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Issa Amadou Tall

Southern Illinois University Carbondale, itall@math.siu.edu

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State Linearization of Control Systems: An Explicit Algorithm

Issa Amadou Tall

Abstract—In this paper we address the problem of linearization of nonlinear control systems using coordinate transformations. Although necessary and sufficient geometric conditions have been provided in the early eighties, the problem of finding the linearizing coordinates is subject to solving a system of partial differential equations and remained open 30 years later. We will provide here a complete solution to the problem by defining an algorithm allowing to compute explicitly the linearizing state coordinates for any nonlinear control system that is indeed linearizable. Each algorithm is performed using a maximum of $n - 1$ steps (n being the dimension of the system) and they are made possible by explicitly solving the Flow-box or straightening theorem. The problem of feedback linearization is addressed in a companion paper. A possible implementation via software like mathematica/matlab/maple using simple integrations, derivations of functions might be considered.

I. INTRODUCTION

IN the late seventies and early eighties the problem of transforming a nonlinear control system, via change of coordinates (and feedback), into a linear one has been introduced and has ever since been referred as linearization (feedback linearization). Of course linear systems constitute the first class of control systems to which feedback classification was applied and a complete picture made possible. The controllability, observability, reachability, and realization of linear systems have been expressed in very simple algebraic terms. Another crucial property of linear controllable systems is that they can be stabilized by linear feedback controllers. Because of the simplicity of their analysis and design; because several physical systems can be modeled using linear dynamics, and due to the observation that some nonlinear phenomena are just hidden linear systems, it is thus not surprising that the linearization problems were (and still are) of paramount importance and have attracted much attention. Uncovering the hidden linear properties of nonlinear control systems turns out to be useful in analyzing the latter systems. The downside of linearization is that some important properties of a nonlinear system, like global controllability, might be lost by the operation. To give a brief description of the linearization problems we will start first by recalling some basic facts about linear systems. We consider linear systems of the form

$$\Lambda : \begin{cases} \dot{x} = Fx + Gu = Fx + \sum_{i=1}^m G_i u_i, \\ y = Hx \end{cases}$$

Issa Amadou Tall is with Southern Illinois University Carbondale, MC 4408, 1245 Lincoln Drive, Carbondale IL, 62901, USA, itall@math.siu.edu.

where $x \in \mathbb{R}^n$, Fx and $G_1, \dots, G_m$ are, respectively, linear and constant vector fields on $\mathbb{R}^n$, Hx a linear vector field on $\mathbb{R}^p$, and $u = (u_1, \dots, u_m)^\top \in \mathbb{R}^m$. To any linear system Λ we can attach two geometric objects, namely, the controllability space

$$\mathcal{C}_n = \text{span} [G \ FG \ \dots \ F^{n-1}G]$$

and the observability space

$$\mathcal{O}_n = \text{span} [H^\top \ (HF)^\top \ \dots \ (HF^{n-1})^\top]^\top.$$

The system Λ is *controllable* (resp. *observable*) if and only if $\dim \mathcal{C}_n = n$ (resp. $\dim \mathcal{O}_n = n$). By a linear change of coordinates $\tilde{x} = Tx$, where T is an $n \times n$ invertible matrix, the system Λ is transformed into a linear equivalent one

$$\tilde{\Lambda} : \begin{cases} \dot{\tilde{x}} = \tilde{F}\tilde{x} + \tilde{G}v, \\ \tilde{y} = \tilde{H}\tilde{x} \end{cases}$$

with $\tilde{F} = TFT^{-1}$, $\tilde{G} = TG$ and $\tilde{H} = HT^{-1}$. The controllability space $\tilde{\mathcal{C}}_n$ and observability space $\tilde{\mathcal{O}}_n$ of the system $\tilde{\Lambda}$ are related to those of Λ by $\tilde{\mathcal{C}}_n = T\mathcal{C}_n$ and $\tilde{\mathcal{O}}_n = \mathcal{O}_n T^{-1}$. Thus the dimensions of $\mathcal{C}_n$ and $\mathcal{O}_n$, (and consequently the controllability and observability) are two invariants of the state classification of linear systems. The problem of classification of linear systems via state transformation is to find a linear change of coordinates $\tilde{x} = Tx$ that maps Λ into a simpler linear system $\tilde{\Lambda}$. In what follows we will address only the single input case except otherwise stated and we will ignore the output. The general case of multi-input systems will be addressed in another paper. It is a classical result of the linear control theory (see, e.g., [1], [6]) that any linear controllable system is state equivalent to the following form:

$$\Lambda_\lambda : \dot{w} = A_\lambda w + bu \triangleq \begin{cases} \dot{w}_1 = \lambda_1 w_1 + w_2 \\ \dot{w}_2 = \lambda_2 w_1 + w_3 \\ \dots \\ \dot{w}_{n-1} = \lambda_{n-1} w_1 + w_n \\ \dot{w}_n = \lambda_n w_1 + u, \end{cases} \quad (\text{I.1})$$

where

$$A_\lambda = \begin{pmatrix} \lambda_1 & 1 & 0 & \dots & 0 \\ \lambda_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{n-1} & 0 & 0 & \dots & 1 \\ \lambda_n & 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

If $\lambda_1 = \dots = \lambda_n = 0$, the matrix A_λ is simply denoted by A and the system Λ_λ is called Brunovsky canonical form Λ_{Br} .

A. Equivalence and State Linearization Problem.

Consider a C^∞ -smooth control-affine system of the form

$$\Sigma : \dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n$$

around the equilibrium that we assume throughout to be $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$, that is, $f(0) = 0$. Let

$$\tilde{\Sigma} : \dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})v, \quad \tilde{x} \in \mathbb{R}^n$$

be another C^∞ -smooth control-affine system. The systems Σ and $\tilde{\Sigma}$ are called *state equivalent* if there exist coordinates change $\tilde{x} = \phi(x)$ that maps Σ into $\tilde{\Sigma}$, that is, such that

$$\begin{cases} d\phi(x) \cdot f(x) &= \tilde{f}(\phi(x)) \\ d\phi(x) \cdot g(x) &= \tilde{g}(\phi(x)). \end{cases}$$

We will briefly write $\phi_*\Sigma = \tilde{\Sigma}$. In the late seventies Krener [7] described, in terms of their associated distributions, necessary and sufficient conditions for two systems Σ and $\tilde{\Sigma}$ to be state equivalent. He then derived as a corollary the answer to the following problem.

S-Linearization Problem. When does there exist a diffeomorphism $w = \phi(x)$ giving rise to new coordinates system $w = (w_1, \dots, w_n)^\top$ in which the transformed system $\phi_*\Sigma$ takes the linear form $\Lambda : \dot{w} = Fw + Gu$, $w \in \mathbb{R}^n$? He proved the following [7] (see also [3], [4], [5], and [11]).

Theorem I.1 A control-affine system $\Sigma : \dot{x} = f(x) + g(x)u$ is locally state equivalent to a linear controllable system $\Lambda : \dot{w} = Fw + Gu$ if and only if

(S1) $\dim \text{span} \{g(x), \text{ad}_f g(x), \dots, \text{ad}_f^{n-1} g(x)\} = n$;

(S2) $[\text{ad}_f^q g, \text{ad}_f^r g] = 0$, $0 \leq q < r \leq n$.

Above the Lie brackets are defined recursively as

$$\text{ad}_f^0 g = g, \text{ad}_f g = [f, g], \dots, \text{ad}_f^l g = [f, \text{ad}_f^{l-1} g], \quad l \geq 2.$$

Notice that if $w = \phi(x)$ linearizes the system Σ , then the following systems of partial differential equations hold

$$\begin{cases} d\phi(x) \cdot f(x) &= F\phi(x) \\ d\phi(x) \cdot g(x) &= G. \end{cases}$$

Although the conditions (S1) and (S2) stated in Theorem I.1 did provide a way of testing the state linearizability of a system, they offer little on how to find the linearizing change of coordinates $\phi(x)$ except for solving the systems of partial differential equations (PDEs) which is, in general, not straightforward. Remark that, even in the single-input case, the solvability of the PDEs is equivalent of finding a function h with $h(0) = 0$ such that

$$L_g h = 0, L_g L_f h = 0, \dots, L_g L_f^{n-2} h = 0, L_g L_f^{n-1} h \neq 0,$$

where for any vector field ν and any function h , we have denoted $L_\nu h = \frac{\partial h}{\partial x} \nu(x)$, and iteratively, $L_\nu^l h = L_\nu(L_\nu^{l-1} h)$. We propose here to give a complete solution without solving the partial differential equations. We will provide an algorithm giving explicit solutions. Recall that we have previously obtained explicit solutions for few subclasses

of control-affine systems, namely *strict feedforward forms*, *strict-feedforward nice* and *feedforward forms*, for which linearizing coordinates were found without solving the corresponding PDEs (see [12], [15]). Indeed, for those subclasses we exhibited algorithms that can be performed using a maximum of $\frac{n(n+1)}{2}$ steps each involving composition and integration of functions only (but not solving PDEs) followed by a sequence of $n+1$ derivations in the case of feedback linearizability. What played a main role in finding those algorithms were the strict feedforward form *structure*, that is, the fact that each component of the system depended only on higher variables. In this paper we consider general control-affine systems for which we provide a linearizing algorithm (see [17] for feedback linearizing algorithm) that can be implemented using a maximum of n steps. These algorithms are based on the explicit solving of the flow-box theorem [16] and are completely different from those outlined in [12], [15] (see also [9], [10]).

B. Notations and Definitions

For simplicity of exposition we consider single-input control systems

$$\Sigma : \dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}.$$

The case of multi-input systems is more involved and will be addressed somewhere else. Let $0 \leq k \leq n-1$ be an integer. We say that Σ is quasi k -linear if

$$g(x) = b, \text{ad}_f g(x) = Ab, \dots, \text{ad}_f^{n-k-1} g(x) = A^{n-k-1}b,$$

where (A, b) is the Brunovský canonical pair. To fix the notation we will denote hereafter the coordinates in which the system is quasi k -linear by the bolded variables $\mathbf{x}_k = (\mathbf{x}_{k1}, \dots, \mathbf{x}_{kn})^\top$ and the system by Σ_k , where $k = \mathbf{k}$. It follows easily that a quasi k -linear system takes the form

$$\Sigma_k : \begin{cases} \dot{\mathbf{x}}_{kj} = F_{kj}(\mathbf{x}_{k1}, \dots, \mathbf{x}_{k,k+1}) \\ \text{if } 1 \leq j \leq k \\ \dot{\mathbf{x}}_{kj} = F_{kj}(\mathbf{x}_{k1}, \dots, \mathbf{x}_{k,k+1}) + \mathbf{x}_{k,j+1} \\ \text{if } k+1 \leq j \leq n-1 \\ \dot{\mathbf{x}}_{kn} = F_{kn}(\mathbf{x}_{k1}, \dots, \mathbf{x}_{k,k+1}) + u. \end{cases}$$

A more compact representation of Σ_k is obtained as

$$\Sigma_k : \dot{\mathbf{x}}_k = F_k(\mathbf{x}_{k1}, \dots, \mathbf{x}_{k,k+1}) + \mathcal{A}_k \mathbf{x}_k + bu, \quad \mathbf{x}_k \in \mathbb{R}^n,$$

where $\mathcal{A}_k \mathbf{x}_k = (0, \dots, 0, \mathbf{x}_{k,k+2}, \mathbf{x}_{k,k+3}, \dots, \mathbf{x}_{kn}, 0)^\top$ is a vector whose last and first k components are zero. If the vector field F_k is affine in the variable $\mathbf{x}_{k,k+1}$, that is, decomposes as

$$F_k(\cdot) = f_k(\mathbf{x}_{k1}, \dots, \mathbf{x}_{kk}) + \mathbf{x}_{k,k+1} g_k(\mathbf{x}_{k1}, \dots, \mathbf{x}_{kk})$$

we then simply write $\Sigma_k = \Sigma_k^{\text{aff}}$.

II. MAIN RESULTS

The first result is as follows and states that any $\mathcal{S}$ -linearizable system can be transformed into a linear form via a sequence of explicit coordinates changes each giving rise to an quasi $\mathbf{k}$ -linear system.

Theorem II.1 *Consider a controllable system*

$$\Sigma : \dot{x} = f(x) + g(x)u, x \in \mathbb{R}^n, u \in \mathbb{R}.$$

Assume it is $\mathcal{S}$ -linearizable (denote $\Sigma \triangleq \Sigma_n^{\text{aff}}$ and $x \triangleq \mathbf{x}_n$).

There exists a sequence of explicit coordinates changes $\phi_n(\mathbf{x}_n), \phi_{n-1}(\mathbf{x}_{n-1}), \dots, \phi_1(\mathbf{x}_1)$ that gives rise to a sequence of quasi $\mathbf{k}$ -linear systems $\Sigma_n^{\text{aff}}, \Sigma_{n-1}^{\text{aff}}, \dots, \Sigma_0^{\text{aff}}$ such that $\Sigma_{\mathbf{k}-1}^{\text{aff}} = \phi_{\mathbf{k}*}(\Sigma_{\mathbf{k}}^{\text{aff}})$ for any $1 \leq \mathbf{k} \leq n$.

Moreover, in the coordinates $w \triangleq \phi_1(\mathbf{x}_1)$ the system Σ (actually Σ_0^{aff}) takes the simpler linear form Λ_λ where $\lambda_1, \dots, \lambda_n$ are constant real numbers.

The proof of this theorem relies mainly on the flow-box theorem for which we gave recently explicit solution [16] (see below) and on Theorem I.1 (S2).

Theorem II.2 *Let ν be a smooth vector field on $\mathbb{R}^n$, any integer $1 \leq k \leq n$ such that $\nu_k(0) \neq 0$ and $\sigma_k(x) = 1/\nu_k(x)$. (i) Define $z = \phi(x)$ by its components as following*

$$\begin{aligned} \phi_j(x) &= x_j + \sum_{s=1}^{\infty} \frac{(-1)^s x_k^s}{s!} L_{\sigma_k \nu}^{s-1}(\sigma_k \nu_j)(x) \\ \phi_k(x) &= \sum_{s=1}^{\infty} \frac{(-1)^{s+1} x_k^s}{s!} L_{\sigma_k \nu}^{s-1}(\sigma_k)(x) \end{aligned} \quad (\text{II.1})$$

for any $1 \leq j \leq n, j \neq k$. The diffeomorphism $z = \phi(x)$ satisfies $\phi_*(\nu) = \partial_{z_k}$.

(ii) The diffeomorphism $x = \psi(z)$ given by its components

$$\begin{aligned} \psi_j(z) &= z_j + \sum_{s=1}^{\infty} \frac{z_k^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_k}^i \cdot L_{\nu}^{s-i-1}(\nu_j)(z) \right) \\ \psi_k(z) &= \sum_{s=1}^{\infty} \frac{z_k^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_k}^i \cdot L_{\nu}^{s-i-1}(\nu_k)(z) \right) \end{aligned} \quad (\text{II.2})$$

for any $1 \leq j \leq n, j \neq k$, is the inverse of $z = \phi(x)$, that is, such that $\frac{\partial \psi(z)}{\partial z_k} = \nu(\psi(z))$.

Above, we have adopted the following notation

$$\partial_{z_k} = \frac{\partial}{\partial z_k}, \partial_{z_k} \cdot h = \frac{\partial h}{\partial z_k}, \dots, \partial_{z_k}^i \cdot h = \frac{\partial^i h}{\partial z_k^i}, i \geq 2.$$

A sketch of the proof will be given in Section V along with few examples. For further details we refer to [16]. The following remarks that are of paramount importance here.

R1. The expressions above are not series around the origin or in the variable x_k as the coefficients $L_{\sigma_k \nu}^s(\sigma_k \nu_j)(x)$ are evaluated at $x = (x_1, \dots, x_n)$ and might well depend on x_k .

R2. If the vector field ν is independent of some variable x_l ($l \neq k$), then the diffeomorphism $\phi(x)$ is also independent of the variable x_l (except a linear dependence).

R3. If any of the components of $\nu(x)$ is zero, say $\nu_j(x) = 0$, then $\phi_j(x) = x_j$.

III. LINEARIZING COORDINATES

In this section we define an algorithm that shows how to compute the linearizing coordinates for the system. The algorithm stands also as a proof of Theorem II.1.

A. ($\mathcal{S}\mathcal{L}$)-Algorithm. Consider a $\mathcal{S}$ -linearizable system

$$\Sigma : \dot{x} = f(x) + g(x)u, x \in \mathbb{R}^n, u \in \mathbb{R}.$$

Without loss of generality we can assume that $g(0) = b = (0, \dots, 0, 1)^\top$. This algorithm consists of $n - 1$ steps.

Step 1. Set $\Sigma \triangleq \Sigma_n$ and $x \triangleq \mathbf{x}_n = (\mathbf{x}_{n-1}, \dots, \mathbf{x}_n)^\top$. Apply Theorem II.2 with $\nu = g(x)$ to construct a change of coordinates $z = \phi(x)$ given by (II.1), such that $\phi_*(g)(z) = \partial_{z_n}$. Such change of coordinates transforms Σ into

$$\tilde{\Sigma} : \dot{z} = \tilde{f}(z) + \tilde{g}(z)u = (\phi_* f)(z) + (\phi_* g)(z)u, z \in \mathbb{R}^n,$$

where $\tilde{g} = b$. Since Σ (hence $\tilde{\Sigma}$) is $\mathcal{S}$ -linearizable, then Theorem I.1 (S2) is satisfied, which is equivalent to

$$[ad_{\tilde{f}}^q \tilde{g}, ad_{\tilde{f}}^r \tilde{g}] = 0, \quad 0 \leq q, r \leq n - 1. \quad (\text{III.1})$$

Taking $q = 0$ and $r = 1$ we get in particular $[\tilde{g}, ad_{\tilde{f}} \tilde{g}] = 0$ or equivalently (because $\tilde{g} = \partial_{z_n}$)

$$(\mathcal{S}\mathcal{L}_n) \implies \frac{\partial^2 \tilde{f}}{\partial z_n^2} = 0.$$

It follows that $\tilde{f}$ is affine with respect to the variable z_n . Denote $\mathbf{x}_{n-1} \triangleq z$ and $\phi_n \triangleq \phi$ it follows that the change of coordinates $\mathbf{x}_{n-1} = \phi_n(\mathbf{x}_n)$ transforms Σ_n into

$$\Sigma_{n-1} : \dot{\mathbf{x}}_{n-1} = F_{n-1}(\mathbf{x}_{n-1}) + \mathcal{A}_{n-1} \mathbf{x}_{n-1} + bu, \mathbf{x}_{n-1} \in \mathbb{R}^n,$$

where $\mathcal{A}_{n-1} \equiv 0$ and $F_{n-1}(\mathbf{x}_{n-1}) = \tilde{f}(\mathbf{x}_{n-1}) = \phi_{n*}(f)$. Moreover, the vector field $F_{n-1}(\mathbf{x}_{n-1})$ is affine with respect to the variable $\mathbf{x}_{n-1n} \triangleq z_n$, that is, decomposes uniquely as

$$\begin{aligned} F_{n-1}(\mathbf{x}_{n-1}) &= f_{n-1}(\mathbf{x}_{n-11}, \dots, \mathbf{x}_{n-1n-1}) \\ &+ \mathbf{x}_{n-1n} g_{n-1}(\mathbf{x}_{n-11}, \dots, \mathbf{x}_{n-1n-1}). \end{aligned}$$

We deduce that $\Sigma_{n-1} = \Sigma_{n-1}^{\text{aff}}$ and $\Sigma_{n-1}^{\text{aff}} = (\phi_n)_* \Sigma_n^{\text{aff}}$.

Step $n - k$. Assume that Σ_n^{aff} has been taken, via a composition of diffeomorphisms $\mathbf{x}_k = \phi_{k+1} \circ \dots \circ \phi_n(x)$, into the following system

$$\Sigma_k : \dot{\mathbf{x}}_k = F_k(\mathbf{x}_k) + \mathcal{A}_k \mathbf{x}_k + bu, \mathbf{x}_k \in \mathbb{R}^n,$$

where $\mathcal{A}_k \mathbf{x}_k = (0, \dots, 0, \mathbf{x}_{kk+2}, \mathbf{x}_{kk+3}, \dots, \mathbf{x}_{kn}, 0)^\top$, and that the vector field $F_k(\mathbf{x}_k)$ is affine with respect to the variable $\mathbf{x}_{kk+1}$, that is, it decomposes uniquely as

$$F_k(\mathbf{x}_k) = f_k(\mathbf{x}_{k1}, \dots, \mathbf{x}_{kk}) + \mathbf{x}_{kk+1} g_k(\mathbf{x}_{k1}, \dots, \mathbf{x}_{kk}).$$

Once again reset the variable $x \triangleq \mathbf{x}_k$ and denote Σ_k simply by $\Sigma : \dot{x} = f(x) + g(x)u$ with $g(x) = b$ and

$$\begin{aligned} f(x) &= F_k(\mathbf{x}_k) + \mathcal{A}_k \mathbf{x}_k \\ &= f_k(x_1, \dots, x_k) + x_{k+1} g_k(x_1, \dots, x_k) + \mathcal{A}_k x. \end{aligned}$$

Notice that in these coordinates

$$g = \partial_{x_n}, ad_f g = -\partial_{x_{n-1}}, \dots, ad_f^{n-k-1} g = (-1)^{n-k-1} \partial_{x_{k+1}}$$

which implies that $ad_{\tilde{f}}^{m-k}g = (-1)^{n-k}g_{\mathbf{k}}(x_1, \dots, x_k)$. We deduce from Theorem I.1 (S1) that the vector field

$$\nu(x) = g_{\mathbf{k}}(x) = (g_{\mathbf{k}1}(x_1, \dots, x_k), \dots, g_{\mathbf{k}n}(x_1, \dots, x_k))^{\top}$$

is nonsingular in $\mathbb{R}^n$ and depends exclusively on the variables $x_1, \dots, x_k$. By Theorem II.2 we can construct a change of coordinates $z = \phi(x)$ such that $\phi_*(\nu)(z) = \partial_{z_k}$. Moreover the components of ϕ are such that

$$\phi_j(x) = x_j + \varphi_j(x_1, \dots, x_k), \quad 1 \leq j \leq n. \quad (\text{III.2})$$

This change of coordinates transforms Σ into

$$\tilde{\Sigma} : \dot{z} = \tilde{f}(z) + \tilde{g}(z)u = (\phi_*f)(z) + (\phi_*g)(z)u$$

where $\tilde{g}(z) = (\phi_*g)(z) = (0, \dots, 0, 1)^{\top}$ and

$$\tilde{f}(z) = (\phi_*f_{\mathbf{k}})(z) + [z_{k+1} - \varphi_{k+1}(\phi^{-1}(z))](\phi_*g_{\mathbf{k}})(z) + (\phi_*(\mathcal{A}_{\mathbf{k}}x))(z).$$

Because the k first components of $\mathcal{A}_{\mathbf{k}}x$ are zero, then (III.2) implies $(\phi_*(\mathcal{A}_{\mathbf{k}}x))(z) = \mathcal{A}_{\mathbf{k}}z$. We then deduce that

$$\tilde{f}(z) = F_{\mathbf{k}-1}(z) + \mathcal{A}_{\mathbf{k}-1}z$$

where $F_{\mathbf{k}-1}(z) = (\phi_*f_{\mathbf{k}})(z) - \varphi_{k+1}(\phi^{-1}(z))\partial_{z_k}$ depends exclusively on the variables $z_1, \dots, z_k$ and

$$\mathcal{A}_{\mathbf{k}-1}z = z_{k+1}\partial_{z_k} + \mathcal{A}_{\mathbf{k}}z = (0, \dots, 0, z_{k+1}, z_{k+2}, \dots, z_n, 0)^{\top}$$

is such that the $(k-1)$ first components are zero. We can easily check that

$$\tilde{g} = \partial_{z_n}, ad_{\tilde{f}}\tilde{g} = -\partial_{z_{n-1}}, \dots, ad_{\tilde{f}}^{m-k}\tilde{g} = (-1)^{n-k}\partial_{z_k}$$

which implies that $ad_{\tilde{f}}^{m-k+1}\tilde{g} = (-1)^{n-k+1}\partial_{z_k} \cdot F_{\mathbf{k}-1}(z)$. Theorem II.2 (S2) for $r = n - k$ and $q = r + 1$ yields

$$[ad_{\tilde{f}}^{m-k+1}\tilde{g}, ad_{\tilde{f}}^{n-k}\tilde{g}] = \frac{\partial^2 F_{\mathbf{k}-1}(z)}{\partial z_k^2} = 0.$$

Hence the vector field $F_{\mathbf{k}-1}(z)$ is affine with respect to the variable z_k , that is, decomposes uniquely as

$$F_{\mathbf{k}-1}(z) = f_{\mathbf{k}-1}(z_1, \dots, z_{k-1}) + z_k g_{\mathbf{k}-1}(z_1, \dots, z_{k-1}). \quad (\text{III.3})$$

We denote $\mathbf{x}_{\mathbf{k}-1} \triangleq z$ and $\phi_{\mathbf{k}} \triangleq \phi$. Thus the change of coordinates $\mathbf{x}_{\mathbf{k}-1} = \phi_{\mathbf{k}}(\mathbf{x}_{\mathbf{k}})$ brings the system $\Sigma_{\mathbf{k}}^{\text{aff}}$ into

$$\Sigma_{\mathbf{k}-1}^{\text{aff}} : \dot{\mathbf{x}}_{\mathbf{k}-1} = F_{\mathbf{k}-1}(\mathbf{x}_{\mathbf{k}-1}) + \mathcal{A}_{\mathbf{k}-1}\mathbf{x}_{\mathbf{k}-1} + bu,$$

where $\mathcal{A}_{\mathbf{k}-1}\mathbf{x}_{\mathbf{k}-1} = (0, \dots, 0, \mathbf{x}_{\mathbf{k}-1k+1}, \dots, \mathbf{x}_{\mathbf{k}-1n}, 0)^{\top}$ and

$$F_{\mathbf{k}-1}(\mathbf{x}_{\mathbf{k}-1}) = f_{\mathbf{k}-1}(\mathbf{x}_{\mathbf{k}-11}, \dots, \mathbf{x}_{\mathbf{k}-1k-1}) + \mathbf{x}_{\mathbf{k}-1k} g_{\mathbf{k}-1}(\mathbf{x}_{\mathbf{k}-11}, \dots, \mathbf{x}_{\mathbf{k}-1k-1}).$$

Notice that when $k = 1$, condition (III.3) reduces simply to

$$F_0(z) = z_1 \lambda, \quad \text{where } \lambda = (\lambda_1, \dots, \lambda_n)^{\top}.$$

This ends the general step and shows that a sequence of explicit coordinates changes $\phi_n(\mathbf{x}_n), \dots, \phi_1(\mathbf{x}_1)$ can be constructed whose composition $z = \phi_1 \circ \dots \circ \phi_n(\mathbf{x}_n)$ takes the original system Σ into the linear form Λ_{λ} of (I.1).

B. Summary of Algorithm. Start with a system

$$\Sigma : \dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}.$$

Step 0. Normalize the vector field $g \mapsto g = (0, \dots, 0, 1)^{\top}$. Apply a linear change of coordinates to transform the linearization such that $\frac{\partial f}{\partial x}(0) = A_{\lambda}$.

Step $n - k$. If the condition

$$(\mathcal{S}\mathcal{L}_{k+1}) \implies \frac{\partial^2 f}{\partial x_{k+1}^2} = 0$$

fails, the algorithm stops: The system is not $\mathcal{S}$ -linearizable.

If $(\mathcal{S}\mathcal{L}_{k+1})$ holds, then decompose the vector field f as

$$f(x_1, \dots, x_{k+1}) = F(x_1, \dots, x_k) + x_{k+1}\nu(x_1, \dots, x_k).$$

Apply Theorem II.2 to construct a change of coordinates $z = \phi(x) \in \mathbb{R}^n$ that rectifies the nonsingular vector field

$$\nu(x) = \nu_1(x)\partial_{x_1} + \dots + \nu_n(x)\partial_{x_n},$$

that is, such that $\phi_*(\nu)(z) = \partial_{z_k}$. Find the transform $\phi_*\Sigma$ of the system in precedent step. For $k = n - 1, n - 2, \dots, 2$ repeat **Step $n - k$** . End if system is linear or algorithm fails.

IV. EXAMPLES

In what follows we illustrate with few examples.

Example IV.1 Consider a single-input control system

$$\Sigma : \dot{x} = f(x) + g(x)u \triangleq \begin{cases} \dot{x}_1 = x_2 - 2x_2x_3 + x_3^2 + 4x_2x_3u \\ \dot{x}_2 = x_3 - 2x_3u \\ \dot{x}_3 = u \end{cases}$$

with

$$f(x) = (x_2 - 2x_2x_3 + x_3^2, x_3, 0)^{\top}, \quad g(x) = (4x_2x_3, -2x_3, 1)^{\top}.$$

We first rectify the vector field $g(x)$. Denote $\nu(x) = g(x)$ and apply Theorem II.2 with $n = 3$ and $\sigma_3(x) = 1$. Since

$$L_{\nu}(\nu_1) = -8x_3^2 + 4x_2, L_{\nu}^2(\nu_1) = -24x_3, L_{\nu}^3(\nu_1) = -24,$$

we have $L_{\nu}^{s-1}(\nu_1) = 0$ for all $s \geq 5$ and hence

$$\begin{aligned} z_1 = \phi_1(x) &= x_1 + \sum_{s=1}^{\infty} (-1)^s \frac{x_3^s}{s!} (L_{\nu}^{s-1}\nu_1)(x), \\ &= x_1 - 4x_2x_3^2 - 4x_3^4 + 2x_2x_3^2 + 4x_3^4 - x_3^4 \\ &= x_1 - 2x_2x_3^2 - x_3^4. \end{aligned}$$

Likewise, $L_{\nu}(\nu_2) = -2$ and $L_{\nu}^{s-1}(\nu_2) = 0$, $s \geq 3$, yielding

$$\begin{aligned} z_2 = \phi_2(x) &= x_2 + \sum_{s=1}^{\infty} (-1)^s \frac{x_3^s}{s!} (L_{\nu}^{s-1}\nu_2)(x) \\ &= x_2 - x_3(-2x_3) + (1/2!)x_3^2(-2) = x_2 + x_3^2. \end{aligned}$$

We apply the change of coordinates

$$z_1 = x_1 - 2x_2x_3^2 - x_3^4, \quad z_2 = x_2 + x_3^2, \quad z_3 = x_3$$

to transform the original system into

$$\tilde{\Sigma} : \dot{z} = \tilde{f}(z) + \tilde{g}(z)u \triangleq \begin{cases} \dot{z}_1 = z_2 - 2z_2z_3 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = u, \end{cases}$$

where $\tilde{g}(z) = (0, 0, 1)^{\top}$ and $\tilde{f}(z) = (z_2 - 2z_2z_3, z_3, 0)^{\top}$. The vector field $\tilde{f}(z) = (z_2 - 2z_2z_3, z_3, 0)^{\top}$ decomposes

$$\tilde{f}(z) = (z_2, 0, 0)^{\top} + z_3(-2z_2, 1, 0)^{\top}.$$

The next step is to rectify $\nu(x) = (-2z_2, 1, 0)^\top$. Theorem II.2 with $k = 2$ and $\sigma_2(z) = 1$ yields

$$\begin{aligned} w_1 &= z_1 + \sum_{s=1}^{\infty} (-1)^s \frac{z_2^s}{s!} L_\nu^{s-1}(\nu_1)(z) \\ &= z_1 - z_2(-2z_2) + (1/2!)z_2^2(-2) = z_1 + z_2^2 \\ w_2 &= z_2 \\ w_3 &= z_3. \end{aligned}$$

The system is then transformed, under these change of coordinates, to the linear Brunovsky form Λ_{Br} . The linearizing coordinates for the original system are thus obtained as a composition of the two-step coordinate changes

$$\begin{aligned} w_1 &= x_1 - 2x_2x_3^2 - x_3^4 + (x_2 + x_3^2)^2 = x_1 + x_2^2 \\ w_2 &= x_2 + x_3^2 \\ w_3 &= x_3. \end{aligned}$$

Of course, these linearizing coordinates could have been obtained directly or by other methods. The emphasis here is on the applicability of the method to any linearizable system.

Example IV.2 We consider the following example

$$\begin{cases} \dot{x}_1 = x_2 + ((1/2)x_2 - (1/12)x_3x_4)u & \dot{x}_3 = x_4 + x_4u \\ \dot{x}_2 = x_3 + (1/2)x_3u & \dot{x}_4 = u. \end{cases}$$

Because of the strict feedforward structure, we showed in [12] (using a 4-step algorithm) that the change of coordinates

$$\begin{aligned} z_1 &= x_1 - (1/24)(12x_2x_4 - 4x_3x_4^2 + x_4^4) \\ z_2 &= x_2 - (1/2)(x_3x_4 - (1/3)x_4^3) \\ z_3 &= x_3 - (1/2)x_4^2 \\ z_4 &= x_4 \end{aligned} \quad (\text{IV.1})$$

linearizes the system. We can recover such coordinates directly by applying the algorithm given in the proof. Denote by $f(x) = (x_2, x_3, x_4, 0)^\top$ and

$$\nu(x) \triangleq g(x) = ((1/2)x_2 - (1/12)x_3x_4, (1/2)x_3, x_4, 1)^\top.$$

The first step consists of rectifying the control vector field via Theorem II.2. Since $\nu_3 = 1$, hence $\sigma_3 = 1$ we have

$$L_\nu(\nu_1) = (1/2)(x_3/2) - (1/12)(x_4^2 + x_3) = (1/6)x_3 - (1/12)x_4^2,$$

and $L_\nu^2(\nu_1) = \frac{1}{6}x_4 - \frac{1}{6}x_4 = 0$, i.e., $L_\nu^s(\nu_1) = 0, s \geq 2$. Thus

$$\begin{aligned} \phi_1(x) &= x_1 - x_4\nu_1(x) + (1/2)x_4^2L_\nu(\nu_1) \\ &= x_1 - (1/2)x_2x_4 + (1/6)x_3x_4^2 - (1/24)x_4^3. \end{aligned}$$

Also $L_\nu(\nu_2) = \frac{1}{2}x_4$, $L_\nu^2(\nu_2) = \frac{1}{2}$ and $L_\nu^s(\nu_2) = 0, s \geq 3$ implies

$$\begin{aligned} \phi_2(x) &= x_2 - x_4\nu_2(x) + (1/2)x_4^2L_\nu(\nu_2) - (1/6)x_4^3L_\nu^2(\nu_2) \\ &= x_2 - (1/2)x_3x_4 + (1/4)x_4^3 - (1/12)x_4^3 \\ &= x_2 - (1/2)x_3x_4 + (1/6)x_4^3. \end{aligned}$$

Similarly $L_\nu(\nu_3) = 1$ and $L_\nu^{s-1}(\nu_3) = 0, \forall s \geq 3$. Hence

$$\begin{aligned} \phi_3(x) &= x_3 - x_4\nu_3(x) + (1/2)x_4^2L_\nu(\nu_2) \\ &= x_3 - x_4^2 + (1/2)x_4^2 = x_3 - (1/2)x_4^2. \end{aligned}$$

Because $\nu_4(x) = 1$, we get $\phi_4(x) = x_4$ and the change of coordinates (IV.1) rectifies the control vector field g and linearizes the system. Notice that the algorithm described in

[12] allowed only to find (IV.1) by computing one component at a time (holding other components identity), starting from ϕ_3 then ϕ_2 and finally ϕ_1 and updating the system after each step. A composition of different coordinates changes gave (IV.1). However, Theorem II.2 allows to compute those components independently to each other. $\triangleright$

V. APPENDIX

Below we give a brief proof of the constructive approach (Theorem II.2) for rectifying nonsingular vector fields.

Proof: Notice that for any diffeomorphism $z = \phi(x)$ the two following conditions are equivalent.

- (i) $\phi_*(\nu)(z) = \partial_{z_n}$.
- (ii) $L_\nu(\phi_j)(x) = 0$ and $L_\nu(\phi_n)(x) = 1$ for $1 \leq j \leq n-1$.

For that reason we will show that condition (ii) holds. To start let us take $1 \leq j \leq n-1$. It follows directly

$$\begin{aligned} L_\nu(\phi_j)(x) &= L_\nu(x_j) + \sum_{s=1}^{\infty} L_\nu \left(\frac{(-1)^s x_n^s}{s!} L_{\sigma_n \nu}^{s-1}(\sigma_n \nu_j) \right) \\ &= \nu_j(x) + \sum_{s=1}^{\infty} \frac{(-1)^s x_n^s}{s!} L_\nu L_{\sigma_n \nu}^{s-1}(\sigma_n \nu_j) \\ &\quad + \sum_{s=1}^{\infty} \frac{(-1)^s x_n^{s-1}}{(s-1)!} \nu_n(x) L_{\sigma_n \nu}^{s-1}(\sigma_n \nu_j) \\ &= \nu_j(x) + \sum_{s=1}^{\infty} \frac{(-1)^s x_n^s}{s!} \nu_n(x) L_{\sigma_n \nu}^s(\sigma_n \nu_j) \\ &\quad - \nu_j(x) - \sum_{s=1}^{\infty} \frac{(-1)^s x_n^s}{s!} \nu_n(x) L_{\sigma_n \nu}^s(\sigma_n \nu_j) \\ &= 0. \end{aligned}$$

A direct computation shows that

$$\begin{aligned} L_\nu \phi_n(x) &= \sum_{s=1}^{\infty} L_\nu \left(\frac{(-1)^{s-1} x_n^s}{s!} L_{\sigma_n \nu}^{s-1}(\sigma_n) \right) \\ &= \sum_{s=1}^{\infty} \frac{(-1)^{s-1} x_n^s}{s!} L_\nu L_{\sigma_n \nu}^{s-1}(\sigma_n) \\ &\quad + \sum_{s=1}^{\infty} \frac{(-1)^{s-1} x_n^{s-1}}{(s-1)!} \nu_n(x) L_{\sigma_n \nu}^{s-1}(\sigma_n) \\ &= \sum_{s=1}^{\infty} \frac{(-1)^{s-1} x_n^s}{s!} \nu_n(x) L_{\sigma_n \nu}^s(\sigma_n) \\ &\quad + \nu_n(x) \sigma_n(x) + \sum_{s=1}^{\infty} \frac{(-1)^s x_n^s}{s!} \nu_n(x) L_{\sigma_n \nu}^s(\sigma_n) \\ &= \nu_n(x) \sigma_n(x) = 1. \end{aligned}$$

This ends the proof of Theorem II.2 (i). $\square$

The proof of Theorem II.2 (ii) is more involved and we refer to [16] for more details. We illustrate with few examples and justify in Example V.2 that the expressions (II.1)-(II.2) of Theorem II.2 are not Taylor series at the origin.

Example V.1 Consider $\nu(x) = x_3\partial_{x_1} + (x_2 + x_3)\partial_{x_2} + \partial_{x_3}$ in $\mathbb{R}^3$. Here $L_\nu(\nu_1) = 1$ and $L_\nu^{s-1}(\nu_1) = 0$ for $s \geq 3$ and $L_\nu^{s-1}(\nu_2) = x_2 + x_3 + 1$ for all $s \geq 2$. It follows that

$$\begin{aligned} \phi_1(x) &= x_1 + \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_\nu^{s-1}(\nu_1)(x) \\ &= x_1 - x_3\nu_1(x) + (1/2!)x_3^2L_\nu(\nu_1)(x) = x_1 - (1/2)x_3^2 \end{aligned}$$

and

$$\begin{aligned}\phi_2(x) &= x_2 + \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_{\nu}^{s-1}(\nu_2)(x) \\ &= x_2 - x_3 \nu_2(x) + \sum_{s=2}^{\infty} \frac{(-1)^s x_3^s}{s!} (x_2 + x_3 + 1) \\ &= (x_2 + x_3 + 1) e^{-x_3} - 1.\end{aligned}$$

To find the inverse first notice that $\partial_{z_3}^i \cdot L_{\nu}^{s-i-1}(\nu_1)(z) = 0$ if $(i, s) \neq (0, 1)$, which yields

$$\begin{aligned}\psi_1(z) &= z_1 + \sum_{s=1}^{\infty} \frac{z_3^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_3}^i \cdot L_{\nu}^{s-i-1}(\nu_1)(z) \right) \\ &= z_1 + (1/2!) z_3^2 \nu_1(z) = z_1 + (1/2) z_3^2.\end{aligned}$$

From $\partial_{z_3}^i \cdot L_{\nu}^{s-i-1}(\nu_2)(z) = 0$ for all $i \geq 2$, we deduce

$$\begin{aligned}\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_3}^i \cdot L_{\nu}^{s-i-1}(\nu_2)(z) \\ = L_{\nu}^{s-1}(\nu_2)(z) - s \partial_{z_3} \cdot L_{\nu}^{s-2}(\nu_2)(z) = z_2 + z_3 + 1 - s.\end{aligned}$$

By Theorem II.2 (ii) we get the 2nd component of $\psi(z)$ as

$$\begin{aligned}\psi_2(z) &= z_2 + \sum_{s=1}^{\infty} \frac{z_3^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_3}^i \cdot L_{\nu}^{s-i-1}(\nu_2)(z) \right) \\ &= z_2 + \sum_{s=1}^{\infty} \frac{z_3^s}{s!} (z_2 + z_3 + 1) - \sum_{s=1}^{\infty} \frac{z_3^s}{s!} s \\ &= (z_2 + 1) e^{z_3} - z_3 - 1.\end{aligned}$$

It is straightforward to verify that the inverse is

$$\begin{aligned}x_1 = \psi_1(z) &= z_1 + (1/2) z_3^2 \\ x_2 = \psi_2(z) &= (z_2 + 1) e^{z_3} - z_3 - 1 \\ x_3 = \psi_3(z) &= z_3.\end{aligned}$$

Example V.2 Consider the non singular vector field

$$\nu(x) = \lambda(x_3) \partial_{x_1} + \partial_{x_3}, \quad x \in \mathbb{R}^3,$$

where λ is a flat function, that is, λ and all its derivatives are zero at $x_3 = 0$. A well-known example is the function

$$\lambda(0) = 0, \quad \lambda(x_3) = \exp(-1/x_3^2) \text{ if } x_3 \neq 0.$$

It is straightforward to check that $L_{\nu}^{s-1}(\nu_1)(x) = \lambda^{(s-1)}(x_3)$ for all $s \geq 1$, where $\lambda^{(k)}(x_3)$ is the k th derivative of λ . Should (II.1) have been a series around 0 or at $x_k = 0$ the straightening diffeomorphism would have been identity:

$$\begin{cases} \phi_1(x) &= x_1 + \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_{\nu}^{s-1}(\nu_1)(0) = x_1 \\ \phi_2(x) &= x_2 + \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_{\nu}^{s-1}(\nu_2)(0) = x_2 \\ \phi_3(x) &= \sum_{s=1}^{\infty} \frac{(-1)^{s-1} x_3^s}{s!} L_{\nu}^{s-1}(1)(0) = x_3 \end{cases}$$

which is impossible. However we can verify easily that $\phi_1(x) = x_1 - \int_0^{x_3} \lambda(u) du$ which coincides with

$$\phi_1(x) = x_1 + \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} \lambda^{(s-1)}(x_3).$$

Indeed, $\int_0^{x_3} \lambda(u) du = - \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} \lambda^{(s-1)}(x_3)$ because the two functions coincide when $x_3 = 0$ and it is enough to verify that their derivatives are also equal. The derivative of the right hand side gives after simplification

$$- \sum_{s=1}^{\infty} \frac{(-1)^s x_3^{s-1}}{(s-1)!} \lambda^{(s-1)}(x_3) - \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} \lambda^{(s)}(x_3) = \lambda(x_3).$$

Now to find the inverse of the normalizing coordinates, let us apply Theorem II.2 (ii) with $n = 3$ and $k = 3$. First we have $L_{\nu}^s \nu = \lambda^{(s)}(x_3) \partial_{x_1}$ for all $s \geq 1$. We thus have

$$\begin{aligned}\psi(z) &= z + \sum_{s=1}^{\infty} \frac{z_3^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \partial_{z_3}^i \cdot (L_{\nu}^{s-i-1} \nu)(z) \right) \\ &= z + \sum_{s=1}^{\infty} \frac{z_3^s}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^i \right) \lambda^{(s-1)}(z_3) \partial_{z_1} \\ &= \begin{pmatrix} z_1 - \sum_{s=1}^{\infty} \frac{(-1)^s z_3^s}{s!} \lambda^{(s-1)}(z_3) \\ z_2 \\ z_3 \end{pmatrix}\end{aligned}$$

It clearly follows that $\psi(z) = \left(z_1 + \int_0^{z_3} \lambda(s) ds, z_2, z_3 \right)^{\top}$ which was predictable directly by inverting $z = \phi(x)$. $\triangleright$

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