DUALITY FOR CLASSICAL GROUPS, A TALK BY CHRIS JANTZEN

NOTES (AND SOME ADDITIONAL EXPOSITION) BY JOSEPH HUNDLEY

Contents

1. Introduction

The topic of this talk is a certain type of duality which occurs in four distinct representation theoretic contexts. The four contexts are:

(1) representations of finite groups of Lie type
(2) representations of certain $\mathbb{C}$ algebras called **Hecke algebras** which are associated to finite groups of Lie type
(3) representations of p-adic groups
(4) representations of certain $\mathbb{C}$ algebras associated to p-adic groups, which are very similar to the Hecke algebras associated to finite groups of Lie type, and which are also called **Hecke algebras**.

2. Some basics

For purposes of these notes, a **linear algebraic group** is a matrix group defined by a finite set of polynomial equations. In more detail, take a commutative ring A with 1 and consider the polynomial ring $A[X_{ij}]$ in a matrix of indeterminates[1]. Take a finite set S of polynomials in $A[X_{ij}]$, having the property that

$$\{g \in \mathrm{Mat}_{n\times n}(R) \mid f(g) = 0 \forall g \in S\}$$

is a group under matrix multiplication for every commutative ring R containing A. This defines a functor

$$R \mapsto G(R) := \{g \in \mathrm{Mat}_{n\times n}(R) \mid f(g) = 0 \forall g \in S\}.$$

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Prof. Jantzen kindly corrected some errors in the first draft of these notes and added some details in the last section.

[1] i.e., in n^2 indeterminates which we interpret as the entries of an $n \times n$ matrix

from commutative rings containing A to groups. A **linear algebraic group** is a functor of this type.

Remark 2.0.1. The advantage of this definition is that it's easy to state and (I hope) easy to understand. There are several disadvantages:

- It's not standard. To be in line with common usage, I should require A to be a field.
- It's not well-suited to trying to define what it means for two linear algebraic groups to be isomorphic.
- In place of commutative rings containing A, I should really use A-algebras. An A-algebra is a commutative ring R with 1 which is equipped with a homomorphism $A \to R$. This homomorphism permits one to define a "scalar" multiplication of A on R. [2] For $f \in A[X_{ij}]$ and $g = (g_{ij}) \in \mathrm{Mat}_{n\times n}(R)$, define $f(g) \in R$ using scalar multiplication of A on R and the set $G(R)$ still makes sense.

We want to work with groups which are "reductive." I have never found the definition of reductive to be particularly instructive, so I will not include it here. The key fact is that reductive groups have a reasonable structure theory in terms of "root systems." Even this is more than we want to go into here. The standard references are three books titled Linear Algebraic Groups : one by Borel, one by Humphreys, and one by Springer. Humphreys is probably the easiest.

The key input from this classification theory is that it equips any reductive group with a certain family of reductive subgroups called standard Levi subgroups. One can combine this with the basic idea that if I have a group G and a subgroup H, one might try to study the representation theory of G by relating it to that of H.

Let's sketch this out in a bit more detail and then fill in complete details for two examples. Reductive groups are classified by looking at the action of their **maximal tori** on their **Lie algebras**. The Lie algebra is a direct sum of simultaneous eigenspaces for all the elements of the torus. For each eigenspace we have the corresponding eigenvalue, which is a function on the torus. The set of nontrivial eigenvalues is the set of **roots** of the group.

The roots live in a lattice[3] called the root lattice. One can choose a $\mathbb{Z}$-basis for this lattice consisting of roots, with the property that the expression for a root as a linear combination of basis elements never has both a positive and a negative coefficient. The elements of the $\mathbb{Z}$-basis are then called the **simple roots.** The set of simple roots is usually denoted Δ.

For each subset I of Δ there is a **standard parabolic subgroup** P_I. It is the semidirect product of it's **unipotent radical** U_I(a normal subgroup, all of whose elements are unipotent[4]) and its **standard Levi factor** M_I which is reductive.

Let's content ourselves with two examples: GL_n and Sp_{2n}. $GL_n(R)$ is the group of all invertible $n \times n$ matrices.[5] While

$$Sp_{2n}(R) = \{g \in GL_{2n}(R) : gJ'\,{}^{\top}g = J'\}, \qquad \text{where } J' = \begin{pmatrix} & J \\ -J & \end{pmatrix}, \qquad J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

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[2] i.e., a function $\cdot A \times R \to R$ such that $ab(r) = a(br)$, $(a+b)r = ar + br \forall a, b \in A, r, \in R$, $a(r+s) = ar + as \forall a \in A, r, s \in R$ and $1_A r = r\ \forall r \in R$. Given $\varphi : A \to R$ define $\cdot : A \times R \to R$ by $a \cdot r = \varphi(a)r$.

[3] by which I mean "group isomorphic to $\mathbb{Z}^r$ for some r"

[4] an $n \times n$ matrix X is unipotent if its characteristic polynomial is $(\lambda - 1)^n$; equivalently, if $(X - I_n)^n = 0$, where I_n is the identity matrix

[5] In order to define it using polynomial equations, I must identify $f \in GL_n(R)$ with the $n+1 \times n+1$ matrix $\left(\begin{smallmatrix} g & \\ & \det g^{-1} \end{smallmatrix}\right)$. The set of $n+1 \times n+1$ matrices of this form is certainly defined by a finite set of polynomial equations, and in obvious bijection with GL_n.

A reductive group has many maximal tori, but they are all conjugate, so one picks a standard one. For both GL_n and Sp_{2n} the standard maximal torus is the subgroup consisting of all diagonal elements.

The Lie algebra of GL_n is the space of all $n \times n$ matrices. The torus acts on it by conjugation. If E_{ij} is the matrix with a 1 at i, j and zeros elsewhere, then it spans a simultaneous eigenspace on which the torus element $\mathrm{diag}(t_1, \dots, t_n)$ acts by t_i/t_j. In order to describe the lattice these functions live in, it helps to use a funny notation. Let e_i be the function that sends $\mathrm{diag}(t_1, \dots, t_n)$ to t_i. Write t^{e_i} instead of $e_i(t)$ for the value of e_i at t. Then our lattice is simply the $\mathbb{Z}$-span of $\{e_i : 1 \le i \le n\}$ and the eigenvalue $\mathrm{diag}(t_1, \dots, t_n) \mapsto t_i/t_j$ corresponds to $e_i - e_j$. This is nontrivial if $i \neq j$. The usual basis of simple roots is $\Delta = \{e_i - e_{i+1} : 1 \le i < n\}$.

By choosing a subset I of the simple roots we group the integers $1, \dots, n$ into blocks: i and $i+1$ are in the same block if $e_i - e_{i+1}$ is in I. The parabolic subgroup P_I is then the group of $g \in GL_n$ which are block upper triangular with respect to that block structure. It's unipotent radical U_I is the subgroup consisting of elements with identity blocks on the diagonal, and its standard Levi M_I is the subgroup of block diagonal elements. So $M_I \cong GL_{n_1} \times \cdots \times GL_{n_k}$ for some $n_1, \dots, n_k$ which add up to n.

I hope the reader will find that I've included enough definitions to work out the set of roots for Sp_{2n}. I haven't said anything about the general procedure by which a subset of Δ should determine a parabolic subgroup. I will just say that the standard parabolic subgroups of Sp_{2n} also consist of block-upper-triangular matrices. The symmetry of Sp_{2n} induces some symmetries of the blocks. For example, for each integer k with $1 \le k < n$ we have a standard parabolic consisting of all matrices of the form

$$\begin{pmatrix} g & * & * \\ & h & * \\ & & {}_t g^{-1} \end{pmatrix}, \qquad g \in GL_k, h \in Sp_{2(n-k)}.$$

where ${}_t$ denotes the "transpose" over the "other diagonal." As before the unipotent radical is the normal subgroup consisting of elements with identity blocks on the diagonal and the standard Levi subgroup is the subgroup consisting of block diagonal elements. More generally, one has a standard parabolic with the Levi isomorphic to $GL_{n_1} \times \cdots \times GL_{n_r} \times Sp_{2m}$ whenever $n_1 + \cdots + n_r + m = n$. And this describes all standard parabolic subgroups of Sp_{2n}, provided we allow the degenerate case $m = 0$.

3. Finite groups of Lie Type

Let $\mathbb{F}_q$ be a finite field, and let G be a reductive linear algebraic group. Then the group $G(\mathbb{F}_q)$ is a finite group of Lie type. Such a group is called a **finite group of Lie type.**

4. Hecke Algebra of a Finite groups of Lie Type

Let G be a reductive linear algebraic group[6] and let B be the standard parabolic subgroup of G corresponding to the empty set. For GL_n this would be the group of all upper triangular invertible matrices. Then we define the **Hecke algebra** of $G(\mathbb{F}_q)$ by defining a product on the space

$$\mathcal{H}(G(\mathbb{F}_q)//B(\mathbb{F}_q)) : \{f : G(\mathbb{F}_q) \to \mathbb{C} \mid f(b_1 g b_2) = f(g), \qquad \forall g \in G(\mathbb{F}_q),\ b_1, b_2 \in B(\mathbb{F}_q)\}.$$

The product is given by convolution,

$$f_1 * f_2(g) = \sum_{h \in G(\mathbb{F}_q)} f_1(gh^{-1}) f_2(h).$$

A **representation** of $\mathcal{H}(G(\mathbb{F}_q)//B(\mathbb{F}_q))$ is a ring homomorphism into the algebra of endomorphisms of a vector space V.

[6]...for which we have chosen a maximal torus and also a basis of simple roots...

If M_I is a standard Levi subgroup of G, then $B \cap M_I$ is the analogue of B for the group M_I. Call it B_I. We can identify $\mathcal{H}(M_I(\mathbb{F}_q)//B_I(\mathbb{F}_q))$ with the subspace of $\mathcal{H}(G(\mathbb{F}_q)//B(\mathbb{F}_q))$ consisting of functions supported on $P_I(\mathbb{F}_q)$ (which turns out to be a subalgebra).

5. p-adic groups

Let p be a prime and let F be a p-adic field, i.e., a finite extension of $\mathbb{Q}_p$. Let G be a reductive linear algebraic group. The group $G(F)$ is called a ***p*-adic group**. A representation of $G(F)$ is said to be **smooth** if every vector in it is fixed by a compact open subgroup, and **admissible** if, in addition to this, the space of vectors fixed by any particular compact open subgroup is finite dimensional. When considering p-adic groups, the representations we consider are the admissible ones.

6. Hecke algebra of a p-adic group

Keep the notation of the last two sections. Recall that F has a nonarchimedean absolute value $|\ |$, that $\mathfrak{o} := \{x \in F : |x| \leq 1\}$ is a subring of F, and that $\mathfrak{p} := \{x \in F : |x| < 1\}$ is the unique maximal ideal in $\mathfrak{o}$. The field $\mathfrak{o}/\mathfrak{p}$ is finite and has characteristic p, so it is $\mathbb{F}_q$ where $q = p^k$ for some k. [7] The canonical homomorphism $\mathfrak{o} \to \mathfrak{o}/\mathfrak{p} = \mathbb{F}_q$ induces a homomorphism $G(\mathfrak{o}) \to G(\mathbb{F}_q)$. Let $\underline{B}$ denote the preimage of $B(\mathbb{F}_q)$. Then we define the **Hecke algebra** of $G(F)$ by defining a product on the space

$$\mathcal{H}(G(F)//\underline{B}) : \{f : G(F) \to \mathbb{C},\ \text{compactly supported} \mid f(b_1 g b_2) = f(g), \qquad \forall g \in G(F),\ b_1, b_2 \in \underline{B}\}.$$

The product is given by convolution,

$$f_1 * f_2(g) = \int_{G(F)} f_1(gh^{-1}) f_2(h)\, dh$$

The integral is with respect to the Haar measure of $G(F)$. A **representation** of $\mathcal{H}(G(F)//\underline{B})$ is a ring homomorphism into the algebra of endomorphisms of a vector space V. We can identity $\mathcal{H}(M_I(F)//\underline{B}_I)$ with the subalgebra of functions $f \in \mathcal{H}(G(F)//\underline{B})$ which are supported on $P_I(F)$.

7. Duality

In each of these various representation theoretic contexts (finite groups of Lie type, p-adic groups, and Hecke algebras of both types of groups), one has functors of induction and "Jacquet restriction." The induction functor i_{G,M_I} goes from the category of representations attached to M_I to the category of representations attached to G, while the Jacquet restriction functor goes the other way. (Hopefully it's clear what the "category of representations attached to G" is in each of the four contexts.) One might ask whether they are inverses of one another. The answer to that is no. What is true (at least for p-adic groups) is that for π a representation of $G(F)$ and σ a representation of $M_I(F)$, $\mathrm{Hom}_{M_I(F)}(r_{M_I,G}\pi, \sigma)$ is canonically isomorphic to $\mathrm{Hom}_G(\pi, i_{G,M_I}\sigma)$. This is sometimes described as a version of Frobenius reciprocity, or by saying that the functors $r_{M_I,G}$ and i_{G,M_I} are adjoint to one another.

In each the four contexts discussed above, there is a a duality defined by

$$\widehat{\pi}(or\ D\pi) = \sum_{I \subset \Delta} (-1)^{|I|} i_{G,M_I} \circ r_{M_I,G}(\pi),$$

where $\Delta = \{simple\ roots\}$, $i_{G,M_I} =$ induction, $r_{M_I,G} =$Jacquet restriction (semisimplified), and the sum is taken in the Grothendieck group.

[7]In case it's not clear, I'm not saying that any of this is obvious. References for this include most books titled Algebraic Number Theory, including the one by Lang and the one by Cassels and Fröhlich.

For finite groups of Lie type, the functor $r_{M_I,G}$ is defined as follows: $r_{M_I,G}(\pi)$ is the natural action of M_I on $V_\pi/\operatorname{Span}(\{\pi(u)v - v : u \in U_I, v \in V_\pi\})$.

Key points:

- if π is irreducible then so is $\widehat{\pi}$.
- if π is the trivial representation, then $\widehat{\pi}$ is the Steinberg representation.
- respects basic functors (induction, contragredients, Jacquet functors, etc.)
- useful for transferring/combining information

8. Some history

(1) Finite groups of Lie Type:
  - Alvis (1979), Curtis (1980), Kawanaka (1982)
  - Irr $\to$ irr Deligne-Lusztig (1982,1983)

(2) Hecke Algebras
  - Iwahori-Matsumoto (1965) define an algebra involution on an algebra of either of these types. This defines an involution on representations of the algebra.
  - Kato(1993) showed that this induced involution of Hecke algebra representations can be given by the same formula. With $r_{M_I,G}$ being (for the correct choice of Iwahori-Matsumoto involution) simply restriction.

(3) p-adic groups
  - for GL_n Zelevinsky (1980) defined what is now called the Zelevinsky involution. Essentially generated by generalized Steinberg $\leftrightarrow$ generalized trivial.
  - still for GL_n Moeglin-Waldspurger (1986), Knight-Zelevinsky (1996) showed how to calculate $\widehat{\pi}$ from π.
  - Aubert (1995/96), Schneider-Stuhler (1997) generalized to connected reductive p-adic groups, and showed that indeed irr $\to$ irr.

9. A bit more exposition

This section is added to flesh out the picture a bit. Everything in it should be true for p-adic groups, and (because the analogies appear to be quite strong) one would tend to think it likely to hold in the other cases as well.

If $I \subset J$ then $M_I \subset M_J$ and in fact M_I is a standard Levi subgroup of M_J. So, we can put M_J in the role of G and we have functors i_{M_J,M_I} and r_{M_I,M_J}. Moreover, the transitivity properties one would hope for hold: $i_{G,M_J} i_{M_J,M_I}\sigma = i_{G,M_I}\sigma$ and $r_{M_I,M_J} r_{M_J,G}\pi = r_{M_I,G}\pi$.

A representation π is **supercuspidal** if $r_{M_I,G}\pi = 0$ for all proper subsets I of Δ.

For $I \subset \Delta$ a proper subset of Δ, and σ an irreducible representation of $M_I(F)$, the representation $i_{G,M_I}\sigma$ is of finite length. So, in the Grothendieck group, it's a finite sum of irreducible representations. (Indeed, it's very often irreducible itself, but not always).

Speaking loosely, one can say that a representation π "can be obtained via induction from σ" if it is among the summands of $i_{G,M_I}\sigma$. The adjunction formula says that this occurs precisely when σ is a summand of $r_{M_I,G}\pi$. It's then clear from the definitions that a representation can be obtained via induction unless it is supercuspidal.

Further, you can always find I such that $r_{M_I,G}\pi$ is supercuspidal. Indeed, if $r_{M_I,G}\pi$ is not supercuspidal, then $r_{M_J,M_I} r_{M_I,G}\pi = r_{M_J,G}\pi$ is nonzero for some $J \subset I$.

So, the rough classification of representations says that each representation can be obtained via induction from a supercuspidal representation of M_I for some I. (If π is itself supercuspidal, take $I = \Delta$, so $M_I = G$. Induction from G to G is the identity functor.)

Next, one wants, given π to know about the set of pairs (I, σ) such that σ is supercuspidal and π is a summand of $i_{G,M_I}\sigma$. The answer is that the summands of $i_{G,M_I}\sigma$ and $i_{G,M_J}\tau$ are either disjoint or identical, and are the same if and only if M_I is conjugate to M_J, and the representation of M_I obtained by conjugating M_I to M_J and then using τ is equivalent to σ. So each π has a "conjugacy class" of (I, σ)'s. This is called the **supercuspidal support** of π.

As I mentioned above for most pairs (I, σ), the representation $i_{G,M_I}(\sigma)$ is irreducible, and when it's not it has a finite number of subquotients. (Or, in the Grothendieck group, a finite number of summands.)

The next problem is to determine precisely when it is reducible, how many summands it does, and how they may be described. This part is often done in stages.

(1) describe discrete series representations in terms of how to get them via induction from supercuspidal representations
(2) describe tempered representations in terms of how to get them via induction from discrete series representations
(3) describe arbitrary representations in terms of how to get them via induction from tempered representations.

For GL_n this program is fairly complete. So is the characterization of duality in terms of the descriptions obtained.

For classical (i.e., symplectic and special orthogonal groups) this program is nearing completion. One would like to characterize duality in terms of the descriptions obtained.

I think that the classification of representations of G_2 over a nonarchimedean local field (that is, a p-adic field) is fairly complete. Trying to work out what duality looks like in that case could be an interesting project.

Another interesting project is to see if one can use the classification of representations of classical groups can be modified to say anything about representations of non-classical groups which are isogenous to classical groups, such as GSpin groups. In this connection a very simple starting point is the description of reducibility of principal series and degenerate principal series for GSpin groups.

For exceptional groups larger than F_4, it's my impression there is still a fair amount of classification-theory to do.

10. Duality for Classical Groups

Question: Given an irreducible representation π, find $\widehat{\pi}$. Here, we assume $\pi, \widehat{\pi}$ to be given by their Langlands data.

Let's consider the example of symplectic groups. Levi isomorphic to

$$GL(n_1, F) \times \cdots \times GL(n_k, F) \times Sp(2n_0, F).$$

The Langlands classification is a classification of all representations in terms of tempered representations. For symplectic groups, it says that

$$\pi \hookrightarrow |\det|^{x_1}\tau_1 \times \cdots \times |\det|^{x_k}\tau_k \rtimes \tau_0,$$

with all τ_i tempered and $x_i < x_{i+1} < 0$ all i.

Definition 10.0.2. For $X = \{|\det|^{x_i}\rho\}$ with ρ fixed supercuspidal representation of $GL(m, F)$ and $-x_i \notin X \forall i$ and $\rho \cong \widetilde{\rho}$ (contragredient), let $M^*_X(\pi)$ consist of all $\tau \otimes \theta \leq r_{M,G}\pi$ such that τ has supercuspidal support in X and is of maximal rank with this property.

One can show that for an irreducible representation π, $M^*_X(\pi)$ consists of a single term $\tau \otimes \theta$, and that this characterizes π (i.e., $M^*_X(\pi') = \tau \otimes \theta \Rightarrow \pi' \cong \pi$). If one can calculate $M^*_X(\pi)$, one can determine the dual for π as follows:

$$\begin{array}{c} M^*_X(\pi) = \tau \otimes \theta \\ \Downarrow \text{ (properties of duality)} \\ M^*_{-X}(\hat{\pi}) = \tilde{\hat{\tau}} \otimes \hat{\theta}. \end{array}$$

One can calculate $\hat{\tau}$ by results for general linear groups. Operating inductively, we may assume $\hat{\theta}$ is known. Then, we may recover $\hat{\pi}$ from $M^*_{-X}(\hat{\pi})$ (as it determines $\hat{\pi}$). To apply this approach, it is enough to be able to calculate $M^*_X(\pi)$ for X with $|X| = 1$, which is what the speaker is currently working on.

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