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Time-Invariant Quadratic Hamiltonians via Generalized Transformations

Issa Amadou Tall
Southern Illinois University Carbondale, itall@math.siu.edu

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Recommended Citation
Tall, Issa Amadou, "Time-Invariant Quadratic Hamiltonians via Generalized Transformations" (2010). *Miscellaneous (presentations, translations, interviews, etc)*. Paper 43.
http://opensiuc.lib.siu.edu/math_misc/43

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2010 American Control Conference
Marriott Waterfront, Baltimore, MD, USA
June 30-July 02, 2010

Time-invariant quadratic Hamiltonians via generalized transformations

Issa Amadou Tall
Department of Mathematics,
Southern Illinois University Carbondale,
Mail Code 4408, 1245 Lincoln Drive,
Carbondale IL, 62901, USA
Email: itall@math.siu.edu

Abstract— **In this paper we give necessary and sufficient conditions for achieving a quadratic positive definite time-invariant Hamiltonian for time-varying generalized Hamiltonian control systems using canonical transformations. Those necessary and sufficient conditions form a system of partial differential equations that reduces to the matching conditions obtained earlier in the literature for time-invariant systems. Their theoretical solvability is proved via the Cauchy-Kowalevskaya theorem and their practical solvability discussed in some particular cases. Systems with time-invariant positive definite Hamiltonian are known to yield a passive input-output map and can be stabilized by unity feedback, which underlines the importance of achieving the positive definiteness and time-invariancy for the Hamiltonian. We illustrate the results with few examples including the rolling coin.**

I. Introduction and Notations

Generalized Hamiltonian systems also known as port-controlled Hamiltonian systems have been introduced in [7] as a generalization of conventional Hamiltonian systems [9]. Fujimoto and Sugie [3] introduced generalized canonical transformations for time-varying port-controlled Hamiltonian systems that preserve the generalized Hamiltonian structure of the system. Unlike unconstrained (affine) control systems for which any diffeomorphism of the coordinates and any invertible (affine) feedback preserve the structure of the system, the situation is very different for constrained control systems, in particular for generalized Hamiltonian systems. For this reason the class of generalized canonical transformations is not *a priori* defined but only determined from the system by solving a system of partial differential equations. Though the solving of these partial differential equations is not always straightforward, the usefulness of canonical transformations has been demonstrated for time-varying generalized Hamiltonian systems (see [3], [4], [5], [8] and the references therein). The stabilization of Lagrangian and Mechanical systems have been intensively addressed in the literature (see [1], [2], [6], [10], [13] and the references therein). In much cases, a transformation is thought to simplify the problem and symmetries are exploited when present. For time-invariant Hamiltonian systems however, only the change in feedback is worth looking for, and the preserving conditions are referred in the literature as *matching conditions* [1], [13] though a change of coordinates can be applied to render the Hamiltonian positive definite. This paper builds from previous work by Fujimoto et al. [3]. It is centered around the idea that achieving a time-independent positive definite (possibly quadratic) Hamiltonian simplifies much more the stabilization procedure, and is always theoretically possible. As it is well known a zero-state detectable Hamiltonian system with passive Hamiltonian can be stabilized by unity feedback. A comparison with the conditions obtained in [3] and the matching conditions of [1], [13] is also provided. Let first fix some notations. Throughout the paper x will denote a column vector $x = (x_1, \ldots, x_n)^{\mathrm{T}}$, where T denotes the transposition. For any function H we denote $\frac{\partial H}{\partial x}^{\mathrm{T}}$ the column vector $(\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n})^{\mathrm{T}}$. A similar notation extends for vector-valued functions $H(x) = (H_1(x), \ldots, H_m(x))$ for which $\frac{\partial H}{\partial x}^{\mathrm{T}}$ is the matrix $(\frac{\partial H_j}{\partial x_i})_{ij}$. Given two column vectors $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, \ldots, y_m)^{\mathrm{T}}$, the column vector $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} = (x_1, \ldots, x_n, y_1, \ldots, y_m)^{\mathrm{T}}$ will be simply denoted by $(x, y)^{\mathrm{T}}$ with an abuse of notation. *Time-varying generalized Hamiltonian control systems*

$$\begin{cases} \dot{x} = J(x,t)\dfrac{\partial \mathcal{H}(x,t)}{\partial x}^{\mathrm{T}} + g(x,t)u, \\ y = g(x,t)^{\mathrm{T}}\dfrac{\partial \mathcal{H}(x,t)}{\partial x}^{\mathrm{T}}, \end{cases} \tag{I.1}$$

where $x \in \mathbb{R}^n$ is the state-space, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^m$ the output of the system, $J(x,t) \in \mathbb{R}^{n\times n}$ a skew-symmetric matrix function: $(J(x,t)^{\mathrm{T}} = -J(x,t)$, for all $x \in \mathbb{R}^n, t \in \mathbb{R})$, $g(x,t) \in \mathbb{R}^{n\times m}$ a full rank matrix, and $\mathcal{H}(x,t)$ the time-dependent generalized Hamiltonian function. When $J(x,t), \mathcal{H}(x,t)$ and $g(x,t)$ are time-independent, the system (I.1) is simply referred to as a generalized Hamiltonian system [3]. Moreover, if $J(x,t)$ and $g(x,t)$ are constant, say

$$J(x,t) = \mathbb{J} = \begin{bmatrix} 0_n & \mathrm{I}_n \\ -\mathrm{I}_n & 0_n \end{bmatrix} \text{ and } g(x,t) = \mathbb{G} = \begin{bmatrix} 0_n \\ \mathrm{I}_n \end{bmatrix},$$

where 0_n and I_n denote respectively the null and identity n-dimensional matrices, we then rediscover conventional (symplectic) Hamiltonian systems

$$\dot{x} = \mathbb{J}\frac{\partial \mathcal{H}(x)}{\partial x}^{\mathrm{T}} + \mathbb{G}u.$$

The class of generalized Hamiltonian systems was analyzed under the action of the following transformations:

$$\begin{aligned} \bar{x} &= \Phi(x,t), \qquad \bar{\mathcal{H}} = \mathcal{H}(x,t) + U(x,t), \\ \bar{u} &= u + \beta(x,t), \ \bar{y} = y + \alpha(x,t). \end{aligned} \tag{I.2}$$

The set of transformations (I.2) is called *generalized canonical transformations* [3] if it takes the system (I.1) into another generalized Hamiltonian control system

$$\begin{cases} \dot{\bar{x}} = \bar{J}(\bar{x},t)\dfrac{\partial\bar{\mathcal{H}}(\bar{x},t)}{\partial\bar{x}}^{\mathrm{T}} + \bar{g}(\bar{x},t)\bar{u}, \\ \bar{y} = \bar{g}(\bar{x},t)^{\mathrm{T}}\dfrac{\partial\bar{\mathcal{H}}(\bar{x},t)}{\partial\bar{x}}^{\mathrm{T}}. \end{cases} \tag{I.3}$$

Necessary and sufficient conditions in terms of partial differential equations were given in [3] for the equivalence of the systems (I.1) and (I.3) via the transformation (I.2). Those partial differential equations (see (II.2) below) are a mix between the components of the system and the components of the transformation. The authors of [3] (see also [4], [5]) noticed that the skew-symmetric matrix $K(x,t)$ can be used to simplify the structure of the matrix $\bar{J}(\bar{x},t)$. However, the most important for the design and stability is a simplified Hamiltonian function. Can we achieve a positive definite Hamiltonian $\bar{\mathcal{H}}(\bar{x},t)$? Can $\bar{\mathcal{H}}(\bar{x},t)$ be made positive definite and time-invariant? Can $\bar{\mathcal{H}}(\bar{x},t)$ be made quadratic and time-invariant? Those questions have not been addressed explicitly and are the focus of this paper. To address them we propose a slightly different approach than [3]. Instead of finding the diffeomorphism $\Phi(x,t)$ from predetermined scalar functions $U(x,t)$ and $\beta(x,t)$, we set the goal for a time-invariant positive definite Hamiltonian $\bar{\mathcal{H}}(\bar{x},t) = \bar{\mathcal{H}}(\bar{x})$ (possibly quadratic) and then determine $U(x,t)$ and $\beta(x,t)$ along with $\Phi(x,t)$ that achieve such goal. Once the transformation that yields a time invariant positive definite $\bar{\mathcal{H}}(\bar{x})$ is found, all the stabilization procedure follows easily. We will assume all objects to be C^ω analytic though the results are expected to hold in the C^∞ category. Section II contains the main result of the paper. Quadratic time-varying Hamiltonian systems are also discussed and several examples given in Section III.

II. Main Results

In this section we establish our results. We show that generalized transformations can be used to achieve a Hamiltonian function that is quadratic, positive definite, and time-invariant. As a corollary we completely characterize quadratic time-varying Hamiltonian systems showing that they can be transformed into a canonical time-invariant Hamiltonian system via linear change of coordinates and appropriate functions U and β obtained from a system of ODEs.

Theorem II.1 *Consider a system (I.1). There exists a canonical transformation (I.2) that maps the system (I.1) into a new Hamiltonian system (I.3) with a quadratic positive definite Hamiltonian $\bar{\mathcal{H}}(\bar{x},t) = (1/2)\bar{x}^{\mathrm{T}}\bar{x}$. Moreover, the skew-symmetric matrix $\bar{J}$ can be chosen to be constant.*

The proof follows directly from the well-known Cauchy-Kowalevskaya Theorem (see [11], [12]) and Lemma II.2.

Lemma II.2 *The canonical transformation (I.2) maps the system (I.1) into a new Hamiltonian system (I.3) with a quadratic positive definite Hamiltonian $\bar{\mathcal{H}}(\bar{x},t) = (1/2)\bar{x}^{\mathrm{T}}\bar{x}$ if and only if $\Phi(x,t)$ and $\alpha(x,t)$ satisfy*

$$\begin{aligned} \frac{\partial\Phi}{\partial t} &= -\frac{\partial\Phi}{\partial x}J(x,t)\frac{\partial\mathcal{H}(x,t)}{\partial x}^{\mathrm{T}} + \bar{J}(\Phi(x,t),t)\Phi(x,t), \\ \alpha &= g^{\mathrm{T}}(x,t)\frac{\partial\mathcal{H}(x,t)}{\partial x}^{\mathrm{T}}. \end{aligned} \tag{II.1}$$

Moreover, $U(x,t)$ is uniquely determined.

Notice that this lemma has been stated for the simplest case of canonical transformations ($\beta = 0$) but extends to generalized transformations by just adding the term $\frac{\partial\Phi}{\partial x}g(x,t)\beta(x,t)$ in the right side of the first equation of (II.1).

Proof: The proof of this lemma is constructive and can also be adapted from Lemma 3 [3] with a slight modification. Indeed, let $\bar{x} = \Phi(x,t)$ and using the fact that

$$\bar{\mathcal{H}}(\bar{x},t) = (1/2)\bar{x}^{\mathrm{T}}\bar{x} \implies \frac{\partial\bar{\mathcal{H}}(\bar{x},t)}{\partial\bar{x}}^{\mathrm{T}} = \bar{x},$$

we have

$$\begin{aligned} \dot{\bar{x}} &= \frac{\partial\Phi}{\partial x}\dot{x} + \frac{\partial\Phi}{\partial t} = \frac{\partial\Phi}{\partial x}\left[J(x,t)\frac{\partial\mathcal{H}(x,t)}{\partial x}^{\mathrm{T}} + g(x,t)u\right] + \frac{\partial\Phi}{\partial t} \\ &= \frac{\partial\Phi}{\partial x}J(x,t)\frac{\partial\mathcal{H}(x,t)}{\partial x}^{\mathrm{T}} + \frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial x}g(x,t)u \\ &= \bar{J}(\bar{x},t)\frac{\partial\bar{\mathcal{H}}(\bar{x},t)}{\partial\bar{x}}^{\mathrm{T}} + \bar{g}(\bar{x},t)\bar{u} \\ &= \bar{J}(\Phi(x,t),t)\Phi(x,t) + \bar{g}(\Phi(x,t),t)\bar{u}. \end{aligned}$$

The first condition of (II.1) follows by simple comparison ($\bar{u} = u$). When a feedback β is used, the input u is replaced by $\bar{u} - \beta(x,t)$ and the term $\frac{\partial\Phi}{\partial x}g(x,t)\beta(x,t)$ is then generated. Before we give a proof of Theorem II.1 let us make a comparison between our conditions and those in the literature [1], [3], [5], [13]. To outline the differences with the results obtained by Fujimoto et al, briefly recall that they proved the following in [3]: for any scalar function $U(x,t)$ and any vector function $\beta(x,t)$, there exists a pair of functions $\Phi(x,t)$ and $\alpha(x,t)$ that yield a generalized transformation if and only if the partial differential equation

$$\frac{\partial\Phi}{\partial(x,t)}\begin{bmatrix} J(x,t)\frac{\partial U}{\partial x}^{\mathrm{T}} + g\beta + K(x,t)\frac{\partial(\mathcal{H}+U)}{\partial x}^{\mathrm{T}} \\ -1 \end{bmatrix} = 0 \tag{II.2}$$

holds with a skew-symmetric matrix $K(x,t)$. First, the unknowns in (II.2) are the diffeomorphism Φ and the skew-symmetric matrix K but those really depend on the choices of U and β. Second, it looks like that (II.2) has more degrees of freedom (functionals) and hence would be easier solvable than (II.1) but recall that the *a priori* given scalar function $U(x,t)$ and feedback $\beta(x,t)$ must be chosen so as to satisfy the system of partial differential inequalities

$$\begin{aligned} &\text{(i)} \quad \mathcal{H} + U \geq 0 \\ &\text{(ii)} \quad \frac{\partial(\mathcal{H}+U)}{\partial x}\left[J\frac{\partial U}{\partial x}^{\mathrm{T}} + g\beta\right] - \frac{\partial(\mathcal{H}+U)}{\partial t} \geq 0. \end{aligned} \tag{II.3}$$

Apparently, the choices of $U(x,t)$ and $\beta(x,t)$ are not obvious as the solvability of (II.3) is not straightforward. Moreover, we don't have any apprehension on the new Hamiltonian $\bar{\mathcal{H}}$ until U and Φ are obtained (see the comparative Example). In our case, there is only a single

system of partial differential equations to be solved with unknowns the diffeomorphism $\Phi(x,t)$ and function $\beta(x,t)$. Their obtention implies directly that of $\alpha(x,t)$ and U as $U(x,t) = (1/2)\Phi(x,t)^{\mathrm{T}}\Phi(x,t) - \mathcal{H}(x,t)$ but most importantly it implies that the resulting Hamiltonian $\bar{\mathcal{H}}$ is time-independent and quadratic positive definite. Thus the stabilization of the system is guaranteed without no further need of solving partial differential inequalities (we refer to Example III.2 for an illustration). The stabilizing feedback can be constructed easily. Now, both conditions (II.1) and (II.2) (see [3], [5] for more details) can be interpreted as a generalization of the matching conditions obtained in [1], [13]. The matching conditions were obtained for time-invariant hamiltonian systems for which the change of coordinates does not play any role in preserving the Hamiltonian structure; hence $\Phi = \mathrm{Id}$. If we let $x = (q,p)^{\mathrm{T}}$, $\Phi(x,t) = x$, $H(x,t) = (1/2)p^{\mathrm{T}}M^{-1}(q)p + V(q)$, $g(x,t) = (0, G(q))^{\mathrm{T}}$, then (II.1) with the feedback term $g(x,t)\beta(x)$ incorporated (resp. (II.2)), simplify down to the matching conditions.

To complete the proof of Theorem II.1 we recall the well-known Cauchy-Kowalevskaya Theorem stated below for quasilinear first-order systems [11], [12].

Theorem II.3 *Consider the partial differential equations*

$$\frac{\partial \Theta_i}{\partial x_m} = \sum_{k=1}^{m-1}\sum_{j=1}^{q} a_{i,j}^k(\mathbf{p})\frac{\partial \Theta_j}{\partial x_k} + b_i(\mathbf{p}), i = 1,\ldots,q, \quad \text{(II.4)}$$

where $\mathbf{p} = (x_1,\ldots,x_{m-1},\Theta_1,\ldots,\Theta_q)^{\mathrm{T}}$ *with*

$$\Theta_i = 0, \quad i = 1,\ldots,q \quad \text{on} \quad x_m = 0. \quad \text{(II.5)}$$

If the functions $a_{i,j}^k(\mathbf{p})$ and $b_i(\mathbf{p})$ are analytic at the origin, then (II.4) with initial conditions (II.5) has a unique system of solutions Θ_i that are real analytic at the origin.

Observe that Theorem II.3 is valid even if $\mathbf{p}$ depends explicitly on the variable x_m and the constraints (II.5) replaced by

$$\Theta_i = \Theta_i^0(x_1,\ldots,x_{m-1}),\ i = 1,\ldots,q \text{ on } x_m = 0, \quad \text{(II.6)}$$

for arbitrary analytic functions $\Theta_i^0(x_1,\ldots,x_{m-1})$. Indeed, it is enough to take $\tilde{\Theta}_1 = \Theta_1 = x_m$, $a_{1,j}^k = 0$ for all j,k, $b_1 = 1$, and $\tilde{\Theta}_i = \Theta_i + \Theta_i^0$ for $i = 2,\ldots,q$.

Proof of Theorem II.1. The first equation of (II.1) rewrites

$$\frac{\partial \Phi_i}{\partial t} = -\sum_{k=1}^{n} J_k(x,t)\frac{\partial \mathcal{H}(x,t)}{\partial x}^{\mathrm{T}}\frac{\partial \Phi_i}{\partial x_k} + \bar{J}_i(\Phi(x,t),t)\Phi(x,t)$$

where $J_k(x,t)\frac{\partial \mathcal{H}(x,t)}{\partial x}^{\mathrm{T}} = \sum_{j=1}^{n} J_{j,k}(x,t)\frac{\partial \mathcal{H}(x,t)}{\partial x_j}$ and

$$\bar{J}_i(\Phi(x,t),t)\Phi(x,t) = \sum_{j=1}^{n}\bar{J}_{j,i}(\Phi(x,t),t)\Phi_j(x,t), 1 \le i \le n.$$

Taking $m = n+1$, $q = n$ and $t = x_m$ we obtain (II.4) after identifying Φ_i with Θ_i, and taking the analytic functions

$$a_{i,j}^k = -J_k(x,t)\frac{\partial \mathcal{H}(x,t)}{\partial x}^{\mathrm{T}}$$

and $b_i(p_1,\ldots,p_{2n}) = \sum_{j=1}^{n}\bar{J}_{j,i}(p_{n+1},\ldots,p_{2n},p_n)p_{n+j}$. Thus an analytic solution satisfying $\Phi(x,t)|_{t=0} = x$ can be found.

Notice that, as pointed by some reviewer, the proof can also be obtained directly form the flow-box theorem. We will discuss that option in forthcoming paper where we also provide explicit coordinates changes.

Time-varying Quadratic Hamiltonian Systems. A particular class of generalized Hamiltonian systems are *time-varying quadratic Hamiltonian systems*, defined here as systems of the form (I.1) with $J(x,t) = \mathbb{J}$ constant and quadratic Hamiltonian $\mathcal{H}(x,t) = \frac{1}{2}x^{\mathrm{T}}M(t)x$. The following proposition is a direct corollary of Theorem II.1 and is stated for even dimension for simplicity but holds for arbitrary dimension.

Proposition II.4 *Consider a* time-varying quadratic Hamiltonian system*, that is, system (I.1) with quadratic Hamiltonian $\mathcal{H}(x,t) = \frac{1}{2}x^{\mathrm{T}}M(t)x$ and $J(x,t) = \mathbb{J}$. Then the canonical transformation (I.1) with $\Phi(x,t) = \Theta(t)x$ and*

$$U(x,t) = (1/2)x^{\mathrm{T}}\left(\Theta^{\mathrm{T}}(t)\Theta(t) - M(t)\right)x$$
$$\alpha(x,t) = g^{\mathrm{T}}(x,t)\Theta(t)x$$

brings the system into the Hamiltonian system

$$\dot{\bar{x}} = \mathbb{J}\frac{\partial \bar{\mathcal{H}}(\bar{x},t)}{\partial \bar{x}}^{\mathrm{T}} + \bar{g}(\bar{x},t)u = \mathbb{J}\bar{x} + \bar{g}(\bar{x},t)u$$

if and only if holds the ordinary differential equation

$$\Theta(t)\mathbb{J}M(t) + \dot{\Theta}(t) = \mathbb{J}\Theta(t).$$

When the control vector field g is constant, say $g(x,t) = \mathbb{G}$, a change in the input $\bar{u} = u + \beta(x,t) = u + \Omega(t)x$ might be useful. It replaces the above ordinary differential equation by

$$\Theta(t)\left(\mathbb{J}M(t) - \mathbb{G}\Omega(t)\right) + \dot{\Theta}(t) = \mathbb{J}\Theta(t). \quad \text{(II.7)}$$

Remark that the condition (II.7) can be written simply as $\dot{\Theta}(t) = A(t)\Theta(t)$ for some time-dependent matrix $A(t)$ whose entries are functions of those of M, and Ω only. If the matrix $A(t)$ and its integral $B(t) = \int A(t)dt$ are bounded and commute ($AB = BA$), then a global solution is $\Theta(t) = \exp B(t)$. In case $A(t)$ is periodic, Floquet theory can be used to solve the ordinary differential equation.

Although a canonical transformation making the Hamiltonian time-invariant and quadratic positive can always be theoretically found, it is however very important to point out that the change of coordinates (or its inverse) is not necessarily bounded. Moreover, finding explicitly the transformation is not an easy task. We will illustrate that by an example. Before, let us mention that for the class of fully-actuated mechanical systems, a positive definite quadratic Hamiltonian can be achieved due to the feedback linearizability of those systems; but most importantly, the transformation can be explicitly given without solving any system of PDE's.

Fully actuated Systems. Consider the system described in the coordinates $x = (q,p) = (q_1,\ldots,q_n,p_1,\ldots,p_n)^{\mathrm{T}}$ by

$$\left\{\begin{array}{l} \begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix} J_{11} & J_{12}\\ -J_{12} & J_{22}\end{bmatrix}\begin{bmatrix}\frac{\partial \mathcal{H}}{\partial q}^{\mathrm{T}}\\ \frac{\partial \mathcal{H}}{\partial p}^{\mathrm{T}}\end{bmatrix} + \begin{bmatrix}0_n\\ G_2\end{bmatrix}u,\\ \quad y = G_2(q,p,t)^{\mathrm{T}}\frac{\partial \mathcal{H}}{\partial p}^{\mathrm{T}}, \end{array}\right.$$

where $J_{11} = J_{11}(q,p,t)$, $J_{12} = J_{12}(q,p,t)$, $J_{22} = J_{22}(q,p,t)$ and $G_2 = G_2(q,p,t)$ are n-dimensional matrices with J_{12}, G_2

and the Hessian $\frac{\partial^2\mathcal{H}}{\partial p^2}=\left(\frac{\partial^2\mathcal{H}}{\partial p_i\partial p_j}\right)_{i,j}$ invertible. Such systems are feedback linearizable; hence can be transformed by a generalized transformation (I.2) to a new Hamiltonian system with positive definite and quadratic Hamiltonian. What is most important is that such transformation can be explicitly computed. Indeed, define the generalized vector field $\mathcal{X}$ as

$$\mathcal{X}=\left[J_{11}\frac{\partial\mathcal{H}}{\partial q}^{\mathrm{T}}+J_{12}\frac{\partial\mathcal{H}}{\partial p}^{\mathrm{T}}\right]\frac{\partial}{\partial q}+\left[-J_{12}\frac{\partial\mathcal{H}}{\partial q}^{\mathrm{T}}+J_{22}\frac{\partial\mathcal{H}}{\partial p}^{\mathrm{T}}\right]\frac{\partial}{\partial p}+\frac{\partial}{\partial t}.$$

Then the transformation (I.2) where

$$\bar{x}=(\bar{q},\bar{p})\triangleq\left\{\begin{array}{rcl}\bar{q}&=&q\\ \bar{p}&=&J_{11}(q,p,t)\frac{\partial\mathcal{H}}{\partial q}^{\mathrm{T}}+J_{12}(q,p,t)\frac{\partial\mathcal{H}}{\partial p}^{\mathrm{T}}\end{array}\right.$$

and the feedback

$$\beta(q,p,t)=\left[\frac{\partial\bar{p}}{\partial p}\cdot G_2(q,p,t)\right]^{-1}\cdot\left(\bar{q}+L_{\mathcal{X}}\bar{p}\right)$$

with appropriate choices of U and α brings the system into

$$\left\{\begin{array}{rcl}\begin{bmatrix}\dot{\bar{q}}\\ \dot{\bar{p}}\end{bmatrix}&=&\begin{bmatrix}0_n&\mathrm{I}_n\\ -\mathrm{I}_n&0_n\end{bmatrix}\begin{bmatrix}\frac{\partial\bar{\mathcal{H}}}{\partial\bar{q}}^{\mathrm{T}}\\ \frac{\partial\bar{\mathcal{H}}}{\partial\bar{p}}^{\mathrm{T}}\end{bmatrix}+\begin{bmatrix}0_n\\ \bar{G}_2(\cdot)\end{bmatrix}\bar{u},\\ \bar{y}&=&\bar{G}_2(\cdot)^{\mathrm{T}}\frac{\partial\bar{\mathcal{H}}}{\partial\bar{p}}^{\mathrm{T}}\end{array}\right. \tag{II.8}$$

where the Hamiltonian $\bar{\mathcal{H}}(\bar{q},\bar{p},t)=(1/2)(\bar{q}^{\mathrm{T}}\bar{q}+\bar{p}^{\mathrm{T}}\bar{p})$.

Notice that such transformation is not unique. Indeed, replacing $\bar{q}$ by $\tilde{q}=\Phi_1(\bar{q},t)$ and $\bar{p}$ by $\tilde{p}=\frac{\partial\Phi_1}{\partial\bar{q}}\bar{p}+\frac{\partial\Phi_1}{\partial t}$ above will yield a new function $\tilde{\beta}$. This fact is related to symmetries and will be addressed in a forthcoming paper.

III. Examples

We illustrative our results by several examples.

Example III.1 Consider the time-varying system

$$\left\{\begin{array}{rcl}\dot{x}&=&J(x,t)\frac{\partial\mathcal{H}(x,t)}{\partial x}^{\mathrm{T}}+g(x,t)u,\\ y&=&g(x,t)^{\mathrm{T}}\frac{\partial\mathcal{H}(x,t)}{\partial x}^{\mathrm{T}},\ x\in\mathbb{R}^{2n},u\in\mathbb{R}^m\end{array}\right. \tag{III.1}$$

where $J(x,t)=\mathbb{J}$, $g(x,t)=\mathbb{G}=(0_n\ \mathrm{I}_n)^{\mathrm{T}}$ and

$$\mathcal{H}(x,t)=\frac{1}{2}x^{\mathrm{T}}M(t)x\quad\text{with}\quad M(t)=f(t)\begin{bmatrix}0_n&0_n\\ 0_n&\mathrm{I}_n\end{bmatrix}.$$

Above $f(t)$ denotes a continuous positive function, and

$$x=(q,p)=(q_1,\ldots,q_n,p_1,\ldots,p_n)^{\mathrm{T}}.$$

A transformation (I.2) with $\bar{x}=(\bar{q},\bar{p})=(\Phi_1,\Phi_2)$ achieves a positive definite Hamiltonian $\bar{\mathcal{H}}=(1/2)(\bar{q}^{\mathrm{T}}\bar{q}+\bar{p}^{\mathrm{T}}\bar{p})$ if and only if the following system of PDEs is satisfied

$$\left\{\begin{array}{l}\Phi_2=f(t)\frac{\partial\Phi_1}{\partial q}\cdot p+\frac{\partial\Phi_1}{\partial t}-\frac{\partial\Phi_1}{\partial p}\cdot\beta(q,p,t)\\ -\Phi_1=f(t)\frac{\partial\Phi_2}{\partial q}\cdot p+\frac{\partial\Phi_2}{\partial t}-\frac{\partial\Phi_2}{\partial p}\cdot\beta(q,p,t).\end{array}\right. \tag{III.2}$$

This system is always solvable for any continuous function f; however the boundedness of its solutions is not guaranteed. We will set some conditions on f that allow to find solutions satisfying the boundedness property.

Case I. Assume that f is differentiable and that there are two constant $k_2\geq k_1>0$ such that

$$k_1\leq f(t)\leq k_2\ \text{ and }\ |f'(t)|\leq k_2\ \ \forall t\in\mathbb{R}. \tag{III.3}$$

A solution to (III.2) is simply obtained by taking

$$\Phi_1(q,p,t)=q,\ \Phi_2(q,p,t)=fp,\ \beta(q,p,t)=\frac{q+f'p}{f}.$$

Completing with $U(q,p,t)=(1/2)\left[q^{\mathrm{T}}q+(f^2-f)p^{\mathrm{T}}p\right]$ and $\alpha(q,p,t)=fp$ we thus define a transformation (I.2) that maps the original system into (III.5) with $\bar{G}_2=f(t)\mathrm{I}_n$. ▷

Case II. Suppose f is twice continuously differentiable and $|f''(t)|\leq k_2$ in addition of (III.3). The transformation (I.2)

$$\begin{array}{rcl}\Phi_1(q,p,t)&=&\gamma q,\quad\Phi_2(q,p,t)=p+\gamma' q,\\ \alpha(q,p,t)&=&fp,\quad\beta(q,p,t)=(\gamma+\gamma'')q+\gamma' fp\\ U(q,p,t)&=&(\gamma^2+\gamma'^2)q^{\mathrm{T}}q+(1-f)p^{\mathrm{T}}p+2\gamma' q^{\mathrm{T}}p\end{array}$$

with $\gamma=1/f$, brings the original system into a canonical form, i.e., into (III.5) where $\bar{G}_2=\mathrm{I}_n$. ▷

In the two precedent cases the differentiability and boundedness of f and its derivatives were imposed by our willingness to changing the control vector $[0\ G_2]^{\mathrm{T}}$ into $[0\ \bar{G}_2]^{\mathrm{T}}$ with $\bar{G}_2$ as simple as possible. If we drop those requirements we can ease the conditions on f (see below).

Case III. Assume that f is such that $\int_0^\infty f(s)\mathrm{d}s<\infty$. An example of$f$ is $f(t)=\frac{1}{1+t^2}$ for which $\int_0^\infty f(s)\mathrm{d}s=\pi/2$. Since $\mathcal{H}$ is quadratic in the state-space variables, we apply Proposition II.4 by looking for a linear change of coordinates $\bar{x}=(\bar{q},\bar{p})^{\mathrm{T}}=\Phi(x,t)=\Theta(t)x$ where $x=(q,p)^{\mathrm{T}}$ and

$$\Theta(t)=\begin{bmatrix}\Theta_{11}(t)&\Theta_{12}(t)\\ \Theta_{21}(t)&\Theta_{22}(t)\end{bmatrix}.$$

The system of ODEs given by Proposition II.4, which is equivalent to (III.2) with $\beta=0$, rewrites

$$\begin{array}{l}\begin{bmatrix}\dot{\Theta}_{11}&\dot{\Theta}_{12}\\ \dot{\Theta}_{21}&\dot{\Theta}_{22}\end{bmatrix}=\begin{bmatrix}0_n&\mathrm{I}_n\\ -\mathrm{I}_n&0_n\end{bmatrix}\begin{bmatrix}\Theta_{11}&\Theta_{12}\\ \Theta_{21}&\Theta_{22}\end{bmatrix}\\ -f(t)\begin{bmatrix}\Theta_{11}&\Theta_{12}\\ \Theta_{21}&\Theta_{22}\end{bmatrix}\begin{bmatrix}0_n&\mathrm{I}_n\\ -\mathrm{I}_n&0_n\end{bmatrix}\begin{bmatrix}0_n&0_n\\ 0_n&\mathrm{I}_n\end{bmatrix}\end{array}$$

and can further be simplified as

$$\left\{\begin{array}{rcl}\dot{\Theta}_{11}&=&\Theta_{21}\qquad\dot{\Theta}_{12}=\Theta_{22}-f(t)\Theta_{11}\\ \dot{\Theta}_{21}&=&-\Theta_{11}\qquad\dot{\Theta}_{22}=-\Theta_{12}-f(t)\Theta_{21}.\end{array}\right. \tag{III.4}$$

A simple solution is obtained by taking

$$\begin{array}{rcl}\Theta_{11}(t)&=&(\cos t)\mathrm{I}_n,\qquad\Theta_{12}(t)=(\sin t-\mu(t)\cos t)\mathrm{I}_n\\ \Theta_{21}(t)&=&-(\sin t)\mathrm{I}_n,\quad\Theta_{22}(t)=(\cos t+\mu(t)\sin t)\mathrm{I}_n,\end{array}$$

where $\mu(t)=\int_0^t f(s)\mathrm{d}s$. Thus $\bar{x}=\Phi(x,t)=\Theta(t)x$ with

$$U(x,t)=\frac{1}{2}x^{\mathrm{T}}\begin{bmatrix}\mathrm{I}_n&-\mu\mathrm{I}_n\\ -\mu\mathrm{I}_n&\mu^2\mathrm{I}_n\end{bmatrix}x$$

and output $\alpha(x,t)=g^{\mathrm{T}}(x,t)\frac{\partial\mathcal{H}}{\partial x}(x,t)^{\mathrm{T}}=f(t)M(t)x$ transform the system (III.1) into a new Hamiltonian system

$$\left\{\begin{array}{rcl}\begin{bmatrix}\dot{\bar{q}}\\ \dot{\bar{p}}\end{bmatrix}&=&\begin{bmatrix}0_n&\mathrm{I}_n\\ -\mathrm{I}_n&0_n\end{bmatrix}\begin{bmatrix}\frac{\partial\bar{\mathcal{H}}}{\partial\bar{q}}^{\mathrm{T}}\\ \frac{\partial\bar{\mathcal{H}}}{\partial\bar{p}}^{\mathrm{T}}\end{bmatrix}+\begin{bmatrix}\bar{G}_1(\cdot)\\ \bar{G}_2(\cdot)\end{bmatrix}\bar{u},\\ \bar{y}&=&\bar{G}_1(\cdot)^{\mathrm{T}}\frac{\partial\bar{\mathcal{H}}}{\partial\bar{q}}^{\mathrm{T}}+\bar{G}_2(\cdot)^{\mathrm{T}}\frac{\partial\bar{\mathcal{H}}}{\partial\bar{p}}^{\mathrm{T}}\end{array}\right. \tag{III.5}$$

with Hamiltonian $\bar{\mathcal{H}}(\bar{q},\bar{p},t) = (1/2)(\bar{q}^{\mathrm{T}}\bar{q} + \bar{p}^{\mathrm{T}}\bar{p})$ and

$\bar{G}_1(\cdot) = (\sin t - \mu(t)\cos t)\mathrm{I}_n,\ \bar{G}_2(\cdot) = (\cos t + \mu(t)\sin t)\mathrm{I}_n.$

Because the matrix

$$\Theta(t) = \begin{bmatrix} (\cos t)\mathrm{I}_n & (\sin t - \mu(t)\cos t)\mathrm{I}_n \\ -(\sin t)\mathrm{I}_n & (\cos t + \mu(t)\sin t)\mathrm{I}_n \end{bmatrix}$$

and its inverse

$$\Theta^{-1}(t) = \begin{bmatrix} (\cos t + \mu(t)\sin t)\mathrm{I}_n & (-\sin t + \mu(t)\cos t)\mathrm{I}_n \\ (\sin t)\mathrm{I}_n & (\cos t)\mathrm{I}_n \end{bmatrix}$$

are both bounded, there exists two positive constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that the *uniform boundedness property*

$$\lambda_1|x| \le |\bar{x}| \le \lambda_2|x|$$

is satisfied. Thus the original system can be stabilized. ▷

Next we make a comparative study using Example 2 of [3].

Example III.2 Consider the system (Example 2 of [3])

$$\begin{cases} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_n & \mathrm{I}_n \\ -\mathrm{I}_n & 0_n \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial q}^{\mathrm{T}} \\ \frac{\partial \mathcal{H}}{\partial p}^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0_n \\ \mathrm{I}_n \end{bmatrix} u, \\ y = \frac{\partial \mathcal{H}}{\partial p}^{\mathrm{T}} \end{cases}$$

where $\mathcal{H}(q,p,t) = (1/2)(1 + a\sin t)p^{\mathrm{T}}p$ and a is a constant real-valued parameter ($|a| < 1$). A two step approach was used to transform this system into the generalized system

$$\begin{cases} \begin{bmatrix} \dot{\hat{q}} \\ \dot{\hat{p}} \end{bmatrix} = \begin{bmatrix} 0_n & \frac{1}{\sqrt{c-2a\sin t}}\mathrm{I}_n \\ -\frac{1}{\sqrt{c-2a\sin t}}\mathrm{I}_n & 0_n \end{bmatrix} \begin{bmatrix} \frac{\partial \widehat{\mathcal{H}}}{\partial \hat{q}}^{\mathrm{T}} \\ \frac{\partial \widehat{\mathcal{H}}}{\partial \hat{p}}^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \frac{a\cos t}{\sqrt{c-2a\sin t}} \\ \mathrm{I}_n \end{bmatrix} \hat{u}, \\ \hat{y} = \begin{bmatrix} \frac{a\cos t}{\sqrt{c-2a\sin t}}\mathrm{I}_n & \mathrm{I}_n \end{bmatrix} \begin{bmatrix} \frac{\partial \widehat{\mathcal{H}}}{\partial \hat{q}}^{\mathrm{T}} \\ \frac{\partial \widehat{\mathcal{H}}}{\partial \hat{p}}^{\mathrm{T}} \end{bmatrix} = \frac{\partial \widehat{\mathcal{H}}}{\partial \hat{p}}^{\mathrm{T}} \end{cases}$$

with Hamiltonian $\widehat{\mathcal{H}}(\hat{q},\hat{p},t) = (1/2)(\hat{q}^{\mathrm{T}}\hat{q} + \hat{p}^{\mathrm{T}}\hat{p})$. First, a choice of $U = -(a\sin t/2)p^{\mathrm{T}}p$ and $\alpha = -ap\sin t$ was used to find change of coordinates $(\bar{q},\bar{p})^{\mathrm{T}} = \Phi(q,p,t)$ by solving

$$\frac{\partial \Phi}{\partial(q,p,t)} \left[\begin{bmatrix} -(a\sin t)p \\ 0_n \end{bmatrix} + K \begin{bmatrix} 0_n \\ p \end{bmatrix} \right] = 0,$$

$$-1$$

where K is an unknown skew-symmetric matrix. The system was then transformed into an intermediate Hamiltonian system with $\bar{\mathcal{H}} = (1/2)\bar{p}^{\mathrm{T}}\bar{p}$. Later a scalar periodic and positive function $\theta(t)$ was sought so $U(\bar{q},t) = \frac{\theta(t)}{2}\bar{q}^{\mathrm{T}}\bar{q}$ and $\beta(\bar{q},t) = \theta(t)\bar{q}$ solve the inequality (II.3). Once $\theta(t)$ is found, the coordinates change $(\hat{q},\hat{p})^{\mathrm{T}} = \bar{\Phi}(\bar{q},\bar{p},t)$ were obtained solving another partial differential equation. The choices of U and β here were merely intuitive but seeking them directly from (II.3) would have been a big task. This justifies in part the two-step approach in finding U and β.

Our approach consists of applying directly Proposition II.4. We seek for a linear change of coordinates

$$\bar{x} = (\bar{q},\bar{p})^{\mathrm{T}} = \Phi(q,p,t) = \Theta(t)(q,p)^{\mathrm{T}}$$

and a linear vector function $\beta(q,p,t) = \Omega_1(t)q + \Omega_2(t)p$ such that the ordinary differential equation (II.7) is satisfied.

Following similar steps as in the previous example, we arrive to the system of ordinary differential equations

$$\begin{cases} \dot{\Theta}_{11} = \Theta_{21} + \Omega_1\Theta_{12} \quad \dot{\Theta}_{12} = \Theta_{22} - (1 + a\sin t)\Theta_{11} + \Omega_2\Theta_{12} \\ \dot{\Theta}_{21} = -\Theta_{11} + \Omega_1\Theta_{22} \quad \dot{\Theta}_{22} = -\Theta_{12} - (1 + a\sin t)\Theta_{21} + \Omega_2\Theta_{22}. \end{cases}$$

If we take $\Theta_{12}(t) = 0$ and $\Theta_{22}(t) = \mathrm{I}_n$, we find that

$$\Theta_{11}(t) = \frac{1}{1 + a\sin t}\mathrm{I}_n \text{ and } \Theta_{21}(t) = \frac{-a\cos t}{(1 + a\sin t)^2}\mathrm{I}_n.$$

Reporting back we get $\Omega_2(t) = (1 + a\sin t)\Theta_{21}$ and

$$\Omega_1(t) = \dot{\Theta}_{21} + \Theta_{11} = \frac{a^2(1 + \cos^2 t) + a\sin t}{(1 + a\sin t)^3}\mathrm{I}_n.$$

Notice that $\Theta_{11}, \Theta_{21}, \Omega_1$ and Ω_2 are all bounded functions on the interval $t \in (-\infty, \infty)$ (recall that $|a| < 1$). Thus there exist $\lambda_2 \ge \lambda_1 > 0$ such that the change of coordinates

$$\bar{q} = \frac{1}{1 + a\sin t}q,\ \bar{p} = \frac{-a\cos t}{(1 + a\sin t)^2}q + p \tag{III.6}$$

whose inverse is $q = (1 + a\sin t)\bar{q}$, $p = \frac{a\cos t}{1 + a\sin t}\bar{q} + \bar{p}$ satisfy the uniform boundedness property

$$\lambda_1|(q,p)^{\mathrm{T}}| \le \left|(\bar{q},\bar{p})^{\mathrm{T}}\right| \le \lambda_2\left|(q,p)^{\mathrm{T}}\right|.$$

We easily verify that the transformation $(\bar{q},\bar{p})^{\mathrm{T}} = \Phi(q,p,t)$

$$\Phi(q,p,t) = \Theta(t)(q,p)^{\mathrm{T}}, \qquad \bar{u} = u + \Omega_1(t)q + \Omega_2(t)p,$$
$$\bar{\mathcal{H}} = \mathcal{H}(q,p,t) + U(q,p,t), \quad \bar{y} = y(q,p,t) + \alpha(q,p,t)$$

with $\alpha(q,p,t) = -\frac{a\cos t}{(1 + a\sin t)^2}q - (a\sin t)p$ and

$$U = \frac{1 + a^2 + 2a\sin t}{2(1 + a\sin t)^4}q^{\mathrm{T}}q - \frac{a\sin t}{2}p^{\mathrm{T}}p - \frac{a\cos t}{(1 + a\sin t)^2}q^{\mathrm{T}}p \tag{III.7}$$

brings the system into the canonical form

$$\begin{cases} \begin{bmatrix} \dot{\bar{q}} \\ \dot{\bar{p}} \end{bmatrix} = \begin{bmatrix} 0_n & \mathrm{I}_n \\ -\mathrm{I}_n & 0_n \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{\mathcal{H}}}{\partial \bar{q}}^{\mathrm{T}} \\ \frac{\partial \bar{\mathcal{H}}}{\partial \bar{p}}^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0_n \\ \mathrm{I}_n \end{bmatrix} \bar{u}, \\ \bar{y} = \bar{p} \end{cases}$$

with quadratic Hamiltonian $\bar{\mathcal{H}}(\bar{q},\bar{p},t) = (1/2)(\bar{q}^{\mathrm{T}}\bar{q} + \bar{p}^{\mathrm{T}}\bar{p})$.

The transformed system is zero state detectable with positive definite Hamiltonian $\bar{\mathcal{H}} = (1/2)(\bar{q}^{\mathrm{T}}\bar{q} + \bar{p}^{\mathrm{T}}\bar{p})$. It thus can be stabilized by the unity feedback $\bar{u} = -\bar{y}$ (see [9] and Lemma 1 [3]). Notice that Fujimoto [3] obtained a quadratic positive definite Hamiltonian but the structure matrix $\bar{J}$ and the output $\bar{y}$ remain time-dependent while we obtain a completely time-independent system. The transformation (III.6)-(III.7) obtained via the ODE of Proposition II.4 coincides also with that of **Case II** with $\gamma = 1/(1 + a\sin t)$. ▷

Remark that if the matrix function $\Omega = 0$, that is, $\bar{u} = u$, then no solution satisfying the uniform boundedness property can be found. Indeed, a typical solution would be given by

$$\bar{q} = (\cos t)q + (-t\cos t + 2\sin t + a\cos^2 t)p$$
$$\bar{p} = = -(\sin t)q + (t\sin t + 2\cos t - a\sin t\cos t)p$$

which fails to satisfy the uniform boundedness property.

The previous examples fall in the class of fully actuated systems. Next, we will take a case of underactuated system.

Example III.3 Consider a coin rolling on a horizontal plane [3], where q_1 denotes the heading angle, (q_2, q_3) the orthogonal coordinate of the point of contact between the coin and the horizontal plane, p_1 the angular velocity with respect to the heading angle q_1, and p_2 the rolling angular velocity of the coin. By u_1 and u_2 we denote the acceleration with respect to p_1, and p_2, respectively. The dynamics of the coin satisfy the port-controlled Hamiltonian system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = J(q,p) \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial q}^{\mathrm{T}} \\ \frac{\partial \mathcal{H}}{\partial p}^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0_{3\times 2} \\ \mathrm{I}_2 \end{bmatrix} u$$

where $q = (q_1, q_2, q_3)^{\mathrm{T}}$, $p = (p_1, p_2)^{\mathrm{T}}$, and the Hamiltonian $\mathcal{H}(q,p) = (1/2)p^{\mathrm{T}}p$ with the structure matrix given by

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cos q_1 \\ 0 & 0 & 0 & 0 & \sin q_1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\cos q_1 & -\sin q_1 & 0 & 0 \end{bmatrix}.$$

The identity transformation $\bar{q} = q, \bar{p} = p$ and the feedback $\bar{u}_1 = u_1 + q_1$, $\bar{u}_2 = u_2 + q_2 \cos q_1 + q_3 \sin q_1$ give

$$\begin{bmatrix} \dot{\bar{q}} \\ \dot{\bar{p}} \end{bmatrix} = \bar{J}(\bar{q},\bar{p}) \begin{bmatrix} \frac{\partial \bar{\mathcal{H}}}{\partial \bar{q}}^{\mathrm{T}} \\ \frac{\partial \bar{\mathcal{H}}}{\partial \bar{p}}^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0_{3\times 2} \\ \bar{G}_2(\bar{q},\bar{p},t) \end{bmatrix} u \qquad \text{(III.8)}$$

with quadratic positive definite $\bar{\mathcal{H}}(\bar{q},\bar{p}) = (1/2)(\bar{q}^{\mathrm{T}}\bar{q} + \bar{p}^{\mathrm{T}}\bar{p})$, where $\bar{J}(\bar{q},\bar{p},t) = J(q,p,t)|_{(q,p)=(\bar{q},\bar{p})}$. A stabilizing feedback is $u_1 = -q_1 - p_1, u_2 = -p_2 - q_2 \cos q_1 - q_3 \sin q_1$.

Now, let's take

$$\bar{J} = \left[\begin{array}{c|cc} 0_{3\times 3} & \multicolumn{2}{c}{0_{3\times 2}} \\ \hline 0_{2\times 3} & 0 & 1 \\ & -1 & 0 \end{array}\right].$$

Then a change of coordinates exists if and only if it satisfies the following system of partial differential equations

$$\begin{cases} p_1 \dfrac{\partial \Phi_j}{\partial q_1} + p_2 \cos q_1 \dfrac{\partial \Phi_j}{\partial q_2} + p_2 \sin q_1 \dfrac{\partial \Phi_j}{\partial q_3} + \dfrac{\partial \Phi_j}{\partial t} = 0, \\ p_1 \dfrac{\partial \Phi_4}{\partial q_1} + p_2 \cos q_1 \dfrac{\partial \Phi_4}{\partial q_2} + p_2 \sin q_1 \dfrac{\partial \Phi_4}{\partial q_3} + \dfrac{\partial \Phi_4}{\partial t} = \Phi_5, \\ p_1 \dfrac{\partial \Phi_5}{\partial q_1} + p_2 \cos q_1 \dfrac{\partial \Phi_5}{\partial q_2} + p_2 \sin q_1 \dfrac{\partial \Phi_5}{\partial q_3} + \dfrac{\partial \Phi_5}{\partial t} = -\Phi_4, \end{cases}$$

for $1 \le j \le 3$. A local solution of this system is given by

$$\begin{aligned} \Phi_1(\cdot) &= \cos(p_1 t - q_1) \\ \Phi_2(\cdot) &= q_2 \sin(p_1 t - q_1) + q_3 \cos(p_1 t - q_1) + p_2 \tfrac{-1+\cos p_1 t}{p_1} \\ \Phi_3(\cdot) &= q_2 \cos(p_1 t - q_1) - q_3 \sin(p_1 t - q_1) - p_2 \tfrac{\sin p_1 t}{p_1} \\ \Phi_4(\cdot) &= p_1 \cos t + p_2 \sin t \\ \Phi_5(\cdot) &= -p_1 \sin t + p_2 \cos t \end{aligned}$$

though this solution remains unbounded when $t \longmapsto \infty$. Changing the structure of the matrix and adding feedback might yield bounded solutions but solving the corresponding partial differential equations would not be quite simple. A software might be required or the use of Taylor expansions (were used to find this solution and would be in consideration for future work). Aiming for a nonconstant matrix, J can be simplified further, yet yielding a quadratic positive definite Hamiltonian via a locally time-independent transformation. Indeed, we check that the following change of coordinates

$$\begin{aligned} \bar{q}_1 &= \Phi_1(q,p) = \tan q_1, \\ \bar{q}_2 &= \Phi_2(q,p) = q_2, \qquad \bar{p}_1 = \Phi_4(q,p) = p_1 \sec^2 q_1, \\ \bar{q}_3 &= \Phi_3(q,p) = q_3, \qquad \bar{p}_2 = \Phi_5(q,p) = p_2 \cos q_1, \end{aligned}$$

and feedback

$$\begin{aligned} \bar{u}_1 &= u_1 + 2p_1 \tan q_1 + \sin q_1 \cos q_1 \\ \bar{u}_2 &= u_2 - p_1 p_2 \tan q_1 + \frac{q_2 \cos q_1 + q_3 \sin q_1}{\cos^2 q_1} \end{aligned}$$

take the original system into (III.8) with Hamiltonian $\bar{\mathcal{H}}(\bar{q},\bar{p}) = (1/2)(\bar{q}^{\mathrm{T}}\bar{q} + \bar{p}^{\mathrm{T}}\bar{p})$ and

$$\bar{G}_2(\bar{q},\bar{p},t) = \begin{bmatrix} 1 + \tan^2 q_1 & 0 \\ 0 & \cos q_1 \end{bmatrix}.$$

The new Hamiltonian system as well as the transformation are time-independent. There are other choices of $\bar{J}$ that yield different transformations (I.2) having all the common property of producing a quadratic positive definite Hamiltonian. ▷

Conclusion

A straightforward approach was proposed to achieve a quadratic positive definite Hamiltonian via generalized transformations. The PDEs obtained generalize the matching conditions and their solvability will be discussed in future work using power series obtained for the straightening theorem.

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