AUTOMORPHIC REPRESENTATIONS OF GL_n AND THEIR L FUNCTIONS, A TALK BY SHUICHIRO TAKEDA

NOTES BY JOSEPH HUNDLEY

Contents

1. Disclaimer

Please attribute any errors to the note-taker. And if you have time, send an email about them.

2. The Riemann zeta function

2.1. **Definition as an infinite sum.** For $\mathrm{Re}(s) > 1$, we define a function $\zeta(s)$ by the following infinite sum, which is absolutely convergent for $\mathrm{Re}(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

2.2. **Meromorphic continuation.** The function $\zeta(s)$ admits a meromorphic continuation to all $\mathbb{C}$. That is, there is a meromorphic function $\mathbb{C} \to \mathbb{C}$ which is given by the above formula for $\mathrm{Re}(s) > 1$.

2.3. **Euler product.** The function $\zeta(s)$ also has an Euler product

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \prod_{p \text{ prime}} (1 + p^{-s} + p^{-2s} + p^{-3s} + \dots).$$

The equality between this and the original definiton of ζ amounts to existence and uniqueness of prime factorizations $n^{-s} = (p_1^{e_1} p_2^{e_2} \dots p_k^{e_k})^{-s}$.

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2.4. **Functional equation.** Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Here Γ is the Gamma function. For techinical reasons, $\pi^{-s/2}\Gamma(s/2)$ is called the "archimedean factor" and $\xi(s)$ is called the "complete zeta function." It has the very nice relation:

$$\boxed{\xi(s) = \xi(1-s)}$$

2.5. **Poles.** The function $\zeta(s)$ is analytic except for a simple pole at $s = 1$.

2.6. **Zeros.** Now, $\xi(s) \neq 0$ for $\mathrm{Re}(s) > 1$ (one checks using definition as an infinite sum), so $\xi(s) \neq 0$ for $\mathrm{Re}(s) < 0$ (by functional equation). Hence all zeros of $\xi(s)$ lie in the strip $0 \leq \mathrm{Re}(s) \leq 1$, which is known as the **critical strip.** The **Riemann hypothesis** states that $\xi(s) = 0 \implies \mathrm{Re}(s) = \frac{1}{2}$.

2.7. **Key points.**

(0) Absolute convergence for $\mathrm{Re}(s) \gg 0$,
(1) Meromorphic continuation
(2) Euler product
(3) Functional Equation
(4) Essentially bounded in vertical strips, i.e. $\xi(s)$ is bounded in the vertical strip $\sigma_1 \leq \mathrm{Re}(s) \leq \sigma_1$ for all real numbers σ_1, σ_2, except when $\sigma_1 \leq 1 \leq \sigma_2$, in which case is is bounded in the complement of a neighborhood of the pole.
(5) Location of poles.

3. Dirichlet L function

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^1 := \{z \in \mathbb{C} : |z| = 1\}$ be a group homomorphism. Extend χ to a function $\mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^1 \cup \{0\}$, by declaring that $\chi(n) = 0$ whenever $\gcd(n, N) \neq 1$. Pull it back to a function $\mathbb{Z} \to \mathbb{C}$. We denote this function $\mathbb{Z} \to \mathbb{C}$ by χ as well. Define

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Note that if you take $N = 1$, you will get the sum defining $\zeta(s)$. This function is called a Dirichlet L function. Like the Riemann zeta function, the sum definition is valid for $\mathrm{Re}(s) > 1$, but the function may be defined on the whole complex plane by meromorphic continuation, and has an Euler product

$$L(s, \chi) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{\chi(p)}{p^s}}$$

Let

$$\Lambda(s, \chi) = \pi^{-(s+\varepsilon)/2}\Gamma\left(\frac{s+\varepsilon}{2}\right) L(s, \chi)$$

where ε is the element of $\{0, 1\}$ such that $\chi(-1) = (-1)^\varepsilon$. Then $\Lambda(s, \chi) = (-i)^\varepsilon \tau(\chi) N^{-s} \Lambda(1-s, \overline{\chi})$, with $\tau(\chi) = \text{Gauss sum} = \sum_{n \bmod N} \chi(n) e^{2\pi i n/N}$.

In this case, there are no poles to the analytic continuation. This function is also bounded in vertical strips (and this time one does not have to insert a caveat about a pole).

Remark 3.0.1. Both $\zeta(s)$ and $L(s, \chi)$ are "degree 1" L-functions, in the sense that the denominator of the term corresponding to a prime p in the Euler product is a polynomial in p^{-s} which is of degree ≤ 1 are all primes and exactly 1 at all but finitely many primes.

4. Modular Forms

Let $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, and $SL(2,\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1\}$. For $z \in \mathcal{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

This defines an action of $SL(2,\mathbb{Z})$ on $\mathcal{H}$.

Definition 4.0.2. Take $k \in \mathbb{Z}$. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is called a **modular form of weight** k if it is bounded on $\{x + iy : |x| \le \frac{1}{2},\ y > N\}$ for some (and hence any) $N > 0$, and

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad \left(\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\right) \tag{4.0.3}$$

Remark 4.0.4. The boundedness condition is equivalent to a more natural condition introduced below.

Remark 4.0.5. One should take $k \ge 2$. For k odd one can prove fairly easily that a modular form of weight k is zero. (Plug the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ into (4.0.3).) For $k = 0$ one can prove with a bit of work that a modular form of weight k is constant. For $k < 0$ one can prove with a bit of work that a modular form of weight k is zero. The last two facts depend on the fact that a bounded entire function is constant.

Example 4.0.6 (Ramanujan)**.**

$$\begin{aligned} f(z) &= q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \qquad q = e^{2\pi i z},\ z \in \mathcal{H}. \\ &= \sum_{n=1}^{\infty} \tau(n) q^n, \end{aligned}$$

where $\tau(n)$ is the Ramanujan τ function. (One may take this equation as the definition of τ.) Then f is a modular form of weight 12.

4.1. **Fourier expansion.**

By taking $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in (4.0.3), we deduce that any modular form satisfies $f(z+1) = f(z)$. This gives rise to a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \qquad a_n \in \mathbb{C},\ q \in e^{2\pi i z}.$$

The sum starts at zero because of the boundedness condition. Note that the function $z \mapsto q$ is a bijection between a neighborhood of ∞ in $\mathcal{H}$ and a punctured disk centered at 0. This is the more natural interpretation of the growth condition: it says that f extends to a holomorphic function on the full (unpunctured) disk. I.e., f is “holomorphic at infinity.”

(0) The infinite sum

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large,

(1) It also has meromorphic continuation to $\mathbb{C}$,

(2) The space of modular forms is a vector space which has a nice basis consisting of Hecke eigenforms, and if we take f to be one of these basis vectors, then

$$L(s,f)=\prod_p \frac{1}{1-a_p p^{-s}+p^{k-1-2s}}$$

(3) As usual, in order to get a nice functional equation we need to put the right "archimedean factor" involving Γ. In this case it is

$$\Lambda(s,f)=(2\pi)^{-s}\Gamma(s)L(s,f) \implies \Lambda(s,f)=(-1)^{k/2}\Lambda(k-s,f).$$

(4) Bounded in vertical strips

(5) No poles, provided $a_0=0$.[1] If $a_0=0$, one says that f is **cuspidal**, or that it is **a cusp form**.

Problem 4.1.1. Given a sequence $(a_n)_{n=1}^{\infty}$, when is

$$f(z)=\sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

a modular form?

Theorem 4.1.2 (Hecke)**.** Define

$$L(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}.$$

Then $f(z)$ is a modular form if and only if $L(s)$ satisfies (0)-(4).

Theorem 4.1.3 (Shimura-Taniyama Conjecture, proved by Wiles[2], which implies Fermat's last theorem)**.** For each elliptic curve E over $\mathbb{Q}$ there is a modular form f such that the L function attached to f is equal to the Hasse-Weil L function attached to the elliptic curve. (Which we don't define.)

5. Automorphic representation of GL_n

The concept of an automorphic representation generalizes the concept of a modular form, and also that of a Dirichlet character.

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$. This is a ring which is defined as a restricted topological product of all the topologically distinct completions of $\mathbb{Q}$. One has the usual completion of $\mathbb{Q}$ as $\mathbb{R}$ and a completion $\mathbb{Q}_p$ corresponding to each prime p. Regarding restricted topological products, we content ourselves with two points:

(1) a restricted topological product of locally compact groups is locally compact (and each completion of $\mathbb{Q}$ is locally compact), and

(2) a restricted topological product is larger than the corresponding direct sum and smaller than the corresponding Cartesian product.

$$\mathbb{R}\oplus\bigoplus_p \mathbb{Q}_p \subset \mathbb{A} = \underbrace{\mathbb{R}\times{\prod_p}' \mathbb{Q}_p}_{\text{restricted topological product}} \subset \mathbb{R}\times\prod_p \mathbb{Q}_p$$

We have an embedding $\mathbb{Q}\hookrightarrow\mathbb{A}$ on the diagonal: $a\mapsto(a,a,a\dots,a,)$. We consider

$$GL_n(\mathbb{A})=\left\{A=\left(a_{ij}\right):\det A\in\mathbb{A}^{\times},\ a_{ij}\in\mathbb{A}\right\}.$$

[1] I'm not sure I took notes correctly on this point, but I'm sure this is true. If $a_0\neq 0$, I believe one will get at least one pole, but only finitely many. Possibly only one.

[2] and others, Wiles proved enough of it to deduce Fermat's last theorem

We have $GL_n(\mathbb{Q}) \hookrightarrow GL_n(\mathbb{A})$. The group $GL_n(\mathbb{A})$ is a locally compact group, and has a "nice" measure called the Haar measure. Using it, one can consider

$$\begin{aligned} L^2 &:= L^2\left(GL_n(\mathbb{Q})Z(\mathbb{A})\backslash GL_n(\mathbb{A})\right) \\ &= \left\{ f : GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}) \to \mathbb{C} : \int_{GL_n(\mathbb{Q})Z(\mathbb{A})\backslash GL_n(\mathbb{A})} |f(x)|^2\, dx < \infty \right\}. \end{aligned}$$

The group $GL_n(\mathbb{A})$ acts on L^2 by right translation, i.e.

$$g \cdot f(x) = f(xg) \qquad (x, g \in GL_n(\mathbb{A}), f \in L^2).$$

This gives a representation of $GL_n(\mathbb{A})$.

We study the decomposition of this representation. The theory of Eisenstein series gives a method of cooking up elements of $L^2(Z(\mathbb{A})GL_n(F)\backslash GL_n(\mathbb{A}))$ from elements of

$$\{L^2(Z(\mathbb{A})GL_m(F)\backslash GL_m(\mathbb{A})) : 1 \leq m < n\}.$$

As a representation of $GL_n(\mathbb{A})$, the space L^2 breaks up into three conceptually distinct pieces

$$L^2 = L^2_0 \oplus \underbrace{L^2_{\text{res}} \oplus L^2_{\text{cont}}}_{\text{constructed from } L^2(GL_m) \text{ for } m<n}.$$

Since we don't want to get into the theory of Eisenstein series, we shall not try to define L^2_{res} or L^2_{cont}. For our purposes, it suffices to say that they are cooked up from functions living on smaller groups. So L^2_0 can be thought of as the orthogonal complement of the subspace of L^2 generated by things which can be obtained from smaller building blocks. It is called the space of cusp forms, or the cuspidal spectrum, and it decomposes as a direct sum of irreducible subrepresentations.

Definition 5.0.4. Each constituent $\pi \subset L^2_0$ is called a **cuspidal automorphic representation of** $GL_n(\mathbb{A})$.

Theorem 5.0.5. To each cuspidal modular form f, one can canonically associate a cuspidal automorphic representation π_f of $GL_2(\mathbb{A})$.

Theorem 5.0.6 (Flath, tensor product theorem)**.** Given π an irreducible automorphic representation of $GL_n(\mathbb{A})$, we have a factorization of π

$$\pi \cong \pi_\infty \otimes \bigotimes_p{}' \pi_p,$$

as a certain type of infinite restricted tensor product, where π_∞ is a representation of $GL_n(\mathbb{R})$ and π_p is a representation of $GL_n(\mathbb{Q}_p)$ for each prime p. Moreover, for all but finitely many p, the representation π_p has a $GL_n(\mathbb{Z}_p)$-fixed vector.

Remark 5.0.7. This is by no means obvious, but it is "expected" given the product structure of $GL_n(\mathbb{A}) = GL_n(\mathbb{R}) \times \prod_p' GL_n(\mathbb{Q}_p)$. Like the precise definition of the restricted topological product, the precise nature of the restricted tensor product is something we skip over.

Remark 5.0.8. A representation of $GL_n(\mathbb{Q}_p)$ is said to be **unramified** or **spherical** if it has a $GL_n(\mathbb{Z}_p)$-fixed vector. Such a vector is known to be unique up to scalar if the representation is irreducible.

Theorem 5.0.9 (Satake)**.** There is a natural surjection from $\mathbb{C}^n$ to the set of isomorphism of classes of spherical representations of $GL_n(\mathbb{Q}_p)$, such that the fibers are orbits for the natural action of the symmetric group S_n on $\mathbb{C}^n$. Thus, a spherical representation of $GL_n(\mathbb{Q}_p)$ is determined by n complex numbers $\{a_{p,1}, a_{p,2}, \ldots, a_{p,n}\}$ called the **Satake parameters**.

Definition 5.0.10. Let π be a cuspidal automorphic representation of $GL_n(\mathbb{Q}_p)$ and let S be a finite set of primes containing all those where π_p is not spherical. Define

$$L^S(s,\pi) = \prod_{p\notin S}\prod_{i=1}^{n} \frac{1}{(1-a_{i,p}p^{-s})},$$

$$L^S(s,\pi,\mathrm{Sym}^2\otimes\chi) = \prod_{p\notin S}\prod_{1\le i\le j\le n} \frac{1}{1-a_{p,i}a_{p,j}\chi(p)p^{-s}}.$$

It is possible to define $L(s,\pi)$ and $L(s,\pi,\mathrm{Sym}^2\otimes\chi)$ by filling in the correct terms at the primes where π_p is not spherical. However, discussion of the techniques required to do that takes on a bit far afield.

Theorem 5.0.11 (Takeda)**.** The L function $L(s,\pi,\mathrm{Sym}^2\otimes\chi)$ has no pole for $\mathrm{Re}(s) > 1$.

6. References

This talk used some concepts such as "meromorphic," "pole," and "meromorphic continuation" from complex analysis. A good reference for complex analysis is Ahlfors book *Complex Analysis.* It also contains a definition of the gamma function if that was not familiar and a good deal of material on the Riemann zeta function.

For modular forms, there are a lot of approaches to the subject and a lot of books. From my own experience, I can recommend Iwaniec's *Topics in Classical Automorphic Forms,* as well as Bump's book (below). I suspect that Shimura's books *Modular Forms: Basics and Beyond* and *Arithmetic Theory of Automorphic Functions,* and Stein's *Modular forms, a computational approach* would be good. For a more analytic flavor one might look at Sarnak's book *Some applications of modular forms.* Iwaniec also has a second book. For a more combinatorial flavor, one might try Ken Ono's *The web of modularity.* or the book by Brunier.

For automorphic forms and representations of $GL_n(\mathbb{A})$ and associated L functions, the references of which I am aware are Bump's *Automorphic forms and representations*, Borel's *Introduction to automorphic forms,* Gelbart's *Automorphic forms on Adele groups,* and *Automorphic representations and L functions for the General Linear group, Volumes I and II* by Goldfeld and Hundley.

E-mail address: `jhundley@math.siu.edu`

Math. Department, Mailcode 4408, Southern Illinois University Carbondale, 1245 Lincoln Drive Carbondale, IL 62901