 x(t,(v,\eta),\theta(t_1,\omega))\| \le K_2(\omega)(1+|y(t)|^\epsilon)\cdot \\
&\quad \cdot \Big[1 + K_3(\omega)\int_0^t \{(1+|y(u)|^\epsilon)(1+|x(u,(v,\eta),\theta(t_1,\omega))|^\epsilon)\|D_2 x(u,(v,\eta),\theta(t_1,\omega))\| \\
&\qquad + (1+|y(u)|^\epsilon)\|D_2 x_u(\cdot,(v,\eta),\theta(t_1,\omega))\|\}\, du\Big] \qquad (15)
\end{aligned}$$

for all $(v,\eta) \in M_2, t \in [0,T], \omega \in \Omega$. Now using the relation

$$x(t,(v,\eta),\theta(t_1,\omega)) = \psi(t, y(t), \theta(t_1,\omega)),$$

the estimate (9) and the cocycle property for ψ, it is easy to see that

$$|x(t,(v,\eta),\theta(t_1,\omega))| \le K_4(\omega)[1 + |y(t)|(\log^+ |y(t)|)^\epsilon] \qquad (16)$$

for all $\omega \in \Omega$, and $t, t_1 \in [0,T]$.

Fix $\omega \in \Omega$, and $t, t_1 \in [0,T]$. Then using (6) and (14), we get

$$\begin{aligned}
|y(t)| &\le |v| + K_1(\omega)\int_0^t (1+|y(u)|^\epsilon)\, du \\
&\le K_5(\omega) + |v| + K_1(\omega)\int_0^t |y(u)|^\epsilon\, du \\
&\le K_6(\omega)\Big[1 + |v| + \int_0^t |y(u)|^\epsilon\, du\Big]. \qquad (17)
\end{aligned}$$

Define

$$y^*(t) := \sup_{\substack{0\le u\le t \\ 0\le t_1\le T}} (|y(u)| \vee 1).$$

Then (17) implies that

$$|y^*(t)| \le K_7(\omega)\Big[1 + |v| + \int_0^t |y^*(u)|^\epsilon\, du\Big]. \qquad (18)$$

Now divide both sides of the above inequality by $|y^*(t)|^\epsilon$ to obtain

$$|y^*(t)|^{1-\epsilon} \leq K_8(\omega)[1+|v|] \tag{19}$$

Therefore, (replacing ϵ by $1-\epsilon$ in (19)), we get

$$|y^*(t)| = |y^*(t)|^\epsilon \cdot |y^*(t)|^{1-\epsilon} \leq K_9(\omega)[1+|v|^2]. \tag{20}$$

Let

$$x^*(t) := \sup_{\substack{0\leq u\leq t\\ 0\leq t_1\leq T}} |x(u,(v,\eta),\theta(t_1,\omega))|.$$

Then (16) and (20) imply that

$$|x^*(t)| \leq K_{10}(\omega)[1+|v|^2(\log^+|v|)^\epsilon]. \tag{21}$$

Next let

$$\alpha(t) := \sup_{\substack{0\leq u\leq t\\ 0\leq t_1\leq T}} \|D_2 x(u,(v,\eta),\theta(t_1,\omega))\|.$$

We will estimate $\|D_2 x_t(\cdot,(v,\eta),\omega)\|$ in terms of $\|D_2 x(u,(v,\eta),\omega)\|, 0 \leq u \leq t$. Let $(v,\eta),(v_1,\eta_1) \in M_2, \eta_2 \in L^2([-r,0],\mathbf{R}^d), t\in[0,r], h\in\mathbf{R}, \omega\in\Omega$. Then

$$\begin{aligned}
&| < D_2 x_t(\cdot,(v,\eta),\omega)(v_1,\eta_1),\eta_2 > | \\
&\qquad \leq \left|\lim_{h\to 0}\frac{1}{h}\int_{-r}^0 < [x(t+s,(v,\eta)+h(v_1,\eta_1),\omega) - x(t+s,(v,\eta),\omega)],\eta_2(s) > ds\right| \\
&\qquad \leq \left|\int_{-r}^{-t} < \eta_1(t+s),\eta_2(s) > ds\right| + \left|\int_0^t < [D_2 x(s,(v,\eta),\omega)((v_1,\eta_1)),\eta_2(s-t) > ds\right| \\
&\qquad \leq \|\eta_1\|\cdot\|\eta_2\| + \sqrt{r}\sup_{0\leq s\leq t}\|D_2 x(s,(v,\eta),\omega)((v_1,\eta_1))\|\cdot\|\eta_2\|
\end{aligned} \tag{22}$$

Therefore,

$$\|D_2 x_t(\cdot,(v,\eta),\omega)\|_{L(M_2,L^2)} \leq 1+\sqrt{r}\sup_{0\leq s\leq t}\|D_2 x(s,(v,\eta),\omega)\|_{L(M_2,\mathbf{R}^d)}, \tag{23}$$

for all $t\in[0,r],(v,\eta)\in M_2,\omega\in\Omega$. For $t\geq r$, a similar argument to the above also gives (23).

From (15), (20), (21) and (23), it follows that

$$\alpha(t) \leq K_{11}(1+|v|^{2\epsilon})\left[1+\int_0^t [1+|v|^2(\log^+|v|)^\epsilon]^\epsilon \alpha(u)\,du\right].$$

By Gronwall's lemma, the above inequality implies that

$$\alpha(t) \leq K_{12}(\omega)(1+|v|^\epsilon)e^{K_{13}(\omega)(1+|v|^\epsilon)}. \tag{24}$$

Taking log^+ in the above inequality, it is not difficult to see that, for sufficiently small $\epsilon > 0$,

$$\log^+ \|D_2X(t_2,(v,\eta),\theta(t_1,\omega)))\|_{L(M_2)} \leq \log^+ K_{14}(\omega) + K_{15}(\omega)|v|^\epsilon \tag{25}$$

for all $(v,\eta) \in M_2, \omega \in \Omega, t_1, t_2 \in [0,T]$, where $K_{14} = K_{14}(\epsilon,\cdot,T), K_{15} = K_{15}(\epsilon,\cdot,T)$ have moments of all orders. Observe that the function on the left-hand side of (25) is jointly measurable in $(t_1,t_2,(v,\eta),\omega)$ because of the remark following the proof of Theorem 4.1 ([M-S.3]). Assertion (5) of the lemma now follows from the above inequality by replacing (v,η) with $Y(\theta(t_1,\omega)) = X(t_1,Y(\omega),\omega)$, using (21) and the fact that $E(\|Y\|^\epsilon) < \infty$ for $0 < \epsilon \leq \epsilon_0$.

The perfect cocycle property for $(D_2X(t,Y(\omega),\omega),\theta(t,\omega))$ follows directly by taking Fréchet derivatives at $(v,\eta) = Y(\omega)$ on both sides of the cocycle identity for (X,θ); viz.

$$\begin{aligned} D_2X(t_1+t_2,Y(\omega),\omega) &= D_2X(t_2,X(t_1,Y(\omega),\omega),\theta(t_1,\omega)) \circ D_2X(t_1,Y(\omega),\omega) \\ &= D_2X(t_2,Y(\theta(t_1,\omega)),\theta(t_1,\omega)) \circ D_2X(t_1,Y(\omega),\omega) \end{aligned}$$

for all $\omega \in \Omega, t_1, t_2 \geq 0$. The existence of a fixed discrete spectrum for the linearized cocycle follows directly from the integrability property (5), the compactness of the derivative $D_2X(r,Y(\omega),\omega)$ ([M-S.3], Theorem 4.1 (iii)), and the analysis in [Ru.2], [Mo.2] and [M-S.1]. This completes the proof of the lemma. $\square$

Remark.

If we differentiate the sfde (I) at any $(v, \eta) \in M_2$, then the derivative flow

$$y(t) := \begin{cases} [D_2 X(t, (v, \eta), \omega)(v_1, \eta_1)]^1, & t > 0 \\ \eta_1(t), & -r < t < 0 \end{cases}$$

satisfies the linearized sfde

$$\left. \begin{aligned} dy(t) &= DH(X(t,(v,\eta)))(y(t), y_t)\,dt + DG(X^1(t,(v,\eta)))(y(t))\,dW(t) \\ &\qquad\qquad\qquad\qquad\qquad\qquad t > 0 \\ y(0) &= v_1 \in \mathbf{R}^d, \;\; y_0 = \eta_1 \in L^2([-r,0], \mathbf{R}^d) \end{aligned} \right\} \qquad (II)$$

(cf. [Mo.1], Corollary 2.1.3, p. 136). In (II), the superscript 1 denotes the projection of M_2 onto the first factor $\mathbf{R}^d$. On the other hand, it is not clear whether the anticipating process

$$\tilde{y}(t) := \begin{cases} [D_2 X(t, Y(\omega), \omega)(v_1, \eta_1)]^1, & t > 0 \\ \eta_1(t), & -r < t < 0 \end{cases}$$

satisfies the linear sfde obtained from (II) by replacing (v, η) with $Y(\omega)$. The substitution theorems in [M-S.4], [N] and [M-N-S] do not seem to apply in our present infinite-dimensional setting. Of course, the above difficulty does not arise in the rather special case when $Y(\omega)$ is fixed independently of ω; e.g. $H(0,0) = 0$, $G(0) = 0$.

Definition 2.2.

A stationary point $Y(\omega)$ of (I) is said to be *hyperbolic* if the linearized cocycle $(D_2 X(t, Y(\omega), \omega), \theta(t, \omega))$ has a non-vanishing Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, viz. $\lambda_i \neq 0$ for all $i \geq 1$.

By the integrability property (5) and Theorem 4 [Mo.2], one obtains the sequence of closed finite-codimensional Oseledec spaces

$$\cdots E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = M_2$$

where

$$E_i(\omega) = \{(v,\eta) \in M_2 : \lim_{t\to\infty} \frac{1}{t} \log \|D_2X(t, Y(\omega), \omega)(v,\eta)\| \le \lambda_i\}, \quad i \ge 1,$$

for all $\omega \in \Omega^*$, a sure event in $\mathcal{F}$ satisfying $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$.

Furthermore, we will denote by $\{\mathcal{U}(\omega), \mathcal{S}(\omega) : \omega \in \Omega^*\}$ the unstable and stable subspaces associated with the linearized cocycle (D_2X, θ) as given by ([Mo.2], Section 4, Corollary 2) and ([M-S.1], Theorem 5.3). In particular, one has the $\mathcal{F}$-measurable invariant splitting

$$M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \Omega^*,$$

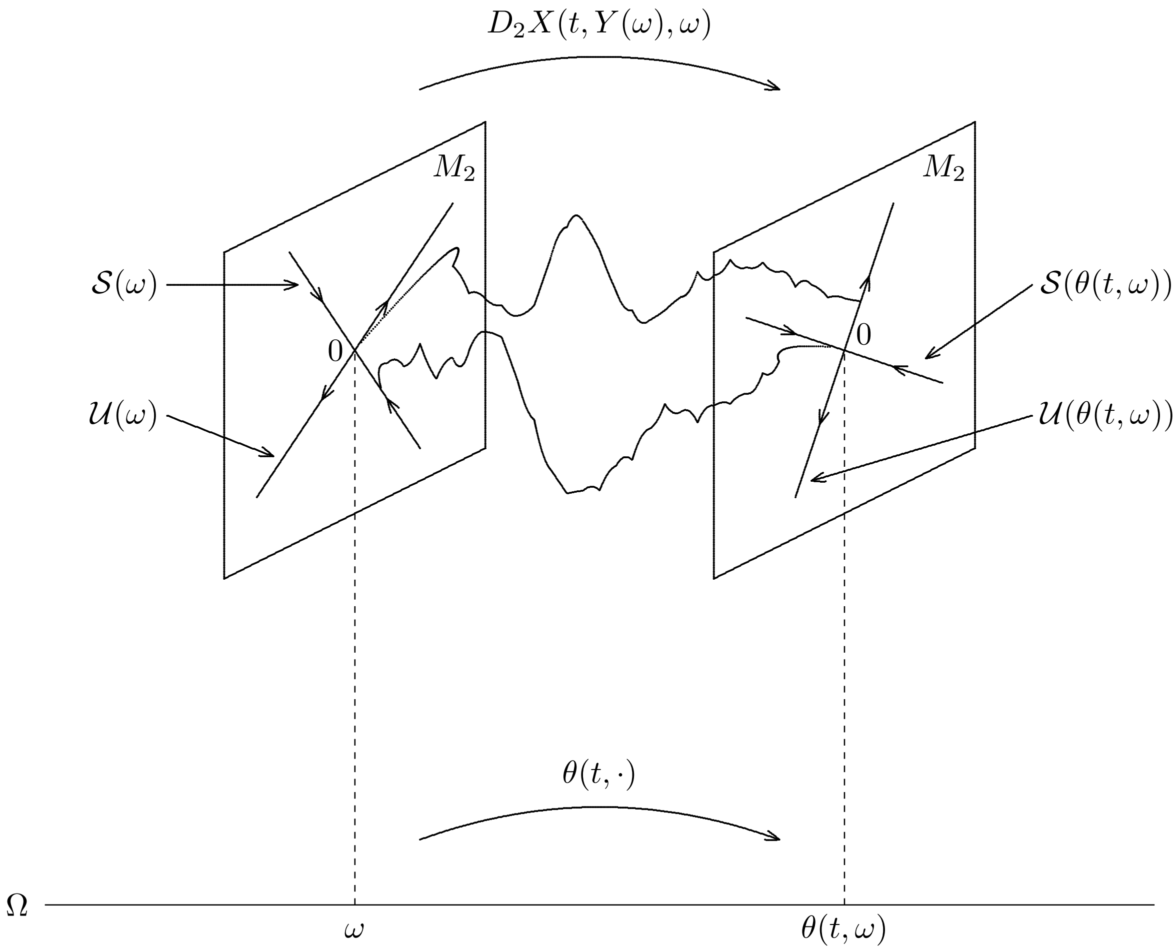
$$D_2X(t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t,\omega)),\ D_2X(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)), \quad t \ge 0,$$

together with the exponential dichotomies

$$\|D_2X(t, Y(\omega), \omega)(v,\eta)\|_{M_2} \ge \|(v,\eta)\|_{M_2} e^{\delta_1 t} \quad \text{for all} \quad t \ge \tau_1^*, (v,\eta) \in \mathcal{U}(\omega),$$

$$\|D_2X(t, Y(\omega), \omega)(v,\eta)\|_{M_2} \le \|(v,\eta)\|_{M_2} e^{-\delta_2 t} \quad \text{for all} \quad t \ge \tau_2^*, (v,\eta) \in \mathcal{S}(\omega),$$

where $\tau_i^* = \tau_i^*(v,\eta,\omega) > 0, i = 1,2$, are random times and $\delta_i > 0, i = 1,2$, are fixed. Note that the unstable subspaces $\mathcal{U}(\omega), \omega \in \Omega^*$, are constructed using long-term behavior of the adjoint linearized cocycle $([D_2X(t,\cdot)]^*, \theta(-t,\cdot))$ for $t \ge 0$ ([Mo.2], Section 4, Corollary 2).



3. Integrability estimates.

In the subsequent sections, we shall prove a local stable manifold theorem for the sfde (I) near a hyperbolic stationary trajectory. This will be achieved by developing further integrability estimates on higher-order Fréchet derivatives of X in the neighborhood of the stationary point, and then applying Ruelle's discrete non-linear infinite-dimensional ergodic theorems ([Ru.2], Theorems 5.1, 6.1, pp. 272-282). In order to do this, we will first assume throughout this section that $Y : \Omega \to M_2$ is an $\mathcal{F}$-measurable hyperbolic stationary point of (I). Next, we introduce the following auxiliary cocycle $Z : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$, which is essentially a "centering" of the semiflow X about the stationary trajectory:

$$Z(t,(v,\eta),\omega) := X(t,(v,\eta) + Y(\omega),\omega) - Y(\theta(t,\omega)) \tag{1}$$

for $t \geq 0, (v, \eta) \in M_2, \omega \in \Omega$.

Lemma 3.1.

(Z, θ) is a perfect cocycle on M_2 and $Z(t, 0, \omega) = 0$ for all $t \geq 0$, and all $\omega \in \Omega$.

Proof.

Let $t_1, t_2 \geq 0, \omega \in \Omega, (v, \eta) \in M_2$. Then by the cocycle property for X, we have

$$\begin{aligned} Z(t_2, Z(t_1, (v, \eta), \omega), \theta(t_1, \omega)) &= X(t_2, Z(t_1, (v, \eta), \omega) + Y(\theta(t_1, \omega)), \theta(t_1, \omega)) - Y(\theta(t_2, \theta(t_1, \omega))) \\ &= X(t_2, X(t_1, (v, \eta) + Y(\omega), \omega), \theta(t_1, \omega)) - Y(\theta(t_2 + t_1, \omega)) \\ &= Z(t_1 + t_2, (v, \eta), \omega). \end{aligned}$$

Therefore, (Z, θ) is a perfect cocycle.

The assertion $Z(t, 0, \omega) = 0$, $t \geq 0, \omega \in \Omega$, follows directly from (1) and Definition 2.1. □

If $\rho \in \mathbf{R}^+$ and $(v, \eta) \in M_2$, recall that $B((v, \eta), \rho)$ is the open ball with center (v, η) and radius ρ in M_2. Denote by $\bar{B}((v, \eta), \rho)$ the corresponding closed ball. For any integer $k \geq 1$ and $\epsilon \in (0, 1)$, recall that $\| \cdot \|_{k,\epsilon}$ is the $C^{k,\epsilon}$-norm on the space $C^{k,\epsilon}(\bar{B}(0, \rho), M_2)$.

The following lemma will be needed for the construction of the stable/unstable manifolds.

Lemma 3.2.

Assume Hypotheses $(SMW)_{k,\delta}$ $(k \geq 1, \delta \in (0, 1])$. Let Y be a stationary point of (I) such that $E(\|Y\|^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$. Then the semiflow X of (I) satisfies

$$\int_\Omega \log^+ \sup_{0 \leq t_1, t_2 \leq T} \|X(t_2, Y(\theta(t_1, \omega)) + (\cdot), \theta(t_1, \omega))\|_{k,\epsilon} \, dP(\omega) < \infty \tag{2}$$

for any fixed $0 < \rho, T < \infty$ and $\epsilon \in (0, \delta)$.

Proof.

We first prove the estimate (2) for $k = 1, \epsilon = 0$. Let $t_1, t_2 \in [0,T]$, $(v,\eta) \in \bar{B}(0,\rho), \omega \in \Omega$, and Y be a stationary point satisfying the hypotheses of the lemma. In this proof, we will use $K_i := K_i(\epsilon, T)$, $i = 1,2,3,\cdots$, to denote random positive constants that have moments of all orders, for a sufficiently small positive ϵ. Unless stated otherwise, all the inequalities in this proof are presumed to hold for *sufficiently small* $\epsilon \in (0, \epsilon_0)$. By inequality (21) of the proof of Lemma 2.1, we get

$$\begin{aligned}
\log^+ \sup_{0 \le t_1, t_2 \le T} &\|X(t_2, Y(\theta(t_1,\omega)) + (v,\eta), \theta(t_1,\omega))\| \\
&\le K_1(\omega)[1 + \log^+ \sup_{0 \le t_1 \le T} \|X(t_1, Y(\omega), \omega) + (v,\eta)\|] \\
&\le K_2(\omega)[1 + \log^+ \|Y(\omega) + (v,\eta)\|] \\
&\le K_3(\omega)[1 + \log^+ \|Y(\omega)\| + \log^+ \|(v,\eta)\|]. \qquad (3)
\end{aligned}$$

Now, from (25) of the proof of Lemma 2.1, we obtain

$$\begin{aligned}
\log^+ &\|D_2 X(t_2, Y(\theta(t_1,\omega)) + (v,\eta), \theta(t_1,\omega)))\|_{L(M_2)} \\
&\le \log^+ K_4(\omega) + K_5(\omega)[\sup_{0 \le t_1 \le T} \|X(t_1, Y(\omega), \omega)\|^\epsilon + |v|^\epsilon] \\
&\le \log^+ K_4(\omega) + K_6(\omega)[\|Y(\omega)\|^\epsilon + |v|^\epsilon]. \qquad (4)
\end{aligned}$$

Take suprema over $(v,\eta) \in \bar{B}(0,\rho)$ in (3) and (4), use the integrability of $\|Y(\cdot)\|^{\epsilon_0}$ and note the fact that K_3, K_4, K_6 have moments of all orders. This immediately gives (2) for $k = 1, \epsilon = 0$.

We next prove (2) for $k > 1, \epsilon = 0$. To do this, define

$$y(t,(v,\eta),\omega) := \psi(t,\cdot,\omega)^{-1}(x(t,(v,\eta),\omega)) = \zeta(t, x(t,(v,\eta),\omega),\omega)$$

for $t \ge 0, (v,\eta) \in M_2, \omega \in \Omega$. Then take Fréchet derivatives of order k with respect to $(v,\eta) \in M_2$ in the following relation

$$x(t,(v,\eta),\theta(t_1,\omega)) = \psi(t,, y(t,(v,\eta),\theta(t_1,\omega)), \theta(t_1,\omega)), \quad t \ge 0, (v,\eta) \in M_2, \omega \in \Omega.$$

Using induction, the chain rule, and the cocycle property for ψ, this implies the following:

$$\begin{aligned}
&\|D_2^{(k)}x(t,(v,\eta),\theta(t_1,\omega))\| \\
&\le K_7(\omega)\times \sum_{\substack{m=2,\cdots,k\\ j_1+j_2+\cdots+j_m=k\\ j_1,j_2,\cdots j_m\ge 1}} \|D_2^{(m)}\psi(t,y(t,(v,\eta),\theta(t_1,\omega)),\theta(t_1,\omega))\|\|D_2^{(j_1)}y(t,(v,\eta),\theta(t_1,\omega))\|\cdots \\
&\cdot \|D_2^{(j_m)}y(t,(v,\eta),\theta(t_1,\omega))\| + \|D_2\psi(t,y(t,(v,\eta),\theta(t_1,\omega)),\theta(t_1,\omega))\|\|D_2^{(k)}y(t,(v,\eta),\theta(t_1,\omega))\| \\
&\le K_8(\omega)[1+|y(t,(v,\eta),\theta(t_1,\omega))|^\epsilon]\Big\{\max_{\substack{1\le j\le k-1\\ 1\le m\le k}} \|D_2^{(j)}y(t,(v,\eta),\theta(t_1,\omega))\|^m + \\
&\qquad + \|D_2^{(k)}y(t,(v,\eta),\theta(t_1,\omega))\|\Big\} \\
&\le K_9(\omega)[1+|v|^{2\epsilon}]\Big\{\max_{\substack{1\le j\le k-1\\ 1\le m\le k}} \|D_2^{(j)}y(t,(v,\eta),\theta(t_1,\omega))\|^m + \|D_2^{(k)}y(t,(v,\eta),\theta(t_1,\omega))\|\Big\}. \qquad (5)
\end{aligned}$$

for $t\ge 0, (v,\eta)\in M_2, \omega\in\Omega$. Therefore,

$$\begin{aligned}
&\|D_2^{(k)}x(t,(v,\eta),\theta(t_1,\omega))\| \\
&\le K_{10}(\omega)[1+|v|^\epsilon]\Big\{\max_{\substack{1\le j\le k-1\\ 1\le m\le k}} \|D_2^{(j)}y(t,(v,\eta),\theta(t_1,\omega))\|^m + \|D_2^{(k)}y(t,(v,\eta),\theta(t_1,\omega))\|\Big\} \qquad (6)
\end{aligned}$$

for all $(v,\eta)\in M_2, \omega\in\Omega, t, t_1\in[0,T]$.

Our next task is to estimate the higher-order Fréchet derivatives of $y(t,(v,\eta),\theta(t_1,\omega))$ appearing on the right hand side of (6) in terms of the corresponding derivatives of $x(u,(v,\eta),\theta(t_1,\omega))$ and $x_u(\cdot,(v,\eta),\theta(t_1,\omega))$ for $0\le u\le t$. In order to do this, we will adopt the following conventions for the sake of brevity:

$$\begin{aligned}
D_2^{(i)}[D_2\psi]^{-1}(v) &:= D_2^{(i)}[D_2\psi(t,\cdot,\theta(t_1,\omega))]^{-1}(v) \\
x(u) &:= x(u,(v,\eta),\theta(t_1,\omega)) \\
x_u &:= x_u(\cdot,(v,\eta),\theta(t_1,\omega)) \\
\zeta(u) &:= \zeta(u,x(u,v,\eta,\theta(t_1,\omega)),\theta(t_1,\omega))
\end{aligned}$$

for $u\ge 0, (v,\eta)\in M_2, \omega\in\Omega$. With the above notation, we claim that there are (deterministic) polynomials $P_l, l=1,2,3$, and $q_l, l=1,2$, such that the terms in each P_l consist of compositions of linear and multilinear maps, the terms in each q_l are compositions of powers of the Fréchet differentiation operators D_1, D_2, and the following relations hold:

$$\begin{aligned} &D_2^{(j)} y(t,(v,\eta),\theta(t_1,\omega)) \\ &\quad = a_j + \int_0^t P_1\Big(D_2^{(i)}[D_2\psi]^{-1}(\zeta(u)), 0 \le i \le j; D_2^{(i)} x(u), 0 \le i \le j-1; \\ &\qquad\qquad D_2^{(i)} x_u, 0 \le i \le j-1; q_1(D_1, D_2) H(x(u), x_u)\Big)\, du \\ &\qquad + \int_0^t P_2\Big([D_2\psi]^{-1}(\zeta(u)), D_2[D_2\psi]^{-1}(\zeta(u)); q_2(D_1) H(x(u), x_u); D_2^{(j)} x(u)\Big)\, du \\ &\qquad + \int_0^t P_3\Big([D_2\psi]^{-1}(\zeta(u)); D_2 H(x(u), x_u); D_2^{(j)} x_u\Big)\, du, \end{aligned} \tag{7}$$

for $t \ge 0, (v,\eta) \in M_2, \omega \in \Omega, j \ge 1$. In the above relations, we further claim that the polynomials P_2, P_3 are linear in the last variable (and do not depend explicitly on $x(u)$ and x_u); $a_j = p_1$ if $j = 1$, where $p_1 : M_2 \to \mathbf{R}^d$ is the projection onto the first factor of $M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d)$; $a_j = 0$ if $j \ge 2$; the differential operator q_1 has order less than or equal to j, and the operator q_2 has order one. To check (7), use induction on $j \ge 1$. First, we check it for $j = 1$. Take Fréchet derivatives of both sides of the following random integral equation:

$$\begin{aligned} y(t,(v,\eta),\theta(t_1,\omega)) = v + \int_0^t &[D_2\psi(u, \zeta(u, x(u,v,\eta,\theta(t_1,\omega)), \theta(t_1\omega)), \theta(t_1,\omega))]^{-1} \cdot \\ &\cdot H(x(u,(v,\eta),\theta(t_1,\omega)), x_u(\cdot,(v,\eta),\theta(t_1,\omega)))\, du, \end{aligned}$$

$t \ge 0, (v,\eta) \in M_2, \omega \in \Omega$, and use the equality:

$$D_2[\psi^{-1}(u,\cdot,\theta(t_1,\omega))](x(u,(v,\eta),\theta(t_1,\omega))) = [D_2\psi(u,\zeta(u),\theta(t_1,\omega))]^{-1},$$

for $u \ge 0, (v,\eta) \in M_2, \omega \in \Omega$, (which follows from the chain rule). This gives

$$\begin{aligned} D_2 y(t,&(v,\eta),\theta(t_1,\omega)) \\ &= p_1 + \int_0^t D_2[D_2\psi]^{-1}(\zeta(u)) \cdot [D_2\psi]^{-1}(\zeta(u)) \cdot D_2 x(u) \cdot H(x(u), x_u)\, du \\ &\qquad + \int_0^t [D_2\psi]^{-1}(\zeta(u)) \cdot D_1 H(x(u), x_u) \cdot D_2 x(u)\, du \\ &\qquad + \int_0^t [D_2\psi]^{-1}(\zeta(u)) \cdot D_2 H(x(u), x_u) \cdot D_2 x_u\, du \end{aligned}$$

for $t \geq 0, (v, \eta) \in M_2, \omega \in \Omega$. The above equation clearly satisfies the general form given in (7) when $j = 1$. To complete the proof, assume (7) holds for some $j \geq 1$. Then differentiating (7)-using the chain and product rules-easily gives a corresponding equation for $D_2^{(j+1)} y(t, (v, \eta), \theta(t_1, \omega))$ with new choices of P_1, q_1, having the same properties as the old ones. Details are left to the reader. This proves our claim (7).

Now take operator norms on both sides of (7). This gives a positive (deterministic) constant K_{11} and non-negative fixed integers $n_l, l = 1, \cdots, 5$, such that

$$
\begin{aligned}
&\|D_2^{(j)} y(t, (v, \eta), \theta(t_1, \omega))\| \\
&\leq 1 + K_{11} \int_0^t \max_{0 \leq i \leq j} \left[\|D_2^{(i)} [D_2 \psi]^{-1} (\zeta(u))\|^{n_1} \vee 1 \right] \cdot \max_{0 \leq i \leq j-1} \left[\|D_2^{(i)} [D_2 \psi]^{-1} (x(u))\|^{n_2} \vee 1 \right] \\
&\qquad \cdot \max_{1 \leq i \leq j-1} \left[\|D_2^{(i)} x_u\|^{n_3} \vee 1 \right] \cdot \max_{1 \leq i \leq j-1} \left[\|D_2^{(i)} x(u)\|^{n_4} \vee 1 \right] du + \\
&+ K_{11} \int_0^t \max_{i=0,1} \left[\|D_2^{(i)} [D_2 \psi]^{-1} (\zeta(u))\|^{n_5} \vee 1 \right] \cdot (\|D_2^{(j)} x(u)\| + \|D_2^{(j)} x_u\|) \, du \qquad (7')
\end{aligned}
$$

for $j = 2, \cdots, k,\ t, t_1 \geq 0, (v, \eta) \in M_2, \omega \in \Omega$.

We next establish the estimate

$$
\|D_2^{(i)} [D_2 \psi(t, \cdot, \theta(t_1, \omega))]^{-1} (v)\| \leq K_{12}(\omega)[1 + |v|^\epsilon], \tag{8}
$$

for all $t, t_1 \in [0, T], \omega \in \Omega, v \in \mathbf{R}^d, 1 \leq i \leq k$. To prove (8), first note the following identity which is a consequence of the cocycle property for ψ and the chain rule:

$$
[D_2 \psi(t, v, \theta(t_1, \omega))]^{-1} = D_2 \psi(t_1, \psi(t_1, \cdot, \omega)^{-1}(v), \omega) \circ [D_2 \psi(t + t_1, \psi(t_1, \cdot, \omega)^{-1}(v), \omega)]^{-1}
$$

for $t, t_1 \geq 0, v \in \mathbf{R}^d, \omega \in \Omega$. Taking Fréchet derivatives with respect to v in the above identity, and making use of the relation

$$
D_2[\psi(t_1, \cdot, \omega)^{-1}](v) = [D_2 \psi(t_1, \psi(t_1, \cdot, \omega)^{-1}(v), \omega)]^{-1}, \quad t_1 \geq 0, v \in \mathbf{R}^d, \omega \in \Omega,
$$

one obtains

$$\begin{aligned}
&D_2^{(i)}[D_2\psi(t,\cdot,\theta(t_1,\omega))]^{-1}(v)\\
&= P_4\Big(D_2^{(j)}\psi(t_1,\psi(t_1,\cdot,\omega)^{-1}(v),\omega), 1\le j\le i; D_2^{(j)}\psi(t+t_1,\psi(t_1,\cdot,\omega)^{-1}(v),\omega), 1\le j\le i;\\
&\qquad [D_2\psi(t_1,\psi(t_1,\cdot,\omega)^{-1}(v),\omega)]^{-1}; [D_2\psi(t+t_1,\psi(t_1,\cdot,\omega)^{-1}(v),\omega)]^{-1}\Big),
\end{aligned}$$

for $t, t_1 \ge 0, v \in \mathbf{R}^d, \omega \in \Omega, 1 \le i \le k$, where P_4 is a fixed polynomial depending on i. Now (8) follows by taking norms in the above identity and using the estimates (9)-(12) in the proof of Lemma 2.1.

We will next prove the following estimates by induction on k:

$$\left.\begin{aligned}
\sup_{\substack{1\le i\le k-1\\ 1\le t,t_1\le T}} \|D_2^{(i)}x(t,(v,\eta),\theta(t_1,\omega))\| \le K_{13}(\omega)[1+|v|^\epsilon]\exp\{K_{14}(\omega)[1+|v|^\epsilon]\}\\
\sup_{\substack{1\le i\le k-1\\ 1\le t,t_1\le T}} \|D_2^{(i)}x_t(\cdot,(v,\eta),\theta(t_1,\omega))\| \le K_{15}(\omega)[1+|v|^\epsilon]\exp\{K_{16}(\omega)[1+|v|^\epsilon]\}.
\end{aligned}\right\} \tag{9^k}$$

for $(v,\eta) \in M_2, \omega \in \Omega$, $k \ge 2$.

From (25) of the proof of Lemma 2.1, it is easy to see that (9^k) holds for $k = 2$. Suppose (9^k) holds for some $k \ge 2$. Then by ($7'$), (8) and (9^k), we obtain

$$\sup_{\substack{1\le j\le k-1\\ 1\le t,t_1\le T}} \|D_2^{(j)}y(t,(v,\eta),\theta(t_1,\omega))\| \le K_{17}(\omega)[1+|v|^\epsilon]\exp\{K_{18}(\omega)[1+|v|^\epsilon]\} \tag{10}$$

Substituting from (10) and ($7'$) into (6), we get

$$\begin{aligned}
\|D_2^{(k)}x(t,(v,\eta),\theta(t_1,\omega))\| &\le K_{19}(\omega)[1+|v|^\epsilon]\exp\{K_{20}(\omega)[1+|v|^\epsilon]\}+\\
&+ K_{21}(\omega)\int_0^t \{\|D_2^{(k)}x(u,(v,\eta),\theta(t_1,\omega))\| + \|D_2^{(k)}x_u(\cdot,(v,\eta),\theta(t_1,\omega))\|)\}\,du
\end{aligned} \tag{11}$$

To complete the induction proof of (9^k), we will relate $D_2^{(k)}x(t,(v,\eta),\theta(t_1,\omega))$ and $D_2^{(k)}x_t(\cdot,(v,\eta),\theta(t_1,\omega))$. It is easy to see that

$$D_2x_t(\cdot,(v,\eta),\omega) = D_2x(t+(\cdot),(v,\eta),\omega) \tag{12}$$

for all $t \geq 0, (v,\eta) \in M_2, \omega \in \Omega$. By repeated Fréchet differentiations, we see that

$$D_2^{(k)} x_t(\cdot, (v,\eta), , \omega) = D_2^{(k)} x(t + (\cdot), (v,\eta), \omega) \tag{13}$$

for all $t \geq 0, (v,\eta) \in M_2, \omega \in \Omega$. This means that

$$D_2^{(k)} x_t(\cdot, (v,\eta), , \omega)((v_1,\eta_1), \cdots, (v_k,\eta_k))(s) = D_2^{(k)} x(t+s, (v,\eta), \omega)((v_1,\eta_1), \cdots, (v_k,\eta_k)) \tag{14}$$

for all $t \geq 0, (v,\eta) \in M_2, \omega \in \Omega, (v_i,\eta_i) \in M_2, 1 \leq i \leq k$, and almost every $s \in [-r, 0]$. The above relation easily implies that

$$\|D_2^{(k)} x_t(\cdot, (v,\eta), \omega)\| \leq 1 + \sqrt{r} \sup_{0 \leq s \leq t} \|D_2^{(k)} x(s, (v,\eta), \omega)\| \tag{15}$$

for all $t \in [0,T], (v,\eta) \in M_2, \omega \in \Omega$ (cf. (23) in the proof of Lemma (2.1)). The norms in the left-hand and right-hand-sides of (15) correspond to the spaces of k-multilinear maps $L^k(M_2, M_2)$ and $L^k(M_2, \mathbf{R}^d)$, respectively. From (15), (11) and Gronwall's lemma, we obtain

$$\sup_{1 \leq t, t_1 \leq T} \|D_2^{(k)} x(t, (v,\eta), \theta(t_1,\omega))\| \leq K_{22}(\omega)[1 + |v|^\epsilon] \exp\{K_{23}(\omega)[1 + |v|^\epsilon]\} \tag{16}$$

Combining (15) and (16) gives

$$\|D_2^{(k)} X(t, (v,\eta), \theta(t_1,\omega))\| \leq K_{24}(\omega)[1 + |v|^\epsilon] \exp\{K_{25}(\omega)[1 + |v|^\epsilon]\} \tag{17}$$

for all $t, t_1 \in [0,T], (v,\eta) \in M_2, \omega \in \Omega$. Therefore (9^{k+1}) holds. This completes the proof of (9^k).

In (17), we may replace (v,η) by $X(t_1, Y(\omega), \omega) + (v,\eta)$, and take $\log^+ \sup_{\substack{0 \leq t, t_1 \leq T \\ (v,\eta) \in \bar{B}(0,\rho)}}$ to obtain

$$\log^+ \sup_{0 \leq t_1, t_2 \leq T} \|X(t_2, Y(\theta(t_1,\omega)) + (\cdot), \theta(t_1,\omega))\|_k \leq \log^+ K_{26}(\omega) + K_{27}(\omega) + K_{28}(\omega) \log^+ \|Y(\omega)\| + K_{29}(\omega) \|Y(\omega)\|^\epsilon. \tag{18}$$

From the remark following the proof of Theorem 4.1 ([M-S.3]), the function on the left-hand side of the above inequality is $\mathcal{F}$-measurable in ω. By hypotheses, the right-hand-side of (18) belongs to $L^1(\Omega, \mathbf{R})$ for $0 < \epsilon \leq \epsilon_0$. Hence the lemma holds for $k \geq 1$, $\epsilon = 0$.

To treat the case $k \geq 1, \epsilon \in (0, \delta)$, let $(v_i, \eta_i) \in \bar{B}(0, \rho), i = 1, 2$, be such that $(v_1, \eta_1) \neq (v_2, \eta_2)$. Using (7), the Hölder properties of ψ, x, y and Hypotheses $(SMW)_{k,\delta}$, we obtain

$$\begin{aligned} &\|D_2^{(k)} y(t, (v_1, \eta_1), \theta(t_1, \omega)) - D_2^{(k)} y(t, (v_2, \eta_2), \theta(t_2, \omega))\| \\ &\qquad \leq K_{30}(\omega) \sum_{j=1}^{m} \|(v_1, \eta_1) - (v_2, \eta_2)\|^{\epsilon^j} + \\ &\qquad\quad + K_{31}(\omega) \int_0^t [\|D_2^{(k)} x(u, (v_1, \eta_1), \theta(t_1, \omega)) - D_2^{(k)} x(u, (v_2, \eta_2), \theta(t_2, \omega))\| \\ &\qquad\quad + \|D_2^{(k)} x_u(\cdot, (v_1, \eta_1), \theta(t_1, \omega)) - D_2^{(k)} x_u(\cdot, (v_2, \eta_2), \theta(t_2, \omega))\|]\, du \end{aligned} \tag{19}$$

for $t, t_1 \geq 0, \omega \in \Omega$, where m is some positive integer. Therefore, choosing a sufficiently small $\epsilon \in (0, \delta)$, dividing both sides of (19) by $\|(v_1, \eta_1) - (v_2, \eta_2)\|^\epsilon$ and taking supremum over all $(v_i, \eta_i) \in \bar{B}(0, \rho), (v_1, \eta_1) \neq (v_2, \eta_2)$, we obtain

$$\begin{aligned} &\|D_2^{(k)} y(t, \cdot, \theta(t_1, \omega))\|_\epsilon \leq K_{32}(\omega) + \\ &\qquad + K_{33}(\omega) \int_0^t [\|D_2^{(k)} x(u, \cdot, \theta(t_1, \omega))\|_\epsilon + \|D_2^{(k)} x_u(\cdot, \cdot, \theta(t_1, \omega))\|_\epsilon]\, du, \end{aligned} \tag{20}$$

for $t, t_1 \geq 0, \omega \in \Omega$. Taking k-th order Fréchet derivatives in the identity

$$x(t, (v, \eta), \theta(t_1, \omega)) = \psi(t, , y(t, (v, \eta), \theta(t_1, \omega)), \theta(t_1, \omega)), \quad t, t_1 \geq 0, (v, \eta) \in M_2, \omega \in \Omega,$$

and using the inequality (20), we get

$$\begin{aligned} &\|D_2^{(k)} x(t, \cdot, \theta(t_1, \omega))\|_\epsilon \\ &\qquad \leq K_{34}(\omega) + K_{35}(\omega) \int_0^t [\|D_2^{(k)} x(u, \cdot, \theta(t_1, \omega))\|_\epsilon + \|D_2^{(k)} x_u(\cdot, \cdot, \theta(t_1, \omega))\|_\epsilon]\, du, \end{aligned} \tag{21}$$

for $t, t_1 \geq 0, \omega \in \Omega$. Now use (21), (15) and Gronwall's lemma in order to obtain the estimate

$$\sup_{0 \leq t, t_1 \leq T} \|D_2^{(k)} X(t, \cdot, \theta(t_1, \omega))\|_\epsilon \leq K_{36}(\omega) e^{K_{37}(\omega)}, \quad \omega \in \Omega.$$

This completes the proof of the lemma. □

4. The Local Stable Manifold Theorem.

In this section, we present a local stable manifold theorem for the sfde (I) (Theorem 4.1 below). This theorem characterizes the local stability/unstability of the stochastic semiflow X of (I) in the neighborhood of a hyperbolic stationary point $Y(\omega) \in M_2, \omega \in \Omega$.

Theorem 4.1. *(The local stable manifold theorem)*

Assume Hypotheses $(SMW)_{k,\delta}$ $(k \geq 1, \delta \in (0,1])$. Let Y be a hyperbolic stationary point of the sfde (I) such that $E(\|Y(\cdot)\|^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$

Suppose the linearized cocycle $(D_2 X(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ of (I) has a Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. Define $\lambda_{i_0} := max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all finite λ_i are positive, set $\lambda_{i_0} = -\infty$. (This implies that $\lambda_{i_0 - 1}$ is the smallest positive Lyapunov exponent of the linearized semiflow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

- *(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,*
- *(ii) $\bar{\mathcal{F}}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0,1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:*

There are $C^{k,\epsilon}$ $(\epsilon \in (0, \delta))$ submanifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

- *(a) For $\lambda_{i_0} > -\infty$, $\tilde{\mathcal{S}}(\omega)$ is the set of all $(v, \eta) \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that*

$$\|X(nr, (v, \eta), \omega) - Y(\theta(nr, \omega))\| \leq \beta_1(\omega)\, e^{(\lambda_{i_0} + \epsilon_1) nr}$$

for all integers $n \geq 0$. *If* $\lambda_{i_0} = -\infty$, *then* $\tilde{\mathcal{S}}(\omega)$ *is the set of all* $(v, \eta) \in \bar{B}(Y(\omega), \rho_1(\omega))$ *such that*

$$\|X(nr, (v, \eta), \omega) - Y(\theta(nr, \omega))\| \leq \beta_1(\omega)\, e^{\lambda nr}$$

for all integers $n \geq 0$ *and any* $\lambda \in (-\infty, 0)$. *Furthermore,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega) - Y(\theta(t, \omega))\| \leq \lambda_{i_0} \tag{1}$$

for all $(v, \eta) \in \tilde{\mathcal{S}}(\omega)$. *Each stable subspace* $\mathcal{S}(\omega)$ *of the linearized semiflow* D_2X *is tangent at* $Y(\omega)$ *to the submanifold* $\tilde{\mathcal{S}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. *In particular, codim* $\tilde{\mathcal{S}}(\omega) =$ *codim* $\mathcal{S}(\omega)$, *is fixed and finite.*

(b) $\limsup_{t\to\infty} \frac{1}{t} \log\Big[\sup\Big\{ \frac{\|X(t, (v_1, \eta_1), \omega) - X(t, (v_2, \eta_2), \omega)\|}{\|(v_1, \eta_1) - (v_2, \eta_2)\|} : (v_1, \eta_1) \neq (v_2, \eta_2),\ (v_1, \eta_1), (v_2, \eta_2) \in \tilde{\mathcal{S}}(\omega) \Big\}\Big] \leq \lambda_{i_0}$.

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ *such that*

$$X(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega)) \tag{2}$$

for all $t \geq \tau_1(\omega)$. *Also*

$$D_2X(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)), \quad t \geq 0. \tag{3}$$

(d) For $\lambda_{i_0-1} < \infty$, $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $(v, \eta) \in \bar{B}(Y(\omega), \rho_2(\omega))$ *with the property that there is a discrete-time "history" process* $y(\cdot, \omega) : \{-nr : n \geq 0\} \to M_2$ *such that* $y(0, \omega) = (v, \eta)$ *and for each integer* $n \geq 1$, *one has* $X(r, y(-nr, \omega), \theta(-nr, \omega)) = y(-(n-1)r, \omega)$ *and*

$$\|y(-nr, \omega) - Y(\theta(-nr, \omega))\|_{M_2} \leq \beta_2(\omega) e^{-(\lambda_{i_0-1} - \epsilon_2)nr}.$$

If $\lambda_{i_0-1} = \infty$, $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $(v,\eta) \in \bar{B}(Y(\omega), \rho_2(\omega))$ *with the property that there is a discrete-time "history" process* $y(\cdot,\omega) : \{-nr : n \geq 0\} \to M_2$ *such that* $y(0,\omega) = (v,\eta)$ *and for each integer* $n \geq 1$,

$$\|y(-nr,\omega) - Y(\theta(-nr,\omega))\|_{M_2} \leq \beta_2(\omega)e^{-\lambda nr},$$

for any $\lambda \in (0,\infty)$. *Furthermore, for each* $(v,\eta) \in \tilde{\mathcal{U}}(\omega)$, *there is a unique continuous-time "history" process also denoted by* $y(\cdot,\omega) : (-\infty, 0] \to M_2$ *such that* $y(0,\omega) = (v,\eta)$, $X(t, y(s,\omega), \theta(s,\omega)) = y(t+s,\omega)$ *for all* $s \leq 0, 0 \leq t \leq -s$, *and*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|y(-t,\omega) - Y(\theta(-t,\omega))\| \leq -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}(\omega)$ *of the linearized semiflow* D_2X *is tangent at* $Y(\omega)$ *to* $\tilde{\mathcal{U}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. *In particular, dim* $\tilde{\mathcal{U}}(\omega)$ *is finite and non-random.*

(e) Let $y(\cdot, (v_i,\eta_i), \omega), i = 1,2$, *be the history processes associated with* $(v_i,\eta_i) = y(0, (v_i,\eta_i), \omega) \in \tilde{\mathcal{U}}(\omega)$, $i = 1,2$. *Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log\Big[\sup\Big\{ \frac{\|y(-t,(v_1,\eta_1),\omega) - y(-t,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1) - (v_2,\eta_2)\|} : (v_1,\eta_1) \neq (v_2,\eta_2),\, (v_i,\eta_i) \in \tilde{\mathcal{U}}(\omega), i = 1,2 \Big\}\Big] \leq -\lambda_{i_0-1}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ *such that*

$$\tilde{\mathcal{U}}(\omega) \subseteq X(t,\cdot,\theta(-t,\omega))(\tilde{\mathcal{U}}(\theta(-t,\omega)))$$

for all $t \geq \tau_2(\omega)$. *Also*

$$D_2X(t,\cdot,\theta(-t,\omega))(\mathcal{U}(\theta(-t,\omega))) = \mathcal{U}(\omega), \quad t \geq 0; \tag{4}$$

and the restriction

$$D_2X(t,\cdot,\theta(-t,\omega))|\mathcal{U}(\theta(-t,\omega)) : \mathcal{U}(\theta(-t,\omega)) \to \mathcal{U}(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto.

(g) *The submanifolds $\tilde{\mathcal{U}}(\omega)$ and $\tilde{\mathcal{S}}(\omega)$ are transversal, viz.*

$$M_2 = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

Assume, in addition, that Hypotheses $(SMW)_{k,\delta}$ are satisfied for every $k \geq 1$ and $\delta \in (0,1]$. Then the local stable and unstable manifolds $\tilde{\mathcal{S}}(\omega)$, $\tilde{\mathcal{U}}(\omega)$ are C^∞.

Remarks.

(i) In the non-delay case $r = 0$, the conclusions of Theorem 4.1 give the stable manifold theorem for sde's when X is replaced by the stochastic flow $\phi : \mathbf{R}^+ \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ associated with the sode

$$\left.\begin{aligned} d\phi(t) &= h(\phi(t))\,dW(t), \quad t > 0 \\ x(0) &= v \in \mathbf{R}^d \end{aligned}\right\} \qquad (III)$$

where h is $C_b^{k,\delta}$ for all $k \geq 1$ and $\delta > 0$ ([M-S.4]). The history process y corresponds to a trajectory of the sode using Kunita's backward stochastic integral. Note, however, that the integrability condition on Y in Theorem 4.1 is stronger than the corresponding one in Theorem 3.1 of [M-S.4].

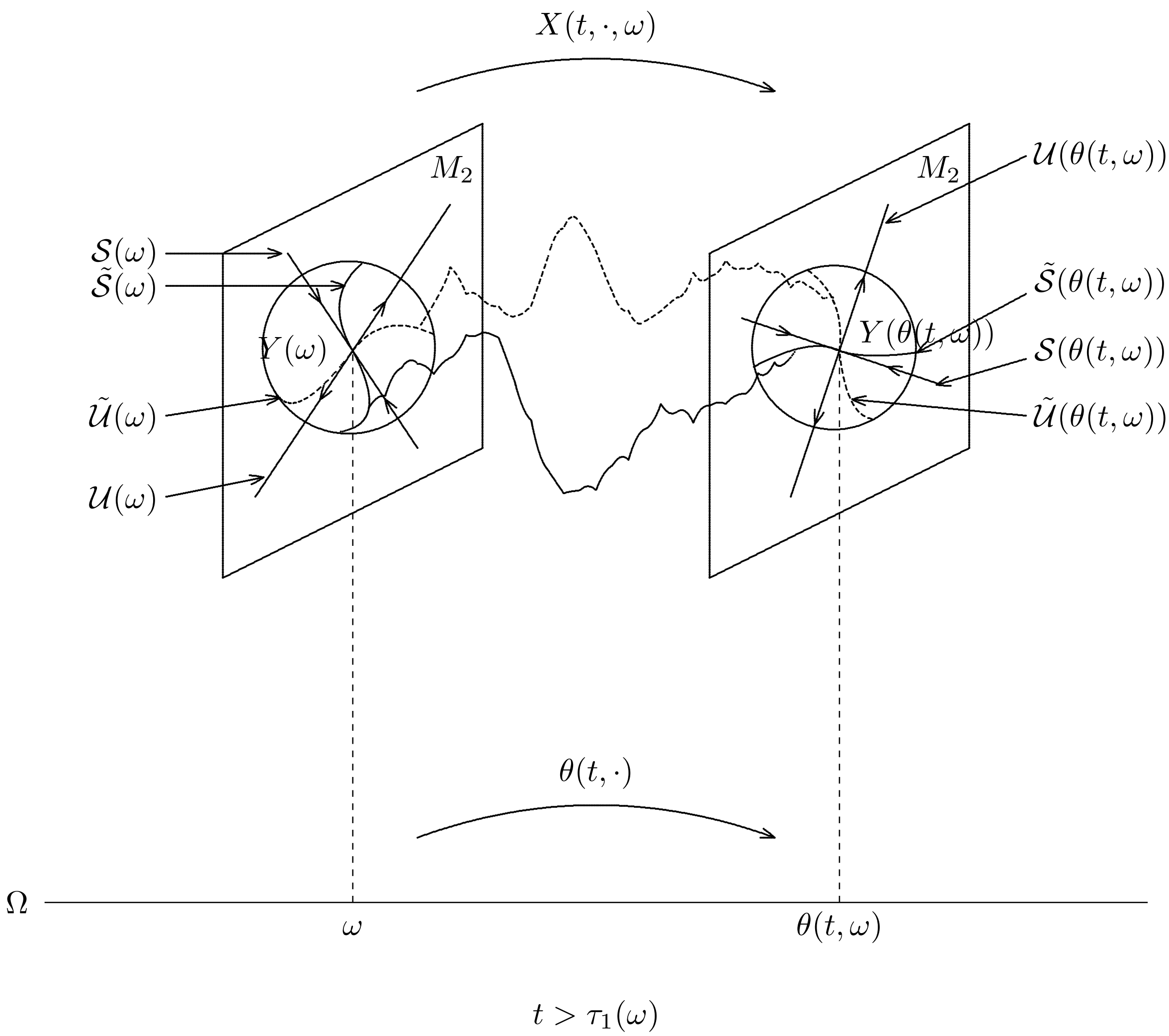
(ii) It is not clear if the conclusions of Theorem 4.1 above are still valid (for $r > 0$) when $\log^+ \|Y(\cdot)\|$ is integrable.

(iii) In view of Section 5(iii) in [M-S.3], one can impose sufficient regularity hypotheses on the coefficients of the sfde

$$\left.\begin{aligned} dx(t) &= H(t, x(t-d_m), \cdots, x(t-d_1), x(t), x_t)\mu(dt) + G(dt, x(t), g(x_t)), \quad t \geq t_0 \geq 0 \\ x(t_0) &= v \in \mathbf{R}^d, \quad x_{t_0} = \eta \in L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\}$$

to establish the existence of local stable and unstable manifolds satisfying the conclusion of Theorem 4.1. However the local stable manifolds are only of class $C^{1,\epsilon}$ ($\epsilon \in (0,1)$) even if H, G, g are C_b^∞.

The figure below summarizes the essential features of Theorem 4.1.



As an important first step in the proof of the stable manifold theorem, we will establish a discrete-time version of the theorem, viz. Proposition 4.1 below. This is an immediate consequence of Ruelle's theorems 5.1, 6.1 [Ru.2]. The rest of the proof in continuous time will be given in the next section. This is done via perfection techniques and interpolation between delay periods.

Proposition 4.1.

Assume the hypotheses and notations of Theorem 4.1. Then all the assertions of Theorem 4.1, with the exception of the invariance (2) and the corresponding invariance for the unstable manifold in (f), are valid when t is replaced by nr for any positive integer n.

Proof.

All real-valued random variables in this proof will be taken to be $\bar{\mathcal{F}}$-measurable.

Consider the cocycle (Z,θ) defined by (1) in Section 3. Define the family of maps $F_\omega : \bar{B}(0,1) \to M_2$, $\omega \in \Omega$, by $F_\omega((v,\eta)) := Z(r,(v,\eta),\omega)$, and let $\tau := \theta(r,\cdot) : \Omega \to \Omega$. Following Ruelle ([Ru.2], p. 272), define $F_\omega^n := F_{\tau^{n-1}(\omega)} \circ \cdots \circ F_{\tau(\omega)} \circ F_\omega$. Then by the cocycle property for Z, we get $F_\omega^n = Z(nr,\cdot,\omega)$ for each $n \geq 1$. Clearly, each F_ω is $C^{k,\epsilon}$ ($\epsilon \in (0,\delta)$) on $\bar{B}(0,1)$ and $(DF_\omega)(0) = D_2X(r,Y(\omega),\omega)$. From Theorem 4.1(iv) in [M-S.3] and the measurability of Y, it follows that the map $\omega \mapsto (DF_\omega)(0)$ is $(\mathcal{F},\mathcal{B}_s(L(M_2)))$-measurable. By (5) of Lemma 2.1, it is clear that $\log^+ \|D_2X(r,Y(\cdot),\cdot)\|_{L(M_2)}$ is integrable. Furthermore, the discrete-time cocycle $((DF_\omega^n)(0),\theta(nr,\omega))$ has a Lyapunov spectrum which coincides with that of the linearized continuous-time cocycle $(D_2X(t,Y(\omega),\omega),\theta(t,\omega))$, viz. $\{-\infty < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. We now apply Theorem 5.1 of Ruelle ([Ru.2], p. 272) under his hypotheses (I). This gives a sure event $\Omega_1^* \in \mathcal{F}$ such that $\theta(n,\cdot)(\Omega_1^*) = \Omega_1^*$ for all $n \in \mathbf{Z}$, $\bar{\mathcal{F}}$-measurable positive random variables $\rho_1, \beta_1 : \Omega_1^* \to (0,1)$, and a random family of $C^{k,\epsilon}$ stable submanifolds $\tilde{\mathcal{S}}_d(\omega)$ of $\bar{B}(0,\rho_1(\omega))$ satisfying the following properties for each $\omega \in \Omega_1^*$:

$$\tilde{\mathcal{S}}_d(\omega) = \{(v,\eta) \in \bar{B}(0,\rho_1(\omega)) : \|Z(nr,(v,\eta),\omega)\|_{M_2} \leq \beta_1(\omega)e^{(\lambda_{i_0}+\epsilon_1)nr} \text{ for all integers } n \geq 0\} \tag{5}$$

in case $\lambda_{i_0} \in (-\infty,0)$. If $\lambda_{i_0} = -\infty$, the second assertion in (a) holds. Each $\tilde{\mathcal{S}}_d(\omega)$ is tangent at 0 to the stable subspace $\mathcal{S}(\omega)$ of the linearized flow D_2X, viz. $T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$.

In particular, codim $\tilde{\mathcal{S}}_d(\omega)$ is finite and non-random. Furthermore, according to ([Ru.2], Theorem 5.1), one has:

$$\limsup_{n\to\infty} \frac{1}{nr} \log\Big[\sup_{\substack{(v_1,\eta_1),(v_2,\eta_2)\in\tilde{\mathcal{S}}_d(\omega)\\ (v_1,\eta_1)\neq(v_2,\eta_2)}} \frac{\|Z(nr,(v_1,\eta_1),\omega) - Z(nr,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1)-(v_2,\eta_2)\|}\Big] \le \lambda_{i_0}. \tag{6}$$

Consider the set $\tilde{\mathcal{S}}(\omega), \omega \in \Omega_1^*$, defined in part (a) of the theorem. Using (5) and the definition of Z, it follows immediately that

$$\tilde{\mathcal{S}}(\omega) = \tilde{\mathcal{S}}_d(\omega) + Y(\omega) \tag{7}$$

for all $\omega \in \Omega_1^*$. Hence $\tilde{\mathcal{S}}(\omega)$ is a $C^{k,\epsilon}$ manifold ($k > 1, \epsilon \in (0,\delta)$). Furthermore, $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$. In particular, codim $\tilde{\mathcal{S}}(\omega) =$ codim $\mathcal{S}(\omega)$ is finite and non-random. From (6) and (7), assertion (b) of Theorem 4.1 holds for $t = nr$.

We next show that assertion (1) in Theorem 4.1 holds when $t = nr$. By (6), we have

$$\limsup_{n\to\infty} \frac{1}{nr} \log \|Z(nr,(v,\eta),\omega)\| \le \lambda_{i_0} \tag{8}$$

for all $\omega \in \Omega_1^*$ and all $(v,\eta) \in \tilde{\mathcal{S}}_d(\omega)$.

To prove the cocycle-invariance (c), apply the Oseledec theorem to the linearized discrete cocycle $(D_2X(nr,Y(\omega),\omega), \theta(nr,\omega))$ ([Mo.2], Theorem 4, Corollary 2). Hence there is a sure $\theta(nr,\cdot)$-invariant event, also denoted by $\Omega_1^* \in \mathcal{F}$, such that $D_2X(nr,Y(\omega),\omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(nr,\omega))$ for all integers $n \ge 0$ and all $\omega \in \Omega_1^*$.

We now show the existence of the local unstable manifolds in (d) of Theorem 4.1 in discrete time $t = nr$. Define the random field $\hat{Z}(nr,(v,\eta),\omega) \in M_2$, $n \in \mathbf{Z}^+, (v,\eta) \in M_2$, by

$$\hat{Z}(nr,(v,\eta),\omega) := X(nr,(v,\eta) + Y(\theta(-nr,\omega)), \theta(-nr,\omega)) - Y(\omega) \tag{9}$$

for all integers $n \ge 0$, $(v,\eta) \in M_2$ and $\omega \in \Omega$. Note that $\hat{Z}(nr,\cdot,\omega) = Z(nr,\cdot,\theta(-nr,\omega))$ for all integers $n \ge 0$ and $\omega \in \Omega$; and each $\hat{Z}(nr,\cdot,\cdot)$ is $(\mathcal{B}(M_2)\otimes\mathcal{F}, \mathcal{B}(M_2))$-measurable, by the remark following the proof of Theorem 4.1 ([M-S.3]). From (4) (Section 2) (with $s = -t = -nr$), it follows immediately that $\hat{Z}(nr, 0, \omega) = 0$ for all integers $n \geq 0$ and $\omega \in \Omega$. We claim that $([D_2\hat{Z}(nr, 0, \omega)]^*, \theta(-nr, \omega),\ n \geq 0)$ is a discrete-time linear cocycle (in $L(M_2)$). To see this we argue as follows. Consider the following identity in $L(M_2)$:

$$D_2X((n+m)r, Y(\omega), \omega) = D_2X(nr, Y(\theta(mr, \omega)), \theta(mr, \omega)) \circ D_2X(mr, Y(\omega), \omega)$$

for all $\omega \in \Omega$ and all integers $n, m \geq 0$. Taking adjoints in the above identity and replacing ω by $\theta(-nr - mr, \omega)$ gives

$$\begin{aligned}&[D_2X(nr + mr, Y(\theta(-nr - mr, \omega)), \theta(-nr - mr, \omega))]^* \\ &\quad = [D_2X(mr, Y(\theta(-nr - mr, \omega)), \theta(-nr - mr, \omega))]^* \circ [D_2X(nr, Y(\theta(-nr, \omega)), \theta(-nr, \omega)]^*\end{aligned}$$

for all $\omega \in \Omega$ and all integers $n, m \geq 0$. Hence

$$[D_2\hat{Z}(nr + mr, 0, \omega)]^* = [D_2\hat{Z}(mr, 0, \theta(-nr, \omega))]^* \circ [D_2\hat{Z}(nr, 0, \omega)]^*$$

for all $\omega \in \Omega$ and all integers $n, m \geq 0$. This proves that $([D_2\hat{Z}(nr, 0, \omega)]^*, \theta(-nr, \omega),\ n \geq 0)$ is a cocycle in $L(M_2)$, as claimed.

Observe that the cocycles $(D_2X(nr, Y(\omega), \omega), \theta(nr, \omega), n \geq 0)$ and $([D_2\hat{Z}(nr, 0, \omega)]^*, \theta(-nr, \omega),\ n \geq 0)$ have the same (discrete) fixed Lyapunov spectrum $\{\cdots \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ with multiplicities. This is because of the integrability property:

$$\begin{aligned}&\int_\Omega \log^+ \|[D_2\hat{Z}(mr, 0, \theta(-nr, \omega))]^*\|_{L(M_2)}\, dP(\omega) \\ &\quad = \int_\Omega \log^+ \|D_2X(mr, Y(\theta(-mr - nr, \omega)), \theta(-mr - nr, \omega)))\|_{L(M_2)}\, dP(\omega) \\ &\quad = \int_\Omega \log^+ \|D_2X(mr, Y(\omega), \omega)\|_{L(M_2)}\, dP(\omega) < \infty, \quad m, n \geq 0,\end{aligned}$$

(cf. (5) of Lemma 2.1)) and the argument in [Ru.2], Section 3.5, p. 261. Note that $\lambda_i \neq 0$ for all $i \geq 1$, by hyperbolicity.

To construct the local unstable manifolds $\tilde{\mathcal{U}}(\omega)$, we will invoke Ruelle's discrete Theorem 6.1, ([Ru.2], p. 280) and its proof. Define the random family of smooth maps $\tilde{F}_\omega : (M_2, 0) \to (M_2, 0), \omega \in \Omega$, by $\tilde{F}_\omega((v,\eta)) := \hat{Z}(r,(v,\eta),\omega)$ for all $(v,\eta) \in M_2$. Then $\tilde{F}_\omega(0) = 0$, and $D\tilde{F}_\omega(0) = D_2X(r, Y(\theta(-r,\omega)), \theta(-r,\omega))$ for all $\omega \in \Omega$. Furthermore, from the above estimates, it follows that the map $\omega \mapsto \log^+ \|[D\tilde{F}_\omega(0)]^*\| = \log^+ \|[D\tilde{F}_\omega(0)]\|$ is in $L^1(\Omega, \mathbf{R}; \mathcal{F})$. Indeed, by the P-preserving property of $\theta(nr,\cdot), n \in \mathbf{Z}$, and Lemma 3.2, it follows that

$$\int_\Omega \log^+ \|\hat{Z}(mr,\cdot,\theta(-nr,\omega))\|_{k,\epsilon}\, dP(\omega) < \infty.$$

Define i_0 as before, so that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized cocycle. Fix $0 < \epsilon_2 < \lambda_{i_0-1}$. In view of the above integrability property, it follows that the sequence $\tilde{T}_n(\omega) := [D_2\hat{Z}(r,0,\theta(-nr,\omega))]^*$, $\theta(-nr,\omega)$, $n \geq 0$, satisfies Condition (S) of [Ru.2]. Therefore Proposition 3.3 in [Ru.2] implies that the sequence $\tilde{T}_n(\omega), n \geq 1$, satisfies Corollary 3.4 ([Ru.2], p. 260) for a.a. ω. This yields a $\theta(-nr,\cdot)$-invariant sure event $\hat{\Omega}_1^* \in \mathcal{F}$ and $\bar{\mathcal{F}}$-measurable random variables $\rho_2, \beta_2 : \hat{\Omega}_1^* \to (0,1)$ with the following properties. Let $\tilde{\mathcal{U}}_d(\omega)$ be the set of all $(v_0,\eta_0) \in \bar{B}(0,\rho_2(\omega))$ with the property that there is a discrete "history" process $u(-nr,\cdot) : \Omega \to M_2, n \geq 0$, such that $u(0,\omega) = (v_0,\eta_0)$, $\hat{Z}(r, u(-(n+1)r,\omega), \theta(-nr,\omega)) = u(-nr,\omega)$ and $\|u(-nr,\omega)\| \leq \beta_2(\omega)e^{-nr(\lambda_{i_0-1}-\epsilon_2)}$ for all $n \geq 0$. For $\lambda_{i_0-1} = \infty$, let $\tilde{\mathcal{U}}_d(\omega)$ be the set of all $(v_0,\eta_0) \in \bar{B}(0,\rho_2(\omega))$ such that there is a history process $u(-nr,\cdot), n \geq 0$, with $u(0,\omega) = (v_0,\eta_0)$ and $\|u(-nr,\omega)\| \leq \beta_2(\omega)e^{-\lambda nr}$ for all $n \geq 0$ and any $\lambda > 0$. The history process $u(-nr,\cdot)$ is uniquely determined by (v_0,η_0) ([Ru.2], p. 281). Furthermore, for every $\omega \in \hat{\Omega}_1^*$, $\tilde{\mathcal{U}}_d(\omega)$ is a $C^{k,\epsilon}$ ($\epsilon \in (0,\delta)$) finite-dimensional submanifold of $\bar{B}(0,\rho_2(\omega))$ with tangent space $U(\omega)$ at 0. Also dim $\tilde{\mathcal{U}}_d(\omega)$ is fixed independently of ω and ϵ_2.

We claim that the set $\tilde{\mathcal{U}}(\omega)$ defined in (d) of Theorem 4.1 coincides with $\tilde{\mathcal{U}}_d(\omega)+Y(\omega)$ for each $\omega \in \hat{\Omega}_1^*$. We first show that $\tilde{\mathcal{U}}_d(\omega) + Y(\omega) \subseteq \tilde{\mathcal{U}}(\omega)$. Let $(v_0,\eta_0) \in \tilde{\mathcal{U}}_d(\omega)$ and u be as above. Set

$$y_0(-nr) := u(-nr) + Y(\theta(-nr,\omega)), \qquad n \geq 0. \tag{10}$$

It is easy to check that y_0 is a discrete history process satisfying the first and second assertions in (d) of the proposition. Hence $(v_0, \eta_0) + Y(\omega) \in \tilde{\mathcal{U}}(\omega)$. Similarly, $\tilde{\mathcal{U}}(\omega) \subseteq \tilde{\mathcal{U}}_d(\omega) + Y(\omega)$ for all $\omega \in \hat{\Omega}_1^*$. Hence $\tilde{\mathcal{U}}(\omega) = \tilde{\mathcal{U}}_d(\omega) + Y(\omega)$ for all $\omega \in \hat{\Omega}_1^*$. This immediately implies that $\tilde{\mathcal{U}}(\omega)$ is a $C^{k,\epsilon}$ $(\epsilon \in (0,\delta))$ finite-dimensional submanifold of $\bar{B}(Y(\omega), \rho_2(\omega))$ and

$$T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = T_0\tilde{\mathcal{U}}_d(\omega) = \mathcal{U}(\omega).$$

for all $\omega \in \hat{\Omega}_1^*$.

Assertion (e) of Theorem 4.1 in discrete time $t = nr$ follows from ([Ru.2], Theorem 6.1).

For $t = nr$ assertion (4) in Theorem 4.1 (f) follows from the Oseledec theorem and the cocycle property for the linearized semiflow; cf. [Mo.2], Corollary 2 (v) of Theorem 4.

The transversality assertion in (g) of Theorem 4.1 is implied by the relations

$$T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega), \quad T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega), \quad M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega)$$

which hold for a.a. ω.

Taking $\Omega^* := \Omega_1^* \cap \hat{\Omega}_1^*$, completes the proof of assertions (a)-(g) of Theorem 4.1 for discrete time $t = nr$, with the exception of of the invariance (2) and the corresponding invariance for the unstable manifold in (f).

Suppose Hypothesis $(SMW)_{k,\delta}$ holds for every $k \geq 1$ and $\delta \in (0,1]$. Then a simple adaptation of the argument in [Ru.2], Section (5.3) (p. 297) gives a $\theta(nr, \cdot)$-invariant sure event in $\mathcal{F}$, also denoted by Ω^*, such that $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞ for all $\omega \in \Omega^*$. The proof of Proposition 4.1 is now complete. $\square$

5. Proof of the local stable manifold theorem.

We devote this section to the proof of Theorem 4.1 in continuous time. A large part of the computations are directed toward perfection arguments, whereby we show that the local stable/unstable manifolds are parametrized by sure events which are invariant under the continuous-time shift $\theta(t,\cdot):\Omega\to\Omega$. The integrability properties of the cocycle (X,θ) (Lemma 3.2) play a crucial role in controlling the excursions of the cocycle within delay periods.

Our first lemma gives "perfect versions" of the ergodic theorem and Kingman's subadditive ergodic theorem. These results are needed in order to construct the shift-invariant sure events appearing in the statement of the local stable manifold theorem (Theorem 4.1). The reader may note that Lemmas 5.1-5.3 hold if $\theta(t,\cdot)$ is any group of measure-preserving ergodic transformations on a probability space $(\Omega,\mathcal{F},P)$, satisfying appropriate measurability properties.

Lemma 5.1.

(i) *Let $\Omega_0\in\bar{\mathcal{F}}$ be a sure event such that $\theta(t,\cdot)(\Omega_0)\subseteq\Omega_0$ for all $t\geq 0$. Then there is a sure event $\Omega_0^*\in\mathcal{F}$ such that $\Omega_0^*\subseteq\Omega_0$ and $\theta(t,\cdot)(\Omega_0^*)=\Omega_0^*$ for all $t\in\mathbf{R}$.*

(ii) *Let $h:\Omega\to\mathbf{R}^+$ be any function such that there exists an $\bar{\mathcal{F}}$-measurable function $g_1\in L^1(\Omega,\mathbf{R}^+;P)$ and a sure event $\Omega_1\in\bar{\mathcal{F}}$ such that $\sup\limits_{0\leq u\leq 1} h(\theta(u,\omega))\leq g_1(\omega)$ for all $\omega\in\Omega_1$. Then there exists a sure event $\Omega^*\in\mathcal{F}$ such that $\theta(t,\cdot)(\Omega^*)=\Omega^*$ for all $t\in\mathbf{R}$, and*

$$\lim_{t\to\infty}\frac{1}{t}h(\theta(t,\omega))=0$$

for all $\omega\in\Omega^*$.

(iii) *Suppose $f:\mathbf{R}^+\times\Omega\to\mathbf{R}\cup\{-\infty\}$ is a process such that for each $t\in\mathbf{R}^+$, $f(t,\cdot)$ is $(\bar{\mathcal{F}},\mathcal{B}(\mathbf{R}\cup\{-\infty\}))$-measurable and the following conditions hold:*

(a) There is an $\bar{\mathcal{F}}$-measurable function $g_2 \in L^1(\Omega, \mathbf{R}^+; P)$ and a sure event $\tilde{\Omega}_1 \in \bar{\mathcal{F}}$ such that $\left[\sup_{0\le u\le 1} f^+(u,\omega) + \sup_{0\le u\le 1} f^+(1-u, \theta(u,\omega))\right] \le g_2(\omega)$ for all $\omega \in \tilde{\Omega}_1$.

(b) $f(t_1+t_2,\omega) \le f(t_1,\omega) + f(t_2, \theta(t_1,\omega))$ for all $t_1, t_2 \ge 0$ and **all** *$\omega \in \Omega$.*

Then there is a sure event $\Omega_2 \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega_2) = \Omega_2$ for all $t \in \mathbf{R}$, and a fixed number $f^* \in \mathbf{R} \cup \{-\infty\}$ such that

$$\lim_{t\to\infty} \frac{1}{t} f(t,\omega) = f^*$$

for all $\omega \in \Omega_2$.

Proof.

The proof of assertion (i) of the lemma is given in Proposition 2.3 ([M-S.4]).

Assertions (ii) and (iii) of the lemma follow from assertion (i) and easy adaptations of the arguments in the proofs of Lemmas 5 and 7 in [Mo.2]. See also Lemma 3.3 in [M-S.4]. □

The following lemma will be needed in order to construct the shift-invariant sure events appearing in the statement of the local stable manifold theorem. The lemma essentially gives a continuous-time "perfect version" of Corollary A.2 of [Ru.2], p. 288.

Lemma 5.2.

Suppose $f : \mathbf{R}^+ \times \Omega \to \mathbf{R} \cup \{-\infty\}$ is a $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(\mathbf{R} \cup \{-\infty\}))$-measurable process satisfying the following conditions:

(a) $\int_\Omega \left[\sup_{0\le t_1,t_2\le T} f^+(t_1, \theta(t_2,\omega))\right] dP(\omega) < \infty, \quad T \in (0,\infty).$

(b) $f(t_1+t_2,\omega) \le f(t_1,\omega) + f(t_2, \theta(t_1,\omega))$ for all $t_1, t_2 \ge 0$ and **all** *$\omega \in \Omega$.*

Then there exists $f^* \in \mathbf{R} \cup \{-\infty\}$ and a sure event $\Omega_3 \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega_3) = \Omega_3$ for all $t \in \mathbf{R}$, and the following hold:

(1) $\lim_{t\to\infty} \frac{1}{t} f(t,\omega) = f^*$, *for all* $\omega \in \Omega_3$.

(2) If $g^* \in \mathbf{R}$ *is a finite number such that* $f^* \le g^*$, *then for every* $\epsilon > 0$, *there exists an* $\bar{\mathcal{F}}$*-measurable function* $K_\epsilon : \Omega_3 \to [0,\infty)$ *with the property that*

$$f(t-s, \theta(s,\omega)) \le (t-s)g^* + \epsilon t + K_\epsilon(\omega)$$

for **all** $\omega \in \Omega_3$ *and whenever* $0 \le s \le t < \infty$. *Furthermore,* K_ϵ *may be chosen such that* $K_\epsilon(\theta(l,\omega)) \le K_\epsilon(\omega) + \epsilon l$ *for all* $l \in [0,\infty)$ *and all* $\omega \in \Omega_3$.

Proof.

By Lemma 5.1 (iii), there exists $f^* \in \mathbf{R} \cup \{-\infty\}$ and a sure event $\Omega_2 \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega_2) = \Omega_2$ for all $t \in \mathbf{R}$ and (1) holds for all $\omega \in \Omega_2$. By hypotheses (a) and Lemma 5.1 (i), there is a sure event $\Omega_0 \subseteq \Omega_2$ such that $\Omega_0 \in \mathcal{F}$, $\theta(t,\cdot)(\Omega_0) = \Omega_0$ for all $t \in \mathbf{R}$, and $\sup_{0\le t_1,t_2\le T} f^+(t_1, \theta(t_2,\omega)) < \infty$ for all $T \ge 0$ and all $\omega \in \Omega_0$. Suppose g^* is a finite real number such that $f^* \le g^*$. Define the process $g : \mathbf{R}^+ \times \Omega \to \mathbf{R}^+$ by

$$g(t,\omega) := \begin{cases} \max\{f(t,\omega) - tg^*, 0\}, & t \ge 0,\ \omega \in \Omega_0, \\ 0 & t \ge 0,\ \omega \notin \Omega_0. \end{cases}$$

It is easy to check that g is non-negative, $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(\mathbf{R}^+))$-measurable and satisfies conditions (a) and (b).

Define the process $g' : \mathbf{R}^+ \times \Omega \to \mathbf{R}^+$ by

$$g'(t,\omega) := \sup_{0\le s\le t} [g(s,\omega) + g(t-s, \theta(s,\omega))], \quad t \ge 0,\ \omega \in \Omega.$$

Using the fact that the projection of a $\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}$-measurable set is $\bar{\mathcal{F}}$-measurable ([Co], p. 281), it follows that g' satisfies the hypotheses of Lemma 5.1 (iii). Therefore, there exists $g'^* \ge 0$, a sure event $\Omega_4 \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega_4) = \Omega_4$ for all $t \in \mathbf{R}$ and $\lim_{t\to\infty} \frac{1}{t} g'(t,\omega) = g'^*$ for all $\omega \in \Omega_4$.

Next, we claim that

$$\lim_{t\to\infty} \frac{1}{t} \sup_{0\le s\le t} g(t-s, \theta(s,\omega)) = 0 \qquad (*)$$

in probability. This claim easily implies $g'^* = 0$. Hence there is a sure event $\Omega_5 \in \mathcal{F}$ such that $\Omega_5 \subseteq \Omega_0 \cap \Omega_4$, $\theta(t,\cdot)(\Omega_5) = \Omega_5$ for all $t \in \mathbf{R}$ and $(*)$ holds for all $\omega \in \Omega_5$. The proof of assertion (2) is completed by setting

$$K_\epsilon(\omega) := \sup_{0 \le s \le t < \infty} [g(t-s, \theta(s,\omega)) - \epsilon t]$$

for all $\omega \in \Omega_5$ and a fixed $\epsilon > 0$. It is easy to see from the above definition that $K_\epsilon : \Omega_5 \to [0,\infty)$ is $(\bar{\mathcal{F}}, \mathcal{B}(\mathbf{R}^+))$-measurable and $K_\epsilon(\theta(l,\omega)) \le K_\epsilon(\omega) + \epsilon l$ for all $l \in [0,\infty)$ and all $\omega \in \Omega_5$.

It remains to establish our claim $(*)$. The process $h : \mathbf{R}^+ \times \Omega \to \mathbf{R}$

$$h(t,\omega) := g(t, \theta(-t,\omega)), \quad t \in \mathbf{R}^+, \ \omega \in \Omega$$

satisfies the conditions of Lemma 5.1 (iii). Therefore

$$\lim_{t\to\infty} \frac{1}{t} h(t,\omega) = 0$$

for almost all $\omega \in \Omega_4$ and hence in probability. Fix $\delta > 0$ and $t_0 > 0$ such that $P(\frac{1}{t}h(t,\cdot) \ge \delta) < \delta$ for all $t \ge t_0$. Suppose $t \ge t_0$, and consider

$$\begin{aligned}\sup_{0\le s\le t} \frac{1}{t} g(t-s,\theta(s,\omega)) &\le \sup_{0\le s\le t-t_0} \frac{1}{t} g(t-s,\theta(s,\omega)) + \sup_{t-t_0\le s\le t} \frac{1}{t} g(t-s,\theta(s,\omega)) \\ &\le \sup_{0\le s\le t-t_0} \frac{1}{t} g(t-s,\theta(-(t-s),\theta(t,\omega))) + \sup_{t-t_0\le s\le t} \frac{1}{t} g(t-s,\theta(s,\omega)).\end{aligned}$$

The first term in the right hand side of the last inequality is less than or equal to δ with probability at least $1-\delta$. The second term converges to 0 in probability by assumption (a). Hence $(*)$ holds and the proof of the lemma is complete. $\square$

For convenience, we shall frequently adopt the following convention:

Definition 5.1.

Let $\{P(\omega) : \omega \in \Omega\}$ be a family of propositions. We say that $P(\omega)$ *holds perfectly in* ω if there is a sure event $\Omega^* \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$ and $P(\omega)$ is true *for every* $\omega \in \Omega^*$.

Our next result is basically a "perfect version" of Proposition 3.2 in [Ru.2], p. 257. The proof uses Lemma 5.2. We denote by $\mathcal{B}_s(L(H))$ the Borel σ-algebra on $L(H)$ generated by the strong topology on $L(H)$, viz. the smallest topology on $L(H)$ for which all evaluations $L(H) \ni A \mapsto A(z) \in H, z \in H$, are continuous.

Lemma 5.3.

Let H be a real separable Hilbert space, $\theta(t,\cdot) : \Omega \to \Omega$ be an ergodic measure-preserving group of transformations on the probability space $(\Omega, \mathcal{F}, P)$. Suppose $(T^t(\omega), \theta(t,\omega))$, $t \geq 0$, is a perfect cocycle of bounded linear operators in H satisfying the following hypotheses:

(i) The process $\mathbf{R}^+ \times \Omega \ni (t,\omega) \mapsto T^t(\omega) \in L(H)$ is $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}_s(L(H)))$-measurable.

(ii) The map $\mathbf{R}^+ \times \Omega \ni (t,\omega) \mapsto \theta(t,\omega) \in \Omega$ is $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{F})$-measurable.

(iii) $E \sup\limits_{0 \leq t_1, t_2 \leq a} \log^+ \|T^{t_2}(\theta(t_1,\cdot))\|_{L(H)} < \infty$ for any finite $a > 0$.

(iv) There is a fixed $t_0 > 0$ such that for each $t \geq t_0$, $T^t(\omega)$ is compact, perfectly in ω.

(v) For any $u \in H$, the map $[0,\infty) \ni t \mapsto T^t(\omega)(u) \in H$ is continuous, perfectly in ω.

Let $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ be the Lyapunov spectrum of $(T^t(\omega), \theta(t,\omega))$, with Oseledec spaces

$$\cdots E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = H.$$

Let $j_0 \geq 1$ be any fixed integer with $\lambda_{j_0} > -\infty$. Let the integer function $r : \{1, 2, \cdots, Q\} \to \{1, 2, \cdots, j_0\}$ "count" the multiplicities of the Lyapunov exponents in the sense that $r(1) =$

1, $r(Q) = j_0$, and for each $1 \leq i \leq j_0$, the number of integers in $r^{-1}(i)$ is the multiplicity of λ_i. Set $V_n(\omega) := E_{j_0+1}(\theta(nt_0, \omega)), n \geq 0$.

Then the sequence $T_n(\omega) := T^{t_0}(\theta((n-1)t_0, \omega)), n \geq 1$, satisfies Condition (S) of [Ru.2] perfectly in ω with $Q = \text{codim}\, E_{j_0+1}(\omega)$. In particular, there is an $\mathcal{F}$-measurable set of Q orthonormal vectors $\{\xi_0^{(1)}(\omega), \cdots, \xi_0^{(Q)}(\omega)\}$ such that $\xi_0^{(k)}(\omega) \in [E_{r(k)}(\omega) \backslash E_{r(k)+1}(\omega)]$ for $k = 1, \cdots, Q$, perfectly in ω, and satisfying the following properties:

Set $\xi_t^{(k)}(\omega) := \dfrac{T^t(\omega)(\xi_0^{(k)}(\omega))}{\|T^t(\omega)(\xi_0^{(k)}(\omega))\|}$, and for any $u \in H$, write

$$u = \sum_{k=1}^{Q} u_t^{(k)}(\omega)\xi_t^{(k)}(\omega) + u_t^{(Q+1)}(\omega), \quad u_t^{(Q+1)}(\omega) \in V_0(\theta(t, \omega)),\ \omega \in \Omega.$$

Then for any $\epsilon > 0$, there is an $\bar{\mathcal{F}}$-measurable random constant $D_\epsilon(\omega) > 0$ such that the following inequalities hold perfectly in ω:

$$|u_t^{(k)}(\omega)| \leq D_\epsilon(\omega) e^{\epsilon t} \|u\|$$

$$\|u_t^{(Q+1)}(\omega)\| \leq D_\epsilon(\omega) e^{\epsilon t} \|u\|$$

$$D_\epsilon(\theta(l, \omega)) \leq D_\epsilon(\omega) e^{\epsilon l}$$

for all $t \geq 0$, $1 \leq k \leq Q$ and for all $l \in [0, \infty)$.

Furthermore, all the random constants in Ruelle's condition (S) may be chosen to be $\bar{\mathcal{F}}$-measurable in ω.

Proof.

We will follow the proof of Proposition 3.2 in [Ru.2], ensuring that the relevant parts of the argument hold perfectly in ω.

For simplicity of notation, we will assume (with no loss of generality) that $t_0 = 1$.

First note that in view of (iii), the perfect cocycle property, Lemma 5.1 and the argument in Theorem 4 ([Mo.2]), it follows that $T_n(\omega)$ satisfies Condition (S1) perfectly in ω. (Observe that Condition 3.4 in [Ru.2] holds perfectly by the ordering of the fixed

Lyapunov spectrum.) Let Ω^* be the perfect event where (S1) holds. Let $\operatorname{codim} V_0(\omega) = Q$, for all $\omega \in \Omega^*$; then, by ergodicity, $\operatorname{codim} V_n(\omega) = \operatorname{codim} E_{j_0+1}(\theta(n,\omega)) = Q$. Hence (S2) holds for all $\omega \in \Omega^*$.

To establish a perfect version of (S3), we will prove the stronger statement that $(T^t(\omega), \theta(t,\omega))$ satisfies (S3) perfectly in ω. Define $\hat{T}^t(\omega) := T^t(\omega)|V_0(\omega), \omega \in \Omega^*, t \geq 0$. Then $\hat{T}^t(\omega)(V_0(\omega)) \subseteq V_0(\theta(t,\omega))$, and

$$\hat{T}^{t_1+t_2}(\omega) = \hat{T}^{t_2}(\theta(t_1,\omega)) \circ \hat{T}^{t_1}(\omega) \tag{1}$$

for all $\omega \in \Omega^*, t \geq 0$. Define $F_t(\omega) := \log \|\hat{T}^t(\omega)\|$, $\omega \in \Omega^*, t \geq 0$. Then (1) implies that $(F_t(\omega), \theta(t,\omega))$ is perfectly subadditive, and (iii) implies that $\sup\limits_{0 \leq t_1, t_2 \leq T} F^+_{t_2}(\theta(t_1, \cdot))$ is integrable for any finite $T > 0$. Hence Lemma 5.1 applies, and we get a fixed number $F^* \in \mathbf{R} \cup \{-\infty\}$ such that

$$\lim_{t \to \infty} \frac{1}{t} F_t(\omega) = F^*$$

perfectly in ω. Let $S = j_0$, and $\mu^{(S+1)} := \lambda_{j_0+1}$, when $\lambda_{j_0+1} > -\infty$; if $\lambda_{j_0+1} = -\infty$, we set $\mu^{(S+1)}$ to be any fixed finite number in $(-\infty, \lambda_{j_0})$. From (3.5), p. 258 in [Ru.2], we see that $F^* \leq \mu^{(S+1)}$. Let $\epsilon > 0$. If $\lambda_{j_0+1} > -\infty$, then by Lemma 5.2(2), we get

$$\log \|\hat{T}^{t-s}(\theta(s,\omega))\| \leq (t-s)\mu^{(S+1)} + \epsilon t + K_\epsilon(\omega), \quad 0 \leq s \leq t < \infty, \tag{2}$$

perfectly in ω, with K_ϵ $\bar{\mathcal{F}}$-measurable. Note that by Lemma 5.2, $K_\epsilon(\omega)$ is finite (perfectly in ω) and satisfies the inequality

$$K_\epsilon(\theta(l,\omega)) \leq K_\epsilon(\omega) + \epsilon l$$

perfectly in ω for all $l \in [0, \infty)$. Putting $t = n, s = m+1$ in (2) where $0 < m < n$ are integers, shows that $T_n(\omega)$ satisfies (S3) perfectly in ω.

Finally, we show that the above sequence also satisfies (S4) perfectly in ω. In the spirit of the preceding analysis, it is sufficient to prove that the continuous-time cocycle

$(T^t(\omega), \theta(t,\omega))$ satisfies (S4) perfectly in ω. Define the family of operators $\check{T}^t(\omega) : H \to V_0(\theta(t,\omega))^\perp \subseteq H$, $\tilde{T}^t(\omega) : H \to V_0(\theta(t,\omega)) \subseteq H$ via the orthogonal decomposition

$$T^t(\omega)(\xi) = \check{T}^t(\omega)(\xi) + \tilde{T}^t(\omega)(\xi) \tag{3}$$

for all $\xi \in H, t \geq 0, \omega \in \Omega^*$, where $\tilde{T}^t(\omega)(\xi) \in V_0(\theta(t,\omega)), \check{T}^t(\omega)(\xi) \in V_0(\theta(t,\omega))^\perp$ are the orthogonal projections of $T^t(\omega)(\xi)$ on $V_0(\theta(t,\omega))$ and $V_0(\theta(t,\omega))^\perp$, respectively. We claim that $(\check{T}^t(\omega), \theta(t,\omega))$ satisfies the perfect cocycle identity in $L(H)$ ([M-S.3], Definition 1.2 (ii)). To see this, fix $\omega \in \Omega, t_1, t_2 \geq 0, \xi \in H$ and consider

$$\begin{aligned} T^{t_1+t_2}(\omega)(\xi) &= T^{t_2}(\theta(t_1,\omega))[T^{t_1}(\omega)(\xi)] \\ &= \check{T}^{t_2}(\theta(t_1,\omega))[\check{T}^{t_1}(\omega)(\xi)] + \check{T}^{t_2}(\theta(t_1,\omega))[\tilde{T}^{t_1}(\omega)(\xi)] + \tilde{T}^{t_2}(\theta(t_1,\omega))[\check{T}^{t_1}(\omega)(\xi)] \\ &\quad + \tilde{T}^{t_2}(\theta(t_1,\omega))[\tilde{T}^{t_1}(\omega)(\xi)]. \end{aligned} \tag{4}$$

Now by the cocycle invariance of $V_0(\omega)$ under $T^t(\omega)$, it follows that $\check{T}^t(\omega)(\xi) = 0$ whenever $\xi \in V_0(\omega)$. Therefore $\check{T}^{t_2}(\theta(t_1,\omega))[\tilde{T}^{t_1}(\omega)(\xi)] = 0$. Thus (4) gives

$$T^{t_1+t_2}(\omega)(\xi) = \check{T}^{t_2}(\theta(t_1,\omega))[\check{T}^{t_1}(\omega)(\xi)] + \tilde{T}^{t_2}(\theta(t_1,\omega))[\check{T}^{t_1}(\omega)(\xi)] + \tilde{T}^{t_2}(\theta(t_1,\omega))[\tilde{T}^{t_1}(\omega)(\xi)] \tag{5}$$

$$= \check{T}^{t_1+t_2}(\omega)(\xi) + \tilde{T}^{t_1+t_2}(\omega)(\xi) \tag{6}$$

for all $\xi \in H$. The first term on the right-hand side of (5) belongs to $V_0(\theta(t_1+t_2,\omega))^\perp$ and the second two terms belong to $V_0(\theta(t_1+t_2,\omega))$. Therefore by uniqueness of the direct-sum representation on the right-hand side of (6), it follows that

$$\check{T}^{t_1+t_2}(\omega)(\xi) = \check{T}^{t_2}(\theta(t_1,\omega))[\check{T}^{t_1}(\omega)(\xi)] \tag{7}$$

for all $\xi \in H$. This proves that $(\check{T}^t(\omega), \theta(t,\omega))$ satisfies the perfect cocycle identity in $L(H)$ ([M-S.3], Definition 1.2 (ii)). To complete the proof of (S4), note first that the integrability property (iii) of the lemma implies that

$$E \sup_{0 \leq t_1, t_2 \leq a} \log^+ \|\check{T}^{t_2}(\theta(t_1,\cdot))\|_{L(H)} < \infty \tag{8}$$

for any finite $a > 0$. Applying the perfect Oseledec theorem to $(T^t(\omega), \theta(t,\omega))$ and $(\check{T}^t(\omega), \theta(t,\omega))$ shows that the following limits exist perfectly in ω for all $\xi \in H$:

$$\lim_{t\to\infty} \frac{1}{t} \log \|\check{T}^t(\omega)(\xi)\| = \check{l}_\xi, \qquad \lim_{t\to\infty} \frac{1}{t} \log \|T^t(\omega)(\xi)\| = l_\xi$$

where $l_\xi, \check{l}_\xi$ are fixed numbers in $\mathbf{R} \cup \{-\infty\}$. Now from (3.6) in ([Ru.2], p. 259), we know that

$$\check{l}_\xi = \lim_{n\to\infty} \frac{1}{n} \log \|\check{T}^n(\omega)(\xi)\| = \lim_{n\to\infty} \frac{1}{n} \log \|T^n(\omega)(\xi)\| = l_\xi$$

for a.a. ω and for all $\xi \in H \backslash V_0(\omega)$. Therefore the equality

$$\lim_{t\to\infty} \frac{1}{t} \log \|\check{T}^t(\omega)(\xi)\| = \lim_{t\to\infty} \frac{1}{t} \log \|T^t(\omega)(\xi)\|$$

holds perfectly in ω for all $\xi \in H \backslash V_0(\omega)$. Hence, relation (3.6) in ([Ru.2], p. 259) may be replaced by the continuous-time "perfect" relation

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{\|\check{T}^t(\omega)(\xi)\|}{\|T^t(\omega)(\xi)\|} = 0 \tag{9}$$

for all $\xi \in H \backslash V_0(\omega)$.

We now complete the proof of the lemma by following the rest of the argument in the proof of Proposition 3.2 in ([Ru.2], p. 259). By ([C-V], Theorem III.6, p. 65) and Gram-Schmidt orthogonalization , we may select a set of Q, $\mathcal{F}$-measurable, orthonormal vectors $\{\xi_0^{(1)}(\omega), \cdots, \xi_0^{(Q)}(\omega)\}$ such that $\xi_0^{(k)}(\omega) \in [E_{r(k)}(\omega) \backslash E_{r(k)+1}(\omega)] \cap V_0(\omega)^\perp$ for $k = 1, \cdots, Q$, perfectly in ω. In the argument in [Ru.2], p. 259, replace (3.6) by (9), n by t, $\xi_n^{(k)}$ by $\xi_t^{(k)}(\omega) := \dfrac{T^t(\omega)(\xi_0^{(k)}(\omega))}{\|T^t(\omega)(\xi_0^{(k)}(\omega))\|}$, V_n by $V_0(\theta(t,\omega))$, and $\eta_n^{(k)}$ by $\eta_t^{(k)}(\omega) := \dfrac{\check{T}^t(\omega)(\xi_0^{(k)}(\omega))}{\|T^t(\omega)(\xi_0^{(k)}(\omega))\|}$. Therefore for $u \in H$, we write

$$u = \sum_{k=1}^{Q} u_t^{(k)}(\omega)\xi_t^{(k)}(\omega) + u_t^{(Q+1)}(\omega), \quad u_t^{(Q+1)}(\omega) \in V_0(\theta(t,\omega)), \tag{10}$$

perfectly in ω for all $t \geq 0$. Furthermore, as in [Ru.2], p. 259, (9) implies that

$$\lim_{t\to\infty} \frac{1}{t} \log |\det(\eta_t^{(1)}(\omega), \cdots, \eta_t^{(Q)}(\omega))| = 0, \tag{11}$$

perfectly in ω.

Finally, we will show that for any $\epsilon > 0$, there is an $\bar{\mathcal{F}}$-measurable non-negative function $D_\epsilon : \Omega \to (0, \infty)$ such that the following inequalities hold perfectly in ω:

$$\left.\begin{aligned} |u_t^{(k)}(\omega)| &\leq D_\epsilon(\omega) e^{\epsilon t} \|u\| \\ \|u_t^{(Q+1)}(\omega)\| &\leq D_\epsilon(\omega) e^{\epsilon t} \|u\| \\ D_\epsilon(\theta(l, \omega)) &\leq D_\epsilon(\omega) e^{\epsilon l} \end{aligned}\right\} \tag{12}$$

for all $t \geq 0$, $1 \leq k \leq Q$ and for all $l \in [0, \infty)$.

To prove the above inequalities, define

$$D_\epsilon(\omega) := 1 + Q \cdot \sup_{0 \leq s \leq t < \infty} e^{-\epsilon t} |\det(\eta_{t-s}^{(1)}(\theta(s,\omega)), \eta_{t-s}^{(2)}(\theta(s,\omega)), \cdots, \eta_{t-s}^{(Q)}(\theta(s,\omega)))|^{-1} \tag{13}$$

perfectly in ω. We will first show that $D_\epsilon(\omega) < \infty$ perfectly in ω. Let $0 \leq s \leq t$. Using the fact that the determinant of the linear operator $\check{T}^{t-s}(\theta(s,\omega))$ is given by $\dfrac{\| \wedge_{k=1}^{Q} \check{T}^{t-s}(\theta(s,\omega))(v_k)\|}{\| \wedge_{k=1}^{Q} v_k\|}$ for *any choice of basis* $\{v_1, \cdots, v_Q\}$ in $V_0(\theta(s,\omega))^\perp$, it is easy to see that

$$\begin{aligned} &|\det(\eta_{t-s}^{(1)}(\theta(s,\omega)), \cdots, \eta_{t-s}^{(Q)}(\theta(s,\omega)))|^{-1} \\ &\quad = \frac{\Pi_{k=1}^{Q} \|T^{t-s}(\theta(s,\omega))(\xi_0^{(k)}(\theta(s,\omega)))\|}{|\det(\check{T}^{t-s}(\theta(s,\omega))(\xi_0^{(1)}(\theta(s,\omega))), \cdots, \check{T}^{t-s}(\theta(s,\omega))(\xi_0^{(Q)}(\theta(s,\omega))))|} \\ &\quad = \frac{\Pi_{k=1}^{Q} [\|T^{t-s}(\theta(s,\omega))(\xi_0^{(k)}(\theta(s,\omega)))\|] \cdot \| \wedge_{k=1}^{Q} [\check{T}^{s}(\omega)(\xi_0^{(k)}(\omega))]\|}{|\det(\check{T}^{t-s}(\theta(s,\omega))(\check{T}^{s}(\omega)(\xi_0^{(1)}(\omega))), \cdots, \check{T}^{t-s}(\theta(s,\omega))(\check{T}^{s}(\omega)(\xi_0^{(Q)}(\omega))))|} \\ &\quad \leq \frac{\Pi_{k=1}^{Q} [\|T^{t-s}(\theta(s,\omega))(\xi_0^{(k)}(\theta(s,\omega)))\| \cdot \|\check{T}^{s}(\omega)(\xi_0^{(k)}(\omega))\|]}{|\det(\check{T}^{t}(\omega)(\xi_0^{(1)}(\omega)), \cdots, \check{T}^{t}(\omega)(\xi_0^{(Q)}(\omega)))|} \\ &\quad = \frac{\Pi_{k=1}^{Q} [\|T^{t-s}(\theta(s,\omega))(\xi_0^{(k)}(\theta(s,\omega)))\| \cdot \|\check{T}^{s}(\omega)(\xi_0^{(k)}(\omega))\|]}{\|[\check{T}^{t}(\omega)|V_0(\omega)^\perp]^{\wedge Q}\|} && (14) \\ &\quad \leq \frac{\|T^{t-s}(\theta(s,\omega))\|^{Q} \cdot \|\check{T}^{s}(\omega)\|^{Q}}{\|[\check{T}^{t}(\omega)|V_0(\omega)^\perp]^{\wedge Q}\|} && (15) \end{aligned}$$

perfectly in ω. The integrability condition (iii) implies that

$$\sup_{0 \leq s \leq t \leq a} \|T^{t-s}(\theta(s,\omega))\|^{Q} \cdot \|\check{T}^{s}(\omega)\|^{Q} < \infty$$

perfectly in ω for any finite $a > 0$. We next show that

$$\sup_{0\le s\le t\le a} |\det(\eta^{(1)}_{t-s}(\theta(s,\omega)),\cdots,\eta^{(Q)}_{t-s}(\theta(s,\omega)))|^{-1} < \infty \tag{16}$$

perfectly in ω for any finite $a > 0$. To prove (16), it suffices to show that

$$\inf_{(t,v_1,\cdots,v_Q)\in S(\omega)} \| \wedge_{k=1}^{Q} [\check{T}^t(\omega)(v_k)]\| > 0 \tag{17}$$

perfectly in ω, where $S(\omega)$ stands for the compact set

$$S(\omega) := \{(t, v_1, \cdots, v_Q) : t \in [0, a], v_k \in V_0(\omega)^{\perp}, \|v_k\| = 1, < v_k, v_l >= 0, 1 \le k < l \le Q\}.$$

To establish (17), note that each map $\check{T}^t(\omega)|V_0(\omega)^{\perp} : V_0(\omega)^{\perp} \to V_0(\theta(t,\omega))^{\perp}$ is injective for each $t \ge 0$ perfectly in ω. This follows easily from the cocycle property and the fact that $\lambda_{j_0} > -\infty$. Indeed,

$$\| \wedge_{k=1}^{Q} [\check{T}^t(\omega)(v_k)]\| > 0 \tag{18}$$

for all $(t, v_1, \cdots, v_Q) \in S(\omega)$. From hypothesis (v) of the lemma, the map

$$[0, a] \times [V_0(\omega)^{\perp}]^Q \ni (t, v_1, \cdots, v_Q) \mapsto \| \wedge_{k=1}^{Q} [\check{T}^t(\omega)(v_k)]\| \in [0,\infty)$$

is jointly continuous. Hence by (18) and the compactness of $S(\omega)$, (17) follows. In view of (15) and (17), one gets (16).

Next, we claim that

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{0\le s\le t} |\det(\eta^{(1)}_{t-s}(\theta(s,\omega)),\cdots,\eta^{(Q)}_{t-s}(\theta(s,\omega)))|^{-1} = 0 \tag{19}$$

perfectly in ω. To prove (19), use (14) to obtain the estimate

$$\begin{aligned} &|\det(\eta^{(1)}_{t-s}(\theta(s,\omega)),\cdots,\eta^{(Q)}_{t-s}(\theta(s,\omega)))|^{-1} \\ &\qquad\le \frac{\Pi_{k=1}^{Q}\{\|[T^{t-s}(\theta(s,\omega))|E_{r(k)}(\theta(s,\omega))]\| \cdot \|[\check{T}^s(\omega)|E_{r(k)}(\omega)]\|\}}{\|[\check{T}^t(\omega)|V_0(\omega)^{\perp}]^{\wedge Q}\|} \end{aligned}$$

for $0 \le s \le t$ perfectly in ω. Take $\dfrac{1}{t}\log \sup\limits_{0\le s\le t}$ on both sides of the above inequality and use Lemma 5.2 (2) to obtain

$$\begin{aligned}
\frac{1}{t}\log &\sup_{0\le s\le t} |\det(\eta^{(1)}_{t-s}(\theta(s,\omega)),\cdots,\eta^{(Q)}_{t-s}(\theta(s,\omega)))|^{-1}\\
&\le \frac{1}{t}\sup_{0\le s\le t}\Big\{\sum_{k=1}^{Q}(\log\|[T^{t-s}(\theta(s,\omega))|E_{r(k)}(\theta(s,\omega)]\| + \log\|[\check{T}^{s}(\omega)|E_{r(k)}(\omega)]\|)\Big\}\\
&\qquad\qquad - \frac{1}{t}\log\|[\check{T}^{t}(\omega)|V_0(\omega)^{\perp}]^{\wedge Q}\|\\
&\le \frac{1}{t}\sup_{0\le s\le t}\Big\{\sum_{k=1}^{Q}(t-s)\lambda_{r(k)} + \epsilon t + K^1_\epsilon(\omega) + \sum_{k=1}^{Q} s\lambda_{r(k)} + \epsilon s + K^2_\epsilon(\omega)\Big\}\\
&\qquad\qquad - \frac{1}{t}\log\|[\check{T}^{t}(\omega)|V_0(\omega)^{\perp}]^{\wedge Q}\|\\
&= \sum_{k=1}^{Q}\lambda_{r(k)} + 2\epsilon + \frac{1}{t}[K^1_\epsilon(\omega)+K^2_\epsilon(\omega)] - \frac{1}{t}\log\|[\check{T}^{t}(\omega)|V_0(\omega)^{\perp}]^{\wedge Q}\|, \quad t>0,
\end{aligned}$$

for arbitrary $\epsilon > 0$ where $K^i_\epsilon(\omega), i = 1,2,$ are finite positive constants (independent of t). The above inequality holds perfectly in ω. Letting $t \to \infty$ in the above inequality, we obtain

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{t}\log &\sup_{0\le s\le t} |\det(\eta^{(1)}_{t-s}(\theta(s,\omega)),\cdots,\eta^{(Q)}_{t-s}(\theta(s,\omega)))|^{-1}\\
&\le \sum_{k=1}^{Q}\lambda_{r(k)} + 2\epsilon - \liminf_{t\to\infty}\frac{1}{t}\log\|[\check{T}^{t}(\omega)|V_0(\omega)^{\perp}]^{\wedge Q}\|\\
&= \sum_{k=1}^{Q}\lambda_{r(k)} + 2\epsilon - \sum_{k=1}^{Q}\lambda_{r(k)}\\
&= 2\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the above inequality implies

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{0\le s\le t}|\det(\eta^{(1)}_{t-s}(\theta(s,\omega)),\cdots,\eta^{(Q)}_{t-s}(\theta(s,\omega)))|^{-1} \le 0 \tag{20}$$

perfectly in ω. The inequality

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\log &\sup_{0\le s\le t}|\det(\eta^{(1)}_{t-s}(\theta(s,\omega)),\cdots,\eta^{(Q)}_{t-s}(\theta(s,\omega)))|^{-1}\\
&\ge \liminf_{t\to\infty}\frac{1}{t}\log|\det(\eta^{(1)}_{t}(\omega),\cdots,\eta^{(Q)}_{t}(\omega))|^{-1} = 0 \qquad (21)
\end{aligned}$$

follows immediately from (11). Combining (20) and (21) yields (19).

Using (16), (19) and (13), it is now easy to see that $D_\epsilon(\omega)$ is finite perfectly in ω.

The reader may check that the last inequality in (12) follows directly from (13).

We next prove the first two inequalities in (12). Consider the equation

$$\breve{u}(\omega) = \sum_{k=1}^{Q} u_t^{(k)}(\omega)\eta_t^{(k)}(\omega), \quad u \in H, t \geq 0.$$

View $\breve{u}(\omega)$, $\eta_t^{(k)}(\omega), 1 \leq k \leq Q$, as column vectors in $\mathbf{R}^Q$ with respect to the basis $\{\xi_0^{(k)}(\theta(t,\omega)) : 1 \leq k \leq Q\}$. Solving the above equation for each $u_t^{(k)}(\omega)$ gives

$$\begin{aligned}
|u_t^{(k)}(\omega)| &= \left| \frac{\det(\eta_t^{(1)}(\omega), \cdots, \eta_t^{(k-1)}(\omega), \breve{u}(\omega), \eta_t^{(k+1)}(\omega), \cdots, \eta_t^{(Q)}(\omega))}{\det(\eta_t^{(1)}(\omega), \cdots, \eta_t^{(Q)}(\omega))} \right| \\
&\leq \frac{\|\breve{u}(\omega)\|}{|\det(\eta_t^{(1)}(\omega), \cdots, \eta_t^{(Q)}(\omega))|} \\
&\leq \frac{[D_\epsilon(\omega) - 1]}{Q} \|u\| e^{\epsilon t} \qquad (22)\\
&\leq D_\epsilon(\omega) \|u\| e^{\epsilon t}, \quad 1 \leq k \leq Q, \ t \geq 0,
\end{aligned}$$

perfectly in ω, by Cramer's rule and (13). Using (10), the triangle inequality and (22), we obtain

$$\|u_t^{(Q+1)}(\omega)\| \leq \|u\| + \sum_{k=1}^{Q} |u_t^{(k)}(\omega)| \leq D_\epsilon(\omega)\|u\| e^{\epsilon t}, \quad t \geq 0,$$

perfectly in ω. This proves that $T_n(\omega)$ satisfies (S4) perfectly in ω, and completes the proof of the proposition. □

The following lemma is used in the discretization argument underlying the proof of the local stable-manifold theorem (Theorem 4.1).

Lemma 5.4.

Assume the hypotheses of Lemma 3.2. Then there is a sure event $\Omega_3 \in \mathcal{F}$ *with the following properties:*

(i) $\theta(t, \cdot)(\Omega_3) = \Omega_3$ *for all* $t \in \mathbf{R}$*,*

(ii) For every $\omega \in \Omega_3$ *and any* $(v, \eta) \in M_2$*, the statement*

$$\limsup_{n \to \infty} \frac{1}{n} \log \|Z(nr, (v, \eta), \omega)\| < 0 \tag{23}$$

implies

$$\limsup_{t \to \infty} \frac{1}{t} \log \|Z(t, (v, \eta), \omega)\| = \limsup_{n \to \infty} \frac{1}{nr} \log \|Z(nr, (v, \eta), \omega)\|. \tag{24}$$

Proof.

Using the integrability condition (2) of Lemma 3.2, the proof of the lemma is exactly analogous to that of Lemma 3.4 in [M-S.4]. □

Proof of Theorem 4.1.

All real-valued random variables in this proof will be taken to be $\bar{\mathcal{F}}$*-measurable.*

It is sufficient to assume that $r > 0$. The case $r = 0$ is handled in [M-S.4], Theorem 3.1.

The proof of Theorem 4.1 will build on Proposition 4.1 and its proof. Recall the notations and assertions of Proposition 4.1 and its proof. Our first task is to show that the sure event $\Omega_1^* \in \mathcal{F}$ can be chosen such that $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$; and for each $\omega \in \Omega_1^*$, the random family of $C^{k,\epsilon}$ discrete-time stable submanifolds $\tilde{\mathcal{S}}_d(\omega)$ of $\bar{B}(0, \rho_1(\omega))$ are given by:

$$\tilde{\mathcal{S}}_d(\omega) = \{(v, \eta) \in \bar{B}(0, \rho_1(\omega)) : \|Z(nr, (v, \eta), \omega)\|_{M_2} \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1) nr} \text{ for all integers } n \geq 0\}, \tag{25}$$

where $\rho_1, \beta_1 : \Omega_1^* \to (0,1)$ are $\bar{\mathcal{F}}$-measurable positive random variables. Each $\tilde{\mathcal{S}}_d(\omega)$ is tangent at 0 to the stable subspace $\mathcal{S}(\omega)$ of the linearized flow D_2X, viz. $T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$. In particular, codim $\tilde{\mathcal{S}}_d(\omega)$ is finite and non-random. Furthermore,

$$\limsup_{n\to\infty} \frac{1}{nr} \log\Bigg[\sup_{\substack{(v_1,\eta_1),(v_2,\eta_2)\in\tilde{\mathcal{S}}_d(\omega)\\(v_1,\eta_1)\neq(v_2,\eta_2)}} \frac{\|Z(nr,(v_1,\eta_1),\omega) - Z(nr,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1)-(v_2,\eta_2)\|}\Bigg] \leq \lambda_{i_0}. \tag{26}$$

We will outline the construction of the $\theta(t,\cdot)$-invariant sure event Ω_1^* referred to above. This will follow from the proof of Theorem 5.1 ([Ru.2], p. 272) coupled with additional perfection arguments given in Lemmas 5.1, 5.2, 5.3. More specifically, and in the notation of [Ru.2], let $T^t(\omega) := D_2Z(rt,0,\omega)$, $f(\omega) := \theta(r,\omega)$, $T_n(\omega) := D_2Z(r,0,\theta((n-1)r,\omega))$, $t \in \mathbf{R}^+, n \in \mathbf{Z}^+$. By the integrability property (2) of Lemma 3.2 and the perfect ergodic theorem (Lemma 5.1 (ii)), one may replace (5.3) in [Ru.2], p. 274) by its continuous-time analogue

$$\lim_{t\to\infty} \frac{1}{t} \log^+ \|Z(r,\cdot,\theta(t,\omega))\|_{1,\epsilon} = 0. \tag{27}$$

The above relation holds perfectly in ω, viz. there is a sure event $\Omega_1^* \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$ and (27) holds for all $\omega \in \Omega_1^*$. In the notation of Theorem 1.1 ([Ru.2], p. 248), set $S = i_0 - 1$, fixed, and $\mu^{(S+1)} = \lambda_{i_0}$, when $\lambda_{i_0} > -\infty$; if $\lambda_{i_0} = -\infty$, we replace $\mu^{(S+1)}$ by any fixed number in $(-\infty, 0)$. In view of the integrability property (2) of Lemma 3.2, and Lemma 5.3 (with $t_0 = r, j_0 = i_0 - 1$), it follows that there is a sure event $\Omega_2^* \in \mathcal{F}$ such that $\Omega_2^* \subseteq \Omega_1^*$, $\theta(t,\cdot)(\Omega_2^*) = \Omega_2^*$ for all $t \in \mathbf{R}$, and the sequence $\{T_n(\omega), V_n(\omega) := E_{i_0}(\theta(nr,\omega)),\ n \geq 1\}$, satisfies Conditions (S) of ([Ru.2], p. 256) for every $\omega \in \Omega_2^*$. Fixing any $\omega \in \Omega_2^*$, we continue to follow the proof of Theorem 5.1 in [Ru.2], pp. 274-278. In particular, Ruelle's "perturbation theorem" (Theorem 4.1, [Ru.2], pp. 262-263) holds for the sequence $T_n(\omega), n \geq 1$, and therefore the results quoted in the previous paragraph hold for $k = 1, \epsilon \in (0,\delta)$. To see that the $C^{k,\epsilon}$ manifolds $(k > 1, \epsilon \in (0,\delta))$

$\tilde{\mathcal{S}}_d(\omega)$ are defined *perfectly in* ω, we follow the inductive argument in [Ru.2], pp. 278-279, by applying the previous analysis to the following perfect cocycle on $M_2 \oplus M_2$:

$$\Big(\check{Z}(t,(v,\eta),(v_1,\eta_1),\omega) := (Z(t,(v,\eta),\omega), D_2Z(t,(v,\eta),\omega)(v_1,\eta_1)), \theta(t,\omega)\Big),$$

for $(v,\eta),(v_1,\eta_1) \in M_2$, $t \geq 0$. The inductive argument yields that $\tilde{\mathcal{S}}_d(\omega)$ is a $C^{k,\epsilon}$ manifold perfectly in ω.

Consider the set $\tilde{\mathcal{S}}(\omega), \omega \in \Omega_1^*$, defined in part (a) of the theorem. Then as in the proof of Proposition 4.1, it follows that $\tilde{\mathcal{S}}(\omega)$ is a $C^{k,\epsilon}$ manifold $(k > 1, \epsilon \in (0,\delta))$ for all $\omega \in \Omega_1^*$, $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$; and codim $\tilde{\mathcal{S}}(\omega) =$ codim $\mathcal{S}(\omega)$ is finite and non-random.

We next show the inequality (1) in (a) of the theorem. By (b) of Proposition 4.1, we have

$$\limsup_{n\to\infty} \frac{1}{nr} \log \|Z(nr,(v,\eta),\omega)\| \leq \lambda_{i_0} \tag{28}$$

for all ω in the shift-invariant sure event Ω_1^* and all $(v,\eta) \in \tilde{\mathcal{S}}_d(\omega)$. Therefore by Lemma 5.4, there is a sure event $\Omega_3^* \subseteq \Omega_2^*$, $\Omega_3^* \in \mathcal{F}$, such that $\theta(t,\cdot)(\Omega_3^*) = \Omega_3^*$ for all $t \in \mathbf{R}$, and

$$\limsup_{t\to\infty} \frac{1}{t} \log \|Z(t,(v,\eta),\omega)\| \leq \lambda_{i_0} \tag{29}$$

for all $\omega \in \Omega_3^*$ and all $(v,\eta) \in \tilde{\mathcal{S}}_d(\omega)$. Now inequality (1) of the theorem follows directly from (29) and the definition of Z ((1) in Section 3).

We next prove assertion (b) of the theorem. Take any $\omega \in \Omega_1^*$. By (26), there is a positive integer $N_0 := N_0(\omega)$ (independent of $(v,\eta) \in \tilde{\mathcal{S}}_d(\omega)$) such that $Z(nr,(v,\eta),\omega) \in \bar{B}(0,1)$ for all $n \geq N_0$. Let Ω_3 be a $\theta(t,\cdot)$-invariant sure event such that

$$\lim_{t\to\infty} \frac{1}{t} \log^+ \sup_{\substack{0\leq u\leq r,\\ (v^*,\eta^*)\in\bar{B}(0,1)}} \|D_2Z(u,(v^*,\eta^*),\theta(t,\omega))\|_{L(M_2)} = 0$$

for all $\omega \in \Omega_3$ (Lemma 5.1 (ii)). Let $\Omega_4^* := \Omega_3^* \cap \Omega_3$. Then $\Omega_4^* \in \mathcal{F}$, is a sure event and $\theta(t,\cdot)(\Omega_4^*) = \Omega_4^*$ for all $t \in \mathbf{R}$. By a similar argument to the one used in the proof of Lemma 3.4 in [M-S.4], it follows that

$$\begin{aligned}
&\sup_{nr \le t \le (n+1)r} \frac{1}{t} \log \Bigg[\sup_{\substack{(v_1,\eta_1) \ne (v_2,\eta_2), \\ (v_1,\eta_1),(v_2,\eta_2) \in \tilde{\mathcal{S}}(\omega)}} \frac{\|X(t,(v_1,\eta_1),\omega) - X(t,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1) - (v_2,\eta_2)\|} \Bigg] \\
&= \sup_{nr \le t \le (n+1)r} \frac{1}{t} \log \Bigg[\sup_{\substack{(v_1,\eta_1) \ne (v_2,\eta_2), \\ (v_1,\eta_1),(v_2,\eta_2) \in \tilde{\mathcal{S}}_d(\omega)}} \frac{\|Z(t,(v_1,\eta_1),\omega) - Z(t,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1) - (v_2,\eta_2)\|} \Bigg] \\
&\le \frac{1}{nr} \log^+ \sup_{\substack{0 \le u \le r, \\ (v^*,\eta^*) \in \bar{B}(0,1)}} \|D_2 Z(u,(v^*,\eta^*),\theta(nr,\omega))\|_{L(M_2)} \\
&\qquad + \frac{n}{(n+1)} \frac{1}{nr} \log \Bigg[\sup_{\substack{(v_1,\eta_1) \ne (v_2,\eta_2), \\ (v_1,\eta_1),(v_2,\eta_2) \in \tilde{\mathcal{S}}_d(\omega)}} \frac{\|Z(nr,(v_1,\eta_1),\omega) - Z(nr,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1) - (v_2,\eta_2)\|} \Bigg]
\end{aligned}$$

for all $\omega \in \Omega_4^*$, all $n \ge N_0(\omega)$ and sufficiently large. Taking $\limsup\limits_{n\to\infty}$ in the above inequality and using (26), immediately gives assertion (b) of the theorem.

To prove the cocycle-invariance statements (c), we begin by the inclusion (3) in the theorem. This is proved by applying the (perfect continuous-time version of the) Oseledec theorem to the linearized cocycle $(D_2X(t,Y(\omega),\omega),\theta(t,\omega))$ ([Mo.2], Theorem 4, Corollary 2). Hence there is a sure $\theta(t,\cdot)$-invariant event, also denoted by $\Omega_1^* \in \mathcal{F}$, such that $D_2X(t,Y(\omega),\omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega))$ for all $t \ge 0$ and all $\omega \in \Omega_1^*$.

We next prove the asymptotic invariance property (2) of the theorem. To this end, we will need to modify the proofs of Theorems 5.1 and 4.1 in [Ru.2], pp. 262-279. We will first show that two random variables ρ_1, β_1 and a sure event (also denoted by) Ω_1^* may be chosen such that $\theta(t,\cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$, and

$$\rho_1(\theta(t,\omega)) \ge \rho_1(\omega) e^{(\lambda_{i_0}+\epsilon_1)t}, \quad \beta_1(\theta(t,\omega)) \ge \beta_1(\omega) e^{(\lambda_{i_0}+\epsilon_1)t} \tag{30}$$

for every $\omega \in \Omega_1^*$ and all $t \ge 0$. For the given choice of ϵ_1, fix $0 < \epsilon_3 < -\epsilon(\lambda_{i_0}+\epsilon_1)/4$. The above inequalities hold in the *discrete* case (when $t = n$, a positive integer) from Theorem 5.1 (c) ([Ru.2], p. 274). We claim that ρ_1 and β_1 may be redefined so that the relations

(30) hold for *continuous* time. To see this, we will modify the definitions of these random variables in the proofs of Theorems 5.1 and 4.1 in [Ru.2]. In the notation of the proof of Theorem 5.1 ([Ru.2], p. 274), we replace the random variable G in (5.4) ([Ru.2], p. 274) by the larger one

$$\tilde{G}(\omega) := \sup_{t\geq 0} \|Z(r,\cdot,\theta(t,\omega))\|_{1,\epsilon}\, e^{(-t\epsilon_3-\lambda\epsilon)}. \tag{31}$$

In (31), $\epsilon\in(0,\delta)$ stands for the Hölder exponent of the semiflow X. By (27) and Lemma 3.2, it is easy to see that $\tilde{G}(\omega)<\infty$ perfectly in ω. Following ([Ru.2], pp. 266, 274), the random variables ρ_1,β_1 may be chosen according to the relations

$$\beta_1 := \left[\frac{\delta_1\wedge\left(\frac{1}{\sqrt{2}A}\right)}{2\tilde{G}}\right]^{\frac{1}{\epsilon}}\wedge 1 \tag{32}$$

$$\rho_1 := \frac{\beta_1}{B_{\epsilon_3}} \tag{33}$$

where A,δ_1 and B_{ϵ_3} are random positive constants that are defined via continuous-time analogues of the relations (4.26), (4.18)-(4.21), (4.24), (4.25) in [Ru.2], pp. 265-267, with η replaced by ϵ_3. In particular, the "ancestry" of A,δ_1 and B_{ϵ_3} in Ruelle's argument may be traced back to the constants $D_{\epsilon_3},K_{\epsilon_3}$ which appear in Lemmas 5.3 and 5.2 of this article. Thus, in order to establish (30), it suffices to observe that, for sufficiently small $\epsilon_3>0$, the following inequalities

$$\left.\begin{aligned} K_{\epsilon_3}(\theta(l,\omega)) &\leq K_{\epsilon_3}(\omega)+\frac{\epsilon_3 l}{2}\\ D_{\epsilon_3}(\theta(l,\omega)) &\leq e^{\frac{\epsilon_3 l}{2}}D_{\epsilon_3}(\omega)\\ \tilde{G}(\theta(l,\omega)) &\leq e^{\epsilon_3 l}\tilde{G}(\omega)\end{aligned}\right\} \tag{34}$$

hold perfectly in ω for all $l\geq 0$. The first inequality in (34) follows from Lemma 5.2 (2), while the second inequality is a consequence of Lemma 5.3. The third inequality in (34) follows directly from (31). In view of (32) and (33), (30) holds. This completes the proof of (30).

We are now ready to prove the asymptotic invariance property (2) in (c) of the theorem. Use (b) to obtain a sure event $\Omega_5^* \subseteq \Omega_4^*$ such that $\theta(t,\cdot)(\Omega_5^*) = \Omega_5^*$ for all $t \in \mathbf{R}$, and for any $0 < \epsilon' < \epsilon_1$ and $\omega \in \Omega_5^*$, there exists $\beta^{\epsilon'}(\omega) > 0$ (independent of (v,η)) with

$$|X(t,(v,\eta),\omega) - Y(\theta(t,\omega))| \leq \beta^{\epsilon'}(\omega) e^{(\lambda_{i_0}+\epsilon')t} \tag{35}$$

for all $(v,\eta) \in \tilde{\mathcal{S}}(\omega)$, $t \geq 0$. Fix any real $t \geq 0$, $\omega \in \Omega_5^*$ and $(v,\eta) \in \tilde{\mathcal{S}}(\omega)$. Let n be a non-negative integer. Then the cocycle property and (35) imply that

$$\begin{aligned} |X(nr, X(t,(v,\eta),\omega), \theta(t,\omega)) - Y(\theta(nr,\theta(t,\omega)))| &= |X(nr+t,(v,\eta),\omega) - Y(\theta(nr+t,\omega))| \\ &\leq \beta^{\epsilon'}(\omega) e^{(\lambda_{i_0}+\epsilon')(nr+t)} \\ &\leq \beta^{\epsilon'}(\omega) e^{(\lambda_{i_0}+\epsilon')t} e^{(\lambda_{i_0}+\epsilon_1)nr}. \end{aligned} \tag{36}$$

If $\omega \in \Omega_5^*$, then it follows from (30),(35), (36) and the definition of $\tilde{\mathcal{S}}(\theta(t,\omega))$ that there exists $\tau_1(\omega) > 0$ such that $X(t,(v,\eta),\omega) \in \tilde{\mathcal{S}}(\theta(t,\omega))$ for all $t \geq \tau_1(\omega)$. This proves the invariance property (2) and completes the proof of assertion (c) of the theorem.

We now prove assertion (d) of the theorem, regarding the existence of the local unstable manifolds $\tilde{\mathcal{U}}(\omega)$ *perfectly* in ω . Define the random field $\hat{Z} : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ by

$$\hat{Z}(t,(v,\eta),\omega) := X(t,(v,\eta) + Y(\theta(-t,\omega)), \theta(-t,\omega)) - Y(\omega) \tag{37}$$

for all $t \geq 0$, $(v,\eta) \in M_2$, $\omega \in \Omega$. Observe that $\hat{Z}(t,\cdot,\omega) = Z(t,\cdot,\theta(-t,\omega))$, $t \geq 0$, $\omega \in \Omega$; and $\hat{Z}$ is $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(M_2) \otimes \mathcal{F}, \mathcal{B}(M_2))$-measurable, by the remark following the proof of Theorem 4.1 in ([M-S.3]). From (4) (Section 2) (with $s = -t$), it follows immediately that $\hat{Z}(t,0,\omega) = 0$ for all $t \geq 0$, $\omega \in \Omega$. Using the fact that $(D_2X(t,Y(\omega),\omega), \theta(t,\omega))$ is an $L(M_2)$-valued perfect cocycle, it is easy to see that $([D_2\hat{Z}(t,0,\omega)]^*, \theta(-t,\omega),\, t \geq 0)$ is a perfect linear cocycle (in $L(M_2)$).

We next show that the cocycles $(D_2X(t,Y(\omega),\omega),\theta(t,\omega),t\geq 0)$ and $([D_2\hat{Z}(t,0,\omega)]^*,$ $\theta(-t,\omega),\ t\geq 0)$ have the same Lyapunov spectrum with multiplicities. First, we need to verify the integrability condition

$$\int_\Omega \log^+ \sup_{0\leq t_1,t_2\leq T} \|[D_2\hat{Z}(t_2,0,\theta(-t_1,\omega))]^*\|_{L(M_2)}\,dP(\omega)<\infty \tag{38}$$

for any fixed $T\in(0,\infty)$. To prove (38), use (5) of Lemma 2.1 and the P-preserving property of $\theta(t,\cdot)$ in order to obtain the following relations:

$$\begin{aligned}
&\int_\Omega \log^+ \sup_{0\leq t_1,t_2\leq T} \|[D_2\hat{Z}(t_2,0,\theta(-t_1,\omega))]^*\|_{L(M_2)}\,dP(\omega)\\
&\quad=\int_\Omega \log^+ \sup_{0\leq t_1,t_2\leq T} \|D_2X(t_2,Y(\theta(-t_2-t_1,\omega)),\theta(-t_2-t_1,\omega))\|_{L(M_2)}\,dP(\omega)\\
&\quad\leq\int_\Omega \log^+ \sup_{0\leq t_1\leq 2T,\,0\leq t_2\leq T} \|D_2X(t_2,Y(\theta(t_1,\omega)),\theta(t_1,\omega))\|_{L(M_2)}\,dP(\omega)\\
&\quad\leq\int_\Omega \log^+ \sup_{0\leq t_1\leq T,\,0\leq t_2\leq T} \|D_2X(t_2,Y(\theta(t_1,\omega)),\theta(t_1,\omega))\|_{L(M_2)}\,dP(\omega)\\
&\quad+\int_\Omega \log^+ \sup_{T\leq t_1\leq 2T,\,0\leq t_2\leq T} \|D_2X(t_2,Y(\theta(t_1-T,\omega)),\theta(t_1-T,\omega))\|_{L(M_2)}\,dP(\omega)\\
&\quad=2\int_\Omega \log^+ \sup_{0\leq t_1,t_2\leq T} \|D_2X(t_2,Y(\theta(t_1,\omega)),\theta(t_1,\omega))\|_{L(M_2)}\,dP(\omega)<\infty.
\end{aligned}$$

In view of the integrability property (38), it follows that the linear cocycle $([D_2\hat{Z}(t,0,\omega)]^*,\theta(-t,\omega),\ t\geq 0)$ has a fixed discrete Lyapunov spectrum which coincides with that of $(D_2X(t,Y(\omega),\omega),\theta(t,\omega))$, viz. $\{\cdots\lambda_{i+1}<\lambda_i<\cdots<\lambda_2<\lambda_1\}$ where $\lambda_i\neq 0$ for all $i\geq 1$, by hyperbolicity. See [Ru.2], Section 3.5, p. 261.

To establish a *perfect* version of the local unstable manifolds $\tilde{\mathcal{U}}(\omega)$, we begin with the estimate

$$\int_\Omega \log^+ \sup_{0\leq t_1,t_2\leq r} \|\hat{Z}(t_2,\cdot,\theta(-t_1,\omega))\|_{k,\epsilon}\,dP(\omega)<\infty,$$

which follows from the P-preserving property of $\theta(t,\cdot), t\in\mathbf{R}$, and Lemma 3.2. Define i_0 as before, so that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized cocycle. Fix $0<\epsilon_2<\lambda_{i_0-1}$. In view of the above integrability property, it follows from Lemma 5.3 that the sequence $\tilde{T}_n(\omega) := [D_2\hat{Z}(r,0,\theta(-nr,\omega))]^*$, $\theta(-nr,\omega)$, $n \geq 0$, satisfies Condition (S) of [Ru.2] perfectly in ω. Therefore Proposition 3.3 in [Ru.2] implies that the sequence $\tilde{T}_n(\omega), n \geq 1$, satisfies Corollary 3.4 ([Ru.2], p. 260) perfectly in ω. Now one can adapt the proof of Theorem 6.1 ([Ru.2], p. 280) along similar lines to the preceding arguments in this proof. This yields a $\theta(-t,\cdot)$-invariant sure event $\hat{\Omega}_1^* \in \mathcal{F}$ and $\bar{\mathcal{F}}$-measurable random variables $\rho_2, \beta_2 : \hat{\Omega}_1^* \to (0,1)$ with the following properties. For $\lambda_{i_0-1} < \infty$, let $\tilde{\mathcal{U}}_d(\omega)$ be the set of all $(v_0,\eta_0) \in \bar{B}(0,\rho_2(\omega))$ with the property that there is a discrete "history" process $u(-nr,\cdot) : \Omega \to M_2, n \geq 0$, such that $u(0,\omega) = (v_0,\eta_0)$, $\hat{Z}(r,u(-(n+1)r,\omega),\theta(-nr,\omega)) = u(-nr,\omega)$ and $\|u(-nr,\omega)\| \leq \beta_2(\omega)e^{-nr(\lambda_{i_0-1}-\epsilon_2)}$ for all $n \geq 0$. When $\lambda_{i_0-1} = \infty$, take $\tilde{\mathcal{U}}_d(\omega)$ to be the set of all $(v_0,\eta_0) \in M_2$ with the property that there is a discrete history process $u(-nr,\cdot) : \Omega \to M_2, n \geq 0$, such that $u(0,\omega) = (v_0,\eta_0)$, and $\|u(-nr,\omega)\| \leq \beta_2(\omega)e^{-\lambda nr}$ for all $n \geq 0$ and arbitrary $\lambda > 0$. The history process $u(-nr,\cdot)$ is uniquely determined by (v_0,η_0) ([Ru.2], p. 281). Furthermore, for every $\omega \in \hat{\Omega}_1^*$, $\tilde{\mathcal{U}}_d(\omega)$ is a $C^{k,\epsilon}$ ($\epsilon \in (0,\delta)$) finite-dimensional submanifold of $\bar{B}(0,\rho_2(\omega))$ with tangent space $U(\omega)$ at 0. Also $\dim \tilde{\mathcal{U}}_d(\omega)$ is fixed independently of ω and ϵ_2; and the following estimates hold perfectly in ω for all $t \geq 0$:

$$\rho_2(\theta(-t,\omega)) \geq \rho_2(\omega)e^{-(\lambda_{i_0-1}-\epsilon_2)t}, \quad \beta_2(\theta(-t,\omega)) \geq \beta_2(\omega)e^{-(\lambda_{i_0-1}-\epsilon_2)t}. \tag{39}$$

The first two assertions in (d) of the theorem follow by the same argument as the one used in the proof of Proposition 4.1 (d).

To prove the third assertion in part (d) of the theorem, let $(v,\eta) \in \tilde{\mathcal{U}}(\omega)$ and write $(v,\eta) = (v_0,\eta_0) + Y(\omega)$ where $(v_0,\eta_0) \in \mathcal{U}_d(\omega)$. Recall that y_0 is defined by

$$y_0(-nr) := u(-nr) + Y(\theta(-nr,\omega)), \qquad n \geq 0. \tag{40}$$

We will prove that y_0 extends to a continuous-time history process $y(\cdot,\omega) : (-\infty,0] \to M_2$ such that $y(0,\omega) = (v,\eta)$, and$y(\cdot,\omega)$ satisfies the third assertion in (d) of the theorem. To do this, we use the cocycle property of X to interpolate within the delay periods

$[-(n+1)r, -nr]$, $n \geq 0$. Let $s \in (-(n+1)r, -nr)$. and write $s = \alpha - (n+1)r$ for some $\alpha \in (0, r)$. Define

$$y(s, \omega) := X(s + (n+1)r, y_0(-(n+1)r, \omega), \theta(-(n+1)r, \omega)).$$

Clearly $y(0, \omega) = (v_0, \eta_0) + Y(\omega) = (v, \eta)$. Fix $s \in (-(n+1)r, -nr)$ as above and let $0 < t \leq -s$. Then there is a positive integer $m < n$ such that $s + t \in [-(m+1)r, -mr]$. Using the perfect cocycle property for X and the above definition of y, the reader may check that

$$y(t + s, \omega) = X(t, y(s, \omega), \theta(s, \omega)). \tag{41}$$

(Note that if we put $s = -t$ in (41), we get $X(t, y(-t, \omega), \theta(-t, \omega)) = (v, \eta)$ for all $t \geq 0$.)

Next we show that

$$\limsup_{t \to \infty} \frac{1}{t} \log \|y(-t, \omega) - Y(\theta(-t, \omega))\| \leq -\lambda_{i_0 - 1} \tag{42}$$

perfectly in ω. From Theorem 6.1 (b) in [Ru.2], we have

$$\limsup_{n \to \infty} \frac{1}{nr} \log \|y(-nr, \omega) - Y(\theta(-nr, \omega))\|_{M_2} \leq -\lambda_{i_0 - 1} \tag{43}$$

perfectly in ω. For each $t \in (nr, (n+1)r)$, write $-t = \alpha - (n+1)r$ for some $\alpha \in (0, r)$. Then by the definition of y and the Mean Value Theorem, we have

$$\begin{aligned}
&\|y(-t, \omega) - Y(\theta(-t, \omega))\|_{M_2} \\
&\quad = \|X(\alpha, y(-(n+1)r, \omega), \theta(-(n+1)r, \omega)) - X(\alpha, Y(\theta(-(n+1)r, \omega), \theta(-(n+1)r, \omega))\|_{M_2} \\
&\quad \leq \sup_{\substack{(v^*, \eta^*) \in \bar{B}(0,1), \\ \alpha \in (0, r)}} \|D_2 X(\alpha, (v^*, \eta^*) + Y(\theta(-(n+1)r, \omega)), \theta(-(n+1)r, \omega))\|_{L(M_2)} \\
&\qquad\qquad \times \|y(-(n+1)r, \omega) - Y(\theta(-(n+1)r, \omega))\|_{M_2}
\end{aligned}$$

perfectly in ω. Therefore

$$\limsup_{t\to\infty} \frac{1}{t} \log \|y(-t,\omega) - Y(\theta(-t,\omega))\|_{M_2}$$
$$\leq \limsup_{n\to\infty} \frac{1}{nr} \log^+ \sup_{\substack{(v^*,\eta^*)\in \bar{B}(0,1),\\ \alpha\in(0,r)}} \|D_2X(\alpha,(v^*,\eta^*)+Y(\theta(-(n+1)r,\omega)),\theta(-(n+1)r,\omega))\|_{L(M_2)}$$
$$+ \limsup_{n\to\infty} \frac{1}{nr} \log \|y(-(n+1)r,\omega) - Y(\theta(-(n+1)r,\omega)))\|_{M_2}.$$

The first term on the right hand side of the above inequality is zero, perfectly in $\omega \in \Omega$, because of Lemma 5.1 (ii) and the integrability condition (2) of Lemma 3.2. The second term is less than or equal to $-\lambda_{i_0-1}$ because $y(0) \in \tilde{\mathcal{U}}(\omega)$. The uniqueness of the continuous-time history process for a given $(v,\eta) \in \tilde{\mathcal{U}}(\omega)$ follows from that of the discrete-time process, (41) and forward uniqueness of the trajectories of (I). Hence the proof of assertion (d) of the theorem is complete.

The proof of assertion (e) of the theorem uses an interpolation argument similar to the above. The reader may check the details.

We will now verify the asymptotic invariance property in (f), that is

$$\tilde{\mathcal{U}}(\omega) \subseteq X(t,\cdot,\theta(-t,\omega))(\tilde{\mathcal{U}}(\theta(-t,\omega))), \quad t \geq \tau_2(\omega) \tag{44}$$

perfectly in ω for some $\tau_2(\omega) > 0$. To do this, let $(v,\eta) \in \tilde{\mathcal{U}}(\omega)$. Then by assertions (d), (e) of the theorem and inequalities (39), there exists a (unique) history process $y(-t,\omega), t \geq 0$, and a random time $\tau_2(\omega) > 0$ such that $y(0,\omega) = (v,\eta), y(-t,\omega) \in \bar{B}(Y(\theta(-t,\omega)),\rho_2(\theta(-t,\omega)))$ for all $t \geq \tau_2(\omega)$, and

$$y(t'-t,\omega) = X(t',y(-t,\omega),\theta(-t,\omega)), \quad 0 < t' \leq t, \tag{45}$$

perfectly in ω. Fix $t_1 \geq \tau_2(\omega)$. Note that by (45) (for $t = t' = t_1$), we have $(v,\eta) = X(t_1,y(-t_1,\omega),\theta(-t_1,\omega))$. We claim that $y(-t_1,\omega) \in \tilde{\mathcal{U}}(\theta(-t_1,\omega)))$ (and in fact $y(-u,\omega) \in \tilde{\mathcal{U}}(\theta(-u,\omega)))$ for all $u \geq \tau_2(\omega)$). To see this, define the process
$y_1(-t,\omega) := y(-t-t_1,\omega)$, $t \geq 0$. Then $y_1(\cdot,\omega)$ is a history process with $y_1(0,\omega) = y(-t_1,\omega)$

$\in \bar{B}(Y(\omega), \rho_2(\theta(-t_1, \omega)))$. Therefore $y(-t_1, \omega) \in \tilde{\mathcal{U}}(\theta(-t_1, \omega)))$. Since $t_1 \geq \tau_2(\omega)$ is arbitrary, (44) follows. The invariance assertion (4) in (f) of the theorem and the fact that

$$D_2X(t, \cdot, \theta(-t, \omega))|\mathcal{U}(\theta(-t, \omega)) : \mathcal{U}(\theta(-t, \omega)) \to \mathcal{U}(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto, are consequences of the Oseledec theorem and the cocycle property for the linearized semiflow; cf. [Mo.2], Corollary 2 (v) of Theorem 4.

The transversality assertion in (g) of the theorem follows immediately from the relations

$$T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega), \quad T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega), \quad M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega)$$

which hold perfectly in ω.

Taking $\Omega^* := \Omega_1^* \cap \hat{\Omega}_1^*$, completes the proof of assertions (a)-(g) of the theorem.

Suppose Hypothesis $(SMW)_{k,\delta}$ holds for every $k \geq 1$ and $\delta \in (0, 1]$. Then a simple adaptation of the argument in [Ru.2], Section (5.3) (p. 297) gives a $\theta(t, \cdot)$-invariant sure event in $\mathcal{F}$, also denoted by Ω^*, such that $\tilde{\mathcal{S}}(\omega)$ and $\tilde{\mathcal{U}}(\omega)$ are C^∞ for all $\omega \in \Omega^*$. The proof of Theorem 4.1 is now complete. □

6. Appendix.

The following result is due to Itô and Nisio ([I-N]). It gives sufficient conditions for the existence of stationary solutions of the sfde (I). In [I-N], various conditions on the coefficients H, G of the sfde (I) are given which guarantee the existence (and uniqueness) of stationary solutions of (I) hence of stationary points ([I-N], Theorems 4, 5, 6, 7, 12, 13). More specifically, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}}, P)$ be the standard filtered Wiener space, with $\Omega := C(\mathbf{R}, \mathbf{R}^p; 0)$, the space of all continuous paths $\omega : \mathbf{R} \to \mathbf{R}^p$ such that $\omega(0) = 0$, given the topology of uniform convergence on compacta and the Borel σ-algebra $\mathcal{F}$. For each t, let $\mathcal{F}_t$ be the P-completed σ-algebra generated by all evaluations $\Omega \ni \omega \mapsto \omega(u) - \omega(v) \in \mathbf{R}^p$, $v \leq u \leq t$. Denote by $\theta : \mathbf{R} \times \Omega \to \Omega$ the canonical two-sided Wiener shift

$$\theta(t, \omega)(s) = \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in \Omega,$$

and by $W : \mathbf{R} \times \Omega \to \mathbf{R}^p$ the p-dimensional Brownian motion:

$$W(t, \omega) := \omega(t), \quad \omega \in \Omega, t \in \mathbf{R}.$$

Define $\tilde{\Omega} := C(\mathbf{R}, \mathbf{R}^d) \times C(\mathbf{R}, \mathbf{R}^p; 0)$. Denote by $\tilde{\mathcal{F}} := \mathcal{B}(C(\mathbf{R}, \mathbf{R}^d)) \otimes \mathcal{B}(C(\mathbf{R}, \mathbf{R}^p; 0))$ the Borel σ-algebra of $\tilde{\Omega}$. Define the processes $x^\infty : \mathbf{R} \times \tilde{\Omega} \to \mathbf{R}^d$ and $W^\infty : \mathbf{R} \times \tilde{\Omega} \to \mathbf{R}^p$ by

$$x^\infty(t, \tilde{\omega}) := f(t), \; W^\infty(t, \tilde{\omega}) := W(t, \omega) = \omega(t),$$

for all $t \in \mathbf{R}, \; \tilde{\omega} := (f, \omega) \in \tilde{\Omega}$.

Following [I-N], say that x^∞ is a *stationary solution* of (I) if there exists a probability measure P^∞ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ such that the following is true:

(i) W^∞ is p-dimensional standard Brownian motion on $(\tilde{\Omega}, \tilde{\mathcal{F}}, P^\infty)$.

(ii) (x^∞, dW^∞) are *strictly stationarily correlated* in the sense that the law of the process

$$(x^\infty(t, \cdot), W^\infty(u, \cdot) - W^\infty(v, \cdot), t \in \mathbf{R}, v \leq u)$$

is invariant under time-shifts.

(iii) The σ-algebra $\sigma\{x^\infty(u) : u \leq t\} \vee \sigma\{W^\infty(u, \cdot) - W^\infty(v, \cdot), v \leq u \leq t\}$ is independent of $\sigma\{W^\infty(u, \cdot) - W^\infty(v, \cdot), t \leq v \leq u\}$ under P^∞ for each $t \in \mathbf{R}$.

(iv) x^∞ is a two-sided solution of (I) when W is replaced by W^∞:

$$dx^\infty(t) = H(x^\infty(t), x_t^\infty)\, dt + G(x^\infty(t))\, dW^\infty(t), \quad t > s > -\infty. \qquad (I^\infty)$$

The following result is proved in [I-N] for the one-dimensional case $d = 1$. The reader may note the argument in the proof of Theorem 3 ([I-N], p. 25) extends to cover the multidimensional case $d > 1$.

Theorem 6.1. *(K. Itô and M. Nisio (1964))*

Assume that the coefficients H and G of the sfde (I) satisfy Hypotheses $(SMW)_{k,\delta}$. Suppose (I) has a solution $x^{(0,0)} : [-r,\infty)\times\Omega\to\mathbf{R}^d$ which satisfies $\sup_{t\geq 0} E|x^{(0,0)}(t)|^2<\infty$. Then (I) has a stationary solution x^∞ satisfying $E\|(x^\infty(t),x^\infty_t)\|^2_{M_2}<\infty$ for all $t\in\mathbf{R}$.

Proof.

We will use the proof of Theorem 3, p. 25, in [I-N].

First, we will reconcile our set-up with that of [I-N]. The hypotheses on the coefficients of (I) imply that $H: M_2\to\mathbf{R}^d$, $G:\mathbf{R}^d\to L(\mathbf{R}^p,\mathbf{R}^d)$ are globally Lipschitz and H is globally bounded. Define the map $Q: C((-\infty,0],\mathbf{R}^d)\to M_2:=\mathbf{R}^d\times L^2([-r,0],\mathbf{R}^d)$ by

$$Q(f):=(f(0),f|[-r,0]),\quad f\in C((-\infty,0],\mathbf{R}^d).$$

It is easy to see that Q is continuous linear if $C((-\infty,0],\mathbf{R}^d)$ is furnished with the compact-open topology. Define the mappings $\tilde{H}: C((-\infty,0],\mathbf{R}^d)\to\mathbf{R}^d$, $\tilde{G}: C((-\infty,0],\mathbf{R}^d)\to L(\mathbf{R}^p,\mathbf{R}^d)$ by

$$\tilde{H}(f):=H(Q(f)),\quad \tilde{G}(f):=G(f(0))$$

for all $f\in C((-\infty,0],\mathbf{R}^d)$. Therefore, $\tilde{H},\tilde{G}$ are continuous on $C((-\infty,0],\mathbf{R}^d)$, and there are positive (deterministic) constants M_1, M_2 such that the following inequality holds:

$$|\tilde{H}(f)|^2+|\tilde{G}(f)|^2\leq M_1+M_2|f(0)|^2$$

for all $f\in C((-\infty,0],\mathbf{R}^d)$. This means that $\tilde{H},\tilde{G}$ satisfy Condition $(A.2'')$ (p. 25) of [I-N]. Now consider the unique solution $x^{(0,0)}:[-r,\infty)\times\Omega\to\mathbf{R}^d$ of the sfde

$$\left.\begin{aligned} dx^{(0,0)}(t)&=H(x^{(0,0)}(t),x_t^{(0,0)})\,dt+G(x^{(0,0)}(t))\,dW(t),\quad t>0,\\ x^{(0,0)}(t)&=0\quad -r\leq t\leq 0.\end{aligned}\right\}\qquad (I)$$

Define $\tilde{x}:\mathbf{R}\times\Omega\to\mathbf{R}^d$ by

$$\tilde{x}(t):=\begin{cases} x^{(0,0)}(t), & t>0,\\ 0 & t\leq 0\end{cases}$$

Clearly $\tilde{x}$ is $(\mathcal{F}_t)_{t\in\mathbf{R}}$-adapted and sample-continuous. Following [I-N], define $\pi_t : C(\mathbf{R}, \mathbf{R}^d) \to C((-\infty, 0], \mathbf{R}^d),\ t \in \mathbf{R}$, by

$$\pi_t(f)(s) := f(t+s), \quad s \le 0$$

for all $f \in C(\mathbf{R}, \mathbf{R}^d)$. Hence $\pi_t(\tilde{x})|[-r, 0] = x_t$ for any $t \in \mathbf{R}$. Furthermore, a straightforward computation shows that

$$\left.\begin{aligned} d\tilde{x}(t) &= \tilde{H}(\pi_t(\tilde{x}))\,dt + \tilde{G}(\pi_t(\tilde{x}))\,dW(t), \quad t > 0, \\ \tilde{x}(t) &= 0, \quad -\infty < t \le 0. \end{aligned}\right\} \qquad (\tilde{I})$$

Now by hypothesis, $\sup_{t\ge 0} E|x(t)|^2 < \infty$. Therefore, $\sup_{t\ge 0} E|\tilde{x}(t)|^2 < \infty$. Hence the sfde

$$d\tilde{x}(t) = \tilde{H}(\pi_t(\tilde{x}))\,dt + \tilde{G}(\pi_t(\tilde{x}))\,dW^\infty(t), \quad t > s > -\infty, \qquad (\tilde{I}^\infty)$$

admits a stationary solution $x^\infty : \mathbf{R} \times \tilde{\Omega} \to \mathbf{R}^d$ defined on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, P^\infty)$ (cf. [I-N], Theorem 3, p. 25). Now x^∞ is also a stationary solution of (I) when W is replaced by W^∞. To see this, note first that $\pi_t(x^\infty)|[-r, 0] = x_t^\infty$ for all $t \in \mathbf{R}$. Therefore,

$$\begin{aligned} dx^\infty(t) &= \tilde{H}(\pi_t(x^\infty))\,dt + \tilde{G}(\pi_t(x^\infty))\,dW^\infty(t), \\ &= H(Q(\pi_t(x^\infty)))\,dt + G(\pi_t(x^\infty)(0))\,dW^\infty(t), \\ &= H(x^\infty(t), x_t^\infty)\,dt + G(x^\infty(t))\,dW^\infty(t), \end{aligned}$$

for $t > s > -\infty$, P^∞-a.s.. Furthermore, $E\|(x^\infty(t), x_t^\infty)\|_{M_2}^2 < \infty$ for all $t \in \mathbf{R}$. This completes the proof of the theorem. $\square$

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