LATTICE MODEL OF THE WEIL REPRESENTATION AND THE HOWE DUALITY CONJECTURE, A TALK BY SHUICHIRO TAKEDA

NOTES BY JOSEPH HUNDLEY

Contents

1. Theta lifting

Let F be a non-arch local field of characteristic zero and residual characteristic p. (I.e., a finite extension of $\mathbb{Q}_p$).

Let V be an F vector space of dimension $2m$, and let $\langle,\rangle_1$ be a symmetric bilinear form $V \times V \to F$. Let W be an F vector space of dimension $2n$ and let $\langle,\rangle_2$ be a symplectic form on W.

We define $O(V)$ to be the

$$O(V) := \{g \in GL(V) : \langle gv, gv' \rangle_1 = \langle v, v' \rangle_1 \forall v, v' \in V\}$$

and

$$Sp(W) := \{g \in GL(W) : \langle gw, gw' \rangle_2 = \langle w, w' \rangle_2 \forall w, w' \in W\}.$$

Define $\mathbb{W} := V \otimes W$, and equip it with the symplectic form defined on pure tensors by

$$\langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle_1 \langle w, w' \rangle_2, \qquad (v, v' \in V,\ w, w' \in W).$$

Clearly $\dim \mathbb{W} = 4mn$. Moreover $O(V) \times Sp(W) \hookrightarrow Sp(\mathbb{W})$. Now, $Sp(\mathbb{W})$ has a nice representation which is called the Weil representation.

Locally, the theta correspondence is defined by looking at the restriction of the Weil representation to $O(V) \times Sp(W) \hookrightarrow Sp(\mathbb{W})$.

Theorem 1.0.1. There is a unique nontrivial[1] two-fold cover of $Sp(\mathbb{W})$ so that

$$0 \to \{\pm 1\} \to Mp(\mathbb{W}) \to Sp(\mathbb{W}) \to 0$$

and, for each additive character of F, a nice representation of $Mp(\mathbb{W})$ denoted ω_ψ called the Weil representation.

Question: What is so nice about this representation?
We'll get to that. But it will take a moment.

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[1]the trivial two-fold cover is $Sp(\mathbb{W}) \times \{\pm 1\}$; uniqueness is of course up to isomorphism

Theorem 1.0.2. The group $O(V) \times Sp(W)$ splits in $Mp(\mathbb{W})$, i.e., there is a group homomorphism $O(V) \times Sp(W) \hookrightarrow Mp(\mathbb{W})$ such that the composition with the canonical projection $Mp(\mathbb{W}) \to Sp(\mathbb{W})$ is the inclusion $O(V) \times Sp(W) \hookrightarrow Sp(\mathbb{W})$.

Therefore we may view ω_ψ as a representation of $O(V) \times Sp(W)$.

Definition 1.0.3. Let π be an irreducible admissible representation of $O(V)$. We define $\boldsymbol{\Theta_\psi(\pi)}$ the maximal π-isotypic quotient of ω_ψ, i.e., the largest representation $\Theta_\psi(\pi)$ such that a $O(V) \times Sp(W)$ intertwining map $\omega_\psi \to \pi \otimes \Theta_\psi(\pi)$ exists. So, its dual is $\mathrm{Hom}_{O(V)}(\omega_\psi, \pi)$.

Remark 1.0.4. It can happen that $\Theta_\psi(\pi) = 0$.

Conjecture 1.0.5 (Howe duality conjecture)**.** If $\Theta_\psi(\pi) \neq 0$ then $\Theta_\psi(\pi)$ has a unique irreducible quotient.

2. Previous Results

Theorem 2.0.6 (Howe, Waldspurger)**.** The Howe duality conjecture holds if the residual characteristic of F is odd.

Theorem 2.0.7 (Kudla)**.** If π is supercuspidal, then Howe duality holds.

Theorem 2.0.8. If $\dim V$ and $\dim W$ are very small $(2, 4$ maybe $6)$, then Howe duality holds.

3. New results

Definition 3.0.9. Let π_F be the uniformizer of F and $\mathfrak{o}_F$ the ring of integers of F. For each integer k let $I(k)$ be the kernel of the reduction map

$$O(V)(\mathfrak{o}_F) \to O(V)\left(\mathfrak{o}_F/\pi_F^{2k}\mathfrak{o}_F\right).$$

We say that π has level k if

- $\pi^{I(k)} \neq 0$
- $\pi^K = 0$ for any open compact subgroup which properly contains $I(k)$.

Remark 3.0.10. It is certainly possible for a representation to have no level. That is, there is no reason why the set of maximal elements in

$$\{K \text{ compact open} : \pi^K \neq 0\}^2$$

should contain one of the groups $I(k)$.

Theorem 3.0.11 (T)**.** Assume that $\dim V = \dim W$, that $O(V)$ is unramified, and that the residual characteristic is even. Then Howe duality holds as long as π has level $k \geq 1 + e$, where e is the ramification index of 2 in F.

[2]Here, π^K denotes the space of K-fixed vectors of the representation π. The representation π is said to be smooth if every vector is in π^K for some K. Thus, if π is smooth one knows that π^K is nonempty for some K, and even that the spaces π^K (as K varies) exhaust π. When studying representations of p-adic groups, it is very uncommon to consider representations which are not smooth.

4. Construction of the Weil representation ω_ψ.

4.1. **Heisenberg group.** Define $H(\mathbb{W})$ to be the group with $\mathbb{W} \times F$ as the underlying set and an operation defined by

$$(w_1, z_1) \cdot (w_2, z_2) = (w_1 + w_2, z_1 + z_2 + \frac{1}{2}\langle w_1, w_2\rangle).$$

Theorem 4.1.1 (Stone-Von Neumann)**.** *Up to isomorphism, $H(\mathbb{W})$ has a unique irreducible representation ρ_ψ with central character ψ for each nontrivial character ψ of F.*

Now, $Sp(\mathbb{W})$ has a natural action on $H(\mathbb{W})$, because the action of $Sp(\mathbb{W})$ on $\mathbb{W}$ preserves $\langle , \rangle$ and consequently gives a group automorphism of $H(\mathbb{W})$.

Let us now choose a particular realization[3] of the representation ρ_ψ on a vector space S.

For each $g \in Sp(\mathbb{W})$, we define $\rho_\psi^g : H(\mathbb{W}) \to GL(S)$ to be the action defined by

$$\rho_\psi^g(w, z) = \rho_\psi(g \cdot (w, z)).$$

One easily checks that ρ_ψ^g still has central character ψ. So $\rho_\pi^g \cong \rho_\psi$. That is, there exists an operator $M_g : S \to S$ such that

$$\rho_\psi(w, z)M_g = M_g \circ \rho_\psi^g(w, z). \qquad (*)$$

Of course M_g is not unique. It's only unique *up to scalar.*

Definition 4.1.2. $\widetilde{Sp}(\mathbb{W}) = \{(g, M_g) : g \in \widetilde{Sp}(\mathbb{W}),$ and M_g satisfies $(*)\}$

Then $\widetilde{Sp}(\mathbb{W})$ is a group, and has a projection to $Sp(\mathbb{W})$.

The group $Mp(\mathbb{W})$ can be defined as a subgroup of $\widetilde{Sp}(\mathbb{W})$ such that the following diagram commutes

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pm 1 & \longrightarrow & Mp(\mathbb{W}) & \longrightarrow & Sp(\mathbb{W}) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \widetilde{Sp}(\mathbb{W}) & \longrightarrow & Sp(\mathbb{W}) & \longrightarrow & 1
\end{array}$$

Remark 4.1.3. That is, $Mp(\mathbb{W})$ is a subgroup of $\widetilde{Sp}(\mathbb{W})$ such that the intersection with $\mathbb{C}^\times$ (viz., with the kernel of the projection to $Sp(\mathbb{W})$) is $\{\pm 1\}$. It is far from obvious that there should be a subgroup of $\widetilde{Sp}(\mathbb{W})$ with this property. But it is true.

The natural projection $(g, M_g) \mapsto M_g \in GL(S)$ is a representation of $\widetilde{Sp}(\mathbb{W})$ on the same space S used to realize ρ_ψ. We denote this representation ω_ψ.

Upshot: to have a nice realization of ω_ψ, start with a nice realization of ρ_ψ.

5. The lattice model of the Weil representation

Remark 5.0.4. (In this context "model" is roughly interchangeable with "realization." That is "the" Weil representation is really an isomorphism class and its "models" or "realizations" are the elements of the class.)

Let $A \subset \mathbb{W}$ be a lattice, i.e. a free $\mathfrak{o}_F$-module of rank $4mn$. I.e., the $\mathfrak{o}_F$ span of some basis of $\mathbb{W}$. Define $A^\perp = \{w \in \mathbb{W} : \langle a, w\rangle \in \mathfrak{o}_F, \forall a \in A\}$. We say that A is self dual if $A = A^\perp$.

Example: a basis $\{e_1, \ldots, e_{2nm}, f_1, \ldots, f_{2nm}\}$ with $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$ and $\langle e_i, f_j\rangle = \delta_{i,j}$ will give a self-dual lattice.

Note that if A is self dual then $\langle a, a'\rangle \in \mathfrak{o}_F \forall a, a' \in A$.

[3] we have introduced ρ_ψ as an "abstract" object which is unique up to isomorphism; properly speaking what we have is an isomorphism class, and we need now to choose a particular element of it.

Assume now that $\psi\big|_{\mathfrak{o}_F}$ is trivial, and that ψ is not trivial on $\pi_F^{-1}\mathfrak{o}_F$. Consider $A \times F \subset \mathbb{W} \times F$. Define $\psi_A : A \times F \to \mathbb{C}^\times$ by $\psi_A(a, z) = \psi(z)$.

Proposition 5.0.5. ψ_A is a character on $A \times F$ if the residual characteristic of F is odd.

Proof.

$$\psi_A((a_1, z_1)(a_2, z_2)) = \psi(z_1 + z_2 + \frac{1}{2}\langle a_1, a_2\rangle)$$
$$\psi_A(a_1, z_1)\psi_A(a_2, z_2) = \psi(z_1)\psi(z_2).$$

The two are equal provided $\frac{1}{2}$ is a unit in $\mathfrak{o}_F$, and this is the case precisely when the residual characteristic is odd. □

Theorem 5.0.6.

$$\mathrm{ind}_{A\times F}^{\mathbb{W}\times F}\,\psi_A \text{ is a model for } \rho_\psi.$$

This is called the lattice model for ρ_ψ.

Now, if F has even residual characteristic, one can define $\beta : \mathbb{W} \times \mathbb{W} \to F$ by

$$\beta(w_1, w_2) = \langle w_1^+, w_2^-\rangle,$$

where $\mathbb{W} = \mathbb{W}^+ \oplus \mathbb{W}^-$ is a polarization. One can then define $H_\beta(\mathbb{W})$, a modified version of the Heisenberg group with the group law being

$$(w_1, z_1)(w_2, z_2) = (w_1 + w_2, z_2 + \beta(w_1, w_2)).$$

However, the action of $Sp(\mathbb{W})$ on $\mathbb{W}$ does not preserve β, and as a result, does not give rise to an action of $Sp(\mathbb{W})$ on $H_\beta(\mathbb{W})$.

6. References

The paper of Weil that introduces the Weil representation is [W]. It's not very easy to read. The theory is also developed in Bump's book. [B] Another standard reference for the local theta correspondence is some notes of Kudla [K]. Dipendra Prasad also has some expository notes on the subject [P], in which he recommends [MVW] for the p-adic theory and [H1] and [H2] for the real case. His notes also point us to [Wa] for the completion of the proof of theorem 2.0.6 and to [K2] for that of theorem 2.0.7.

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