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SMALL EXTENSIONS OF WITT RINGS

Robert W. Fitzgerald

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We consider certain Witt ring extensions S of a noetherian Witt ring R obtained by adding one new generator. The conditions on the new generator are those known to hold when R is the Witt ring of a field F, S is the Witt ring of a field K and K/F is an odd degree extension. We show that if R is of elementary type then so is S.

The elementary type conjecture is a proposed classification of noetherian Witt rings. A potential source of counter-examples is as follows: start with a field F where WF is known (necessarily of elementary type) then look at noetherian WK for extension fields K of F. Jacob and Ware [3] have shown that WK is again of elementary type when $[K:F]=2$. Here we look at the simplest case of odd degree extensions, again showing WK is of elementary type. We note that WF is noetherian iff $G(F)\equiv F^{\cdot}/F^{\cdot 2}$ is finite. Also when K/F has odd degree then $G(F)\cong F^{\cdot}K^{\cdot 2}/K^{\cdot 2}$ embeds into $G(K)$.

We will in fact work with abstract Witt rings R (as defined by Marshall [4]) with associated group of one dimensional forms $G(R)$. The small extensions considered here are as follows. Let H be a subgroup of $G(R)$. We say a Witt ring S is an H-extension of R if there exists an $\alpha\in G(S)$ such that:

(1) $G(S)=\{1,\alpha\}G(R)$, and
(2) For all $x\in G(R)$ we have:

$$D_S\langle 1,-x\rangle=\begin{cases} D_R\langle 1,-x\rangle, & \text{if } x\notin H\\ \{1,\alpha\}D_R\langle 1,-x\rangle & \text{if } x\in H\end{cases}$$
$$D_S\langle 1,-\alpha x\rangle=\{1,-\alpha x\}\left(D_R\langle 1,-x\rangle\cap H\right).$$

These conditions hold for $R=WF, S=WK$ when K/F is an odd degree extension and $[G(K):G(F)]=2$ by [2,4.7] (we note that [2,4.7] should include the condition that $N_{K/F}(a)=1$). No such field extensions are known. However, there are many examples of H-extensions of abstract Witt rings, which we determine inductively. This can be viewed as a first step in classifying extensions of noetherian Witt rings. It also helps the search for odd degree extensions K/F with $[G(K):G(F)]=2$, while lessening the motivation for such a search.

For any group H, $H^{\cdot}$ denotes $H\setminus\{1\}$. The quaternionic mapping associated to R will be denoted by q. For $x\in G(R)$, $Q(x)=\{q(x,y):y\in G(R)\}$ and for a subgroup H,

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$Q(H) = \{q(h, y) : h \in H, y \in G(R)\}$. The value set of $\langle 1, -x\rangle$ is $D\langle 1, -x\rangle = \{y \in G(R) : q(x, y) = 1\}$. We will often work with several Witt rings at once and write $q_R, Q_R(x)$ and $D_R\langle 1, -x\rangle$ to indicate these objects for R.

R is of local type if $|q(G(R), G(R))| = 2$. We let E_n denote the elementary 2-group of order n. The group ring $R[E_n]$ is again a Witt ring. An element $t \in G(R)$ is rigid if $D\langle 1, t\rangle = \{1, t\}$ and t is birigid if both t and $-t$ are rigid. The basic part of R, $B(R)$, consists of ± 1 and all $x \in G(R)$ with either x or $-x$ not rigid. $B(R)$ is a subgroup of $G(R)$ and $R = R_0[G(R)/B(R)]$, where R_0 is the Witt ring generated by $B(R)$. We express this last statement by writing $R_0 = W(B(R))$.

The product in the category of Witt rings is:

$$R_1 \sqcap R_2 = \{(r_1, r_2) : r_i \in R_i \quad \text{and} \quad \dim r_1 \equiv \dim r_2 \pmod 2\}$$

If $R = R_1 \sqcap R_2$ then $G(R) = G(R_1) \times G(R_2)$ and:

$$D_R\langle (1, 1), (x, y)\rangle = D_{R_1}\langle 1, x\rangle \times D_{R_2}\langle 1, y\rangle.$$

The radical of R is $\mathrm{rad}(R) = \{x \in G(R) : D\langle 1, -x\rangle = G(R)\}$. We say R is degenerate if $\mathrm{rad}(R) \neq 1$ and totally degenerate if $\mathrm{rad}(R) = G(R)$. D_n denotes a totally degenerate Witt ring with square class group of order 2^n. There are two possibilities for D_n depending on whether -1 is a square or not. Specifically, D_n is either a product of n copies of $(\mathbb{Z}/2\mathbb{Z})[E_1]$ or n copies of $\mathbb{Z}/4\mathbb{Z}$. If R is degenerate then there exist uniquely determined n and non-degenerate Witt ring R_0 such that $R = D_n \sqcap R_0$. R_0 is the non-degenerate part of R.

R is of elementary type if it can be built from $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$ and Witt rings of local type by a succession of group ring extensions (for some E_n) and products. The elementary type conjecture is that every noetherian Witt ring is of elementary type.

1. Group ring extensions.

Lemma 1.1. *Let S be an H-extension of R.*

(a) *If $H = 1$ then $S = R[E_1]$, with E_1 generated by α.*
(b) *If $H = G(R)$ then $S = D_1 \sqcap R$, with D_1 generated by α.*

Proof. Suppose first that $H = 1$. Then for all $g \in G(R)$ we have from the definition of an H-extension that $D_S\langle 1, -\alpha g\rangle = \{1, -\alpha g\}(D_R\langle 1, -g\rangle \cap H)$ and so αg is birigid. Thus $B(S) \subset G(R)$ and $G(S) = \{1, \alpha\}G(R)$. So by[4,5.19] $S = R[E_1]$, where E_1 is generated by α.

Next suppose that $H = G(R)$. Then $D_S\langle 1, -\alpha\rangle = \{1, -\alpha\}(D_R\langle 1, -1\rangle \cap H) = G(S)$. Hence $\alpha \in \mathrm{rad}(S)$. Then by [4,pp. 105–106] $S = D_1 \sqcap R$, where D_1 is generated by α. □

When $H \neq 1$, which we will often assume in light of (1.1), we use the following notation

(recall that $B(R)$ is the basic part of R):

$$\begin{aligned} T &= \bigcup_{h\in H^{\cdot}} D_R\langle 1,-h\rangle \\ T_0 &= \bigcup_{h\in H^{\cdot}} (D_R\langle 1,-h\rangle \setminus \{-h\}) \\ B(H) &= H\cap B(R) \\ BT &= \bigcup_{h\in B(H)^{\cdot}} D_R\langle 1,-h\rangle \\ BT_0 &= \bigcup_{h\in B(H)^{\cdot}} (D_R\langle 1,-h\rangle \setminus \{-h\}) \end{aligned}$$

Lemma 1.2. *Let S be an H-extension of R with $|H|>1$. Then:*

(a) $\pm T_0 \subset B(R) \subset \pm T$,
(b) $B(S) = \pm\{1,\alpha\}T$.

Proof. First note that $D_S\langle 1,-\alpha\rangle = \{1,-\alpha\}H$, and $|H|>1$ imply $\alpha\in B(S)$. If $x\in G(R)\setminus \pm T$ then $D_R\langle 1,\pm x\rangle \cap H = \{1\}$ so by the definition of H-extensions, $D_S\langle 1,\pm\alpha x\rangle = \{1,\pm\alpha x\}$. Hence $\alpha x\notin B(S)$, and as $\alpha\in B(S)$, $x\notin B(S)$. That is, x is birigid in S, and so also in R. Thus $x\notin B(R)$ and we have:

$$\begin{aligned} &B(R)\subset \pm T, \\ &B(S)\subset \pm\{1,\alpha\}T \end{aligned}$$

Let $x\in T_0, x\neq -1$ so that for some $h\in H^{\cdot}, x\in D_R\langle 1,-h\rangle$ and $x\neq -h$. Then $D_R\langle 1,-x\rangle$ contains $-x,h$ which are distinct and not equal to 1. So $-x\in B(R)$ and also $x\in B(R)$. If $x=-1$ then again $x\in B(R)$. Thus $T_0\subset B(R)$, and so $\pm T_0\subset B(R)$ completing the proof of (a).

If $x\in T$ with $x\in D_R\langle 1,-h\rangle, h\in H^{\cdot}$ then $D_S\langle 1,-x\rangle = \{1,-\alpha x\}\,(D_R\langle 1,-x\rangle\cap H)$ contains $\{1,-\alpha x,h,-\alpha xh\}$. Thus $-\alpha x\in B(S)$. Again $\alpha\in B(S)$ so $x\in B(S)$. This shows $\pm\{1,\alpha\}T\subset B(S)$, completing the proof of (b). □

Lemma 1.3. *If S is an H-extension of R with $|H|>1$ then $B(R)=\pm BT$ and either:*

(a) *$B(R)=\pm B(H)$ and $B(S)=\pm\{1,\alpha\}H$, or*
(b) *$B(R)=\pm T$ and $B(S)=\{1,\alpha\}B(R)$.*

Proof. If $h\in H\setminus B(H)$ then $D_R\langle 1,-h\rangle = \{1,-h\}$. So:

$$\pm T = \pm BT \cup \pm(H\setminus B(H)). \tag{1.4}$$

Now $B(R)\subset \pm T$ by (1.2)(a) and $B(R)\cap \pm(H\setminus B(H))=\emptyset$ so $B(R)\subset \pm BT$. Conversely,

$$\pm BT = \pm(BT\cup B(H)) \subset \pm T_0 \cup \pm B(H) \subset B(R),$$

which proves the first statement.

Now (1.4) gives:

$$\pm T = \pm BT \cup \pm H = B(R) \cup \pm H,$$

and (1.2)(b) gives:

$$B(S) = \pm\{1,\alpha\}T = \{1,\alpha\}B(R) \cup \pm\{1,\alpha\}H.$$

This expresses the group $B(S)$ as the union of two subgroups, hence either:

(i) $\{1,\alpha\}B(R) \subset \pm\{1,\alpha\}H$, or
(ii) $\pm\{1,\alpha\}H \subset \{1,\alpha\}B(R)$.

In case (i) $B(S) = \pm\{1,\alpha\}H$ and $B(R) \subset \pm\{1,\alpha\}H \cap G(R) = \pm H$. Hence $B(R) = \pm B(H)$. In case (ii) $B(S) = \{1,\alpha\}B(R)$ and $H \subset B(R)$. Then $H = B(H), BT = T$ and by the first statement $B(R) = \pm T$. $\square$

Recall that any Witt ring R can be written as $R_0[G(R)/B(R)]$, where $R_0 = W(B(R))$, the Witt ring generated by $B(R)$. See [4,Chapter 5, Section 7] for details.

Proposition 1.5. *Let $R = R_0[E_n]$, with R_0 basic. Let S be an H-extension of R. Then:*

(a) *If $H = 1$ then $S = R_0[E_{n+1}]$.*
(b) *If $|H| > 1$ and $H \subset G(R_0)$ then $S = S_0[E_n]$, for some Witt ring S_0 that is an H-extension of R_0 (with the same α).*
(c) *If $H \not\subset G(R_0)$ and $-1 \in H$ then $S = (D_1 \sqcap R_0[H/B(H)])\,[G(R)/H]$, and $G(R_0) \subset H$.*
(d) *If $H \not\subset G(R_0)$ and $-1 \notin H$ then $S = (\mathbb{Z} \sqcap R_0[H/B(H)])\,[G(R)/\pm H]$ and $G(R_0) \subset \pm H$.*

Proof. If $H = 1$ then $S = R[E_1]$ by (1.1), which gives (a). So assume $|H| > 1$. Further suppose that $H \subset G(R_0)$ so that $B(H) = H \cap B(R) = H \cap G(R_0) = H$. Then $-H^{\cdot} \subset T \subset G(R_0)$. Hence $\pm H \subset \pm T \subset G(R_0)$. Thus if (1.3)(a) holds, so that $G(R_0) = B(R) = \pm B(H) = \pm H$, then $B(R) = \pm T$ also. So we are always in case (b) of (1.3). Then, since $G(S)/B(S) \cong G(R)/B(R)$, we have $S = S_0[E_n]$, where $S_0 = W(B(S))$. From $H \subset B(R)$ we have that S_0 is an H-extension of R_0.

Next suppose that $H \not\subset G(R_0)$. We still have that $G(R_0) = B(R)$. If $B(R) = \pm T$ then $H \subset B(R)$, contrary to our assumption. Thus we are in Case (a) of (1.3). First say that $-1 \in H$. Note that $-1 \in H \cap B(R) = B(H)$ also. Then $B(R) = B(H)$ and $B(S) = \{1,\alpha\}H$. Thus $S = S_0[G(R)/H]$, for $S_0 = W(\{1,\alpha\}H)$, since $G(S)/B(S) \cong G(R)/H$.

Now $D_{S_0}\langle 1,-\alpha\rangle = D_S\langle 1,-a\rangle \cap \{1,\alpha\}H = \{1,\alpha\}H$. Hence $\alpha \in \mathrm{rad}(S_0)$. Write $S_0 = D_1 \sqcap S_1$, for some Witt ring S_1 and with D_1, generated by α, being $\mathbb{Z}_2[E_1]$ or $\mathbb{Z}_4$ using [4, p. 104]. Note that $S_1 = W(H)$.

If $h \in H \setminus B(H)$ then $D_R\langle 1,-h\rangle = \{1,-h\}$ and $D_S\langle 1,-h\rangle = \{1,\alpha,-h,-\alpha h\}$ so that $D_{S_1}\langle 1,-h\rangle = D_S\langle 1,-h\rangle \cap H = \{1,-h\}$. Similarly, $D_{S_1}\langle h\rangle = \{1,h\}$. And if $h \in B(H)$ then $D_{S_1}\langle 1,-h\rangle = \{1,\alpha\}D_R\langle 1,-h\rangle \cap H = D_R\langle 1,-h\rangle \cap H = D_R\langle 1,-h\rangle \cap B(H) = D_{R_0}\langle 1,-h\rangle$. Thus $S_1 = R_0[H/B(H)]$.

We still suppose $H \not\subset G(R_0)$, so that we are in Case (a) of (1.3), and now say that $-1 \notin H$. Then $S = S_0[G(R)/\pm H]$ as $G(S)/B(S) = \{1,\alpha\}G(R)/\pm\{1,\alpha\}H \cong G(R)/\pm H$. Here $S_0 = W(\pm\{1,\alpha\}H)$.

Now $D_{S_0}\langle 1,-\alpha\rangle = \{1,\alpha\}H$ has index two in $G(S_0) = \pm\{1,\alpha\}H$. Further $\alpha \notin D_S\langle 1,-\alpha\rangle$ else $-1 \in \{1,-\alpha\}H$ and $-1 \in H$. Thus we have an orthogonal decomposition in the sense of [1]:

$$G(S_0) = \{1,\alpha\} \perp D_S\langle 1,-\alpha\rangle. \tag{1.6}$$

Set $S_1 = W(\{1,\alpha\})$ and $S_2 = W(D_S\langle 1,-\alpha\rangle)$. Note that $S_1 = \mathbb{Z}$ as $-1 \notin D_S\langle 1,-\alpha\rangle$. If $h \in \pm H \setminus \pm B(H)$ then $D_R\langle 1,\pm h\rangle = \{1,\pm h\}, D_S\langle 1,-h\rangle = \{1,\alpha,-h,-\alpha h\}$ and $D_S\langle 1,h\rangle = \{1,h\}$. Thus we have $D_{S_1}\langle 1,\pm h\rangle = D_S\langle 1,\pm h\rangle \cap \pm H = \{1,\pm h\}$. So $S_2 = S_3[H/B(H)]$, for some Witt ring S_3. S_2 is indeed a group ring as $H \not\subset G(R_0)$ implies $H \neq B(H)$.

We wish to apply [1,3.4] and deduce that the decomposition (1.6) yields a product of Witt rings. First we need to handle the case where S_2 is decomposable, that is, $S_2 = \mathbb{Z}[E_1]$. In this case $|G(S_2)| = |D_S\langle 1,-\alpha\rangle| = 4$ so that $H = \{1,t\}$, for some $t \notin G(R_0)$ and $G(R_0) = \{\pm 1\}$ as $G(R_0) \subset \pm H$. Now $D_S\langle 1,-t\rangle = \{1,\alpha,-t,-\alpha t\}$ so we consider instead the orthogonal decomposition:

$$\{1,t\} \perp D_S\langle 1,-t\rangle.$$

Now $Q_S(\{1,t\}) = \{1,q(t,-1)\}$ and $Q_S(D_S\langle 1,-t\rangle) = \{1,q(\alpha,-1),q(-t,-1),q(-\alpha t,-1)\}$, using $q(\alpha,-t) = q(\alpha,-1)$ and $q(t,-\alpha t) = 1$. Then $Q_S(\{1,t\}) \cap Q_S(D_S\langle 1,-t\rangle) = 1$. Thus:

$$S_0 = W(\{1,t\}) \sqcap W(D_S\langle 1,-t\rangle).$$

Now $W(\{1,t\}) = \mathbb{Z}$ as $-1 \notin D_S\langle 1,-t\rangle$ and $W(D_S\langle 1,-t\rangle) = R_0[\{1,\alpha\}]$ since α is birigid in $qfst$ and $G(R_0) = \{\pm 1\}$. This gives the desired result of (d).

We now apply [1,3.4] and obtain that either (1.6) yoilds a product or $Q_S(\{1,\alpha\}) = Q_S(D_S\langle 1,-\alpha\rangle)$. But if $t \in H \setminus G(R_0)$ then $q(t,t) = q(t,-1)$ and $q(\alpha,\alpha) = q(\alpha,-1)$ are distinct, since $-1 \notin D_S\langle 1,-\alpha t\rangle$. Hence the decomposition (1.6) in fact yields the product $S_0 = S_1 \sqcap S_2$. We have already seen that $S_1 = \mathbb{Z}$ and that $S_2 = S_3[H/B(H)]$. If $h \in \pm B(H)$ then $D_{S_3}\langle 1,-h\rangle = D_{S_1}\langle 1,-h\rangle = D_R\langle 1,-h\rangle = D_{R_0}\langle 1,-h\rangle$. So $S_3 = R_0$.

□

2. Local type rings.

Notation. For a subset $A \subset G(R)$ set:

$$C_R(A) = \bigcap_{a\in A} D_R\langle 1,-a\rangle.$$

Lemma 2.1. *Suppose that S is an H-extension of R. Suppose $k \in C_R(H) \setminus H$. Then $Q_R(H) \cap Q_R(k) = 1$.*

Proof. Let $\rho \in Q_R(H) \cap Q_R(k)$ so that $\rho = q(h,x) = q(k,y)$ with $h \in H$, and $x,y \in G(R)$. Since $H \subset D_S\langle 1,-\alpha\rangle$ we have that $q(k,y) = q(\alpha x,h)$. By linkage there exists a $t \in G(S)$ such that:

$$q(k,y) = q(k,t) = q(\alpha x,t) = q(\alpha x,h).$$

The first equality gives $ty \in D_S\langle 1, -k\rangle = D_R\langle 1, -k\rangle \subset G(R)$, since $k \notin H$. Hence $t \in G(R)$. The second equality gives:

$$t \in D_S\langle 1, -\alpha xk\rangle \cap G(R) = D_R\langle 1, -xk\rangle \cap H.$$

This implies $t \in D_R\langle 1, -x\rangle$ since $H \subset D_R\langle 1, -k\rangle$. The third equality gives:

$$ht \in D_S\langle 1, -\alpha x\rangle \cap G(R) = D_R\langle 1, -x\rangle \cap H.$$

Hence $h \in D_R\langle 1, -x\rangle$ and $\rho = q(x, h) = 1$. $\square$

The small Witt rings of local type will often be treated separately. The only local type Witt ring with two generators is $\mathbb{Z}$. There are two Witt rings of local type on four generators and both are group rings. If L is local type and $|G(L)| \geq 8$ then L is not a group ring. See [4,Chapter 5, Section 3] for details.

Lemma 2.2. *Suppose $R = L \sqcap R_2$, with L a Witt ring of local type and $|G(L)| \geq 8$. Let π_1 be the projection map of $G(R)$ onto $G(L)$. Let S be an H-extension of R. Then $\pi_1(H) = 1$ or $G(L)$.*

Proof. Set $B = \pi_1(H)$ and write $Q(L) = \{1, \rho\}$. Suppose that $B \neq 1$. If $(u, v) \in H$ with $u \neq 1$ then pick $r \in G(L) \setminus D_L\langle 1, -u\rangle$. We get $q((u, v), (r, 1)) = (\rho, 1)$ and so $(\rho, 1) \in Q(H)$.

Now $H \subset B \times G(R_2)$ so that $C_L(B) \times 1 = C_R(B \times G(R_2)) \subset C_R(H)$. If $C_L(B) = 1$ then $B = G(L)$ and we are done. So suppose there exists $1 \neq z \in C_L(B)$. Set $h = (z, 1) \in C_R(H)$. Then $Q(h) = \{1, (\rho, 1)\} \subset Q(H)$. By (2.1) we must have $h \in H$ and so $C_L(B) \times 1 \subset H$. In particular, $C_L(B) \subset B$.

Continue to let $h = (z, 1)$ where $1 \neq z \in C_L(B)$. We claim there exist an $h_1 \in H$ and an $x \in D_R\langle 1, -h\rangle$ such that $q(x, h_1) = (\rho, 1)$. Suppose not. We consider any $x = (a, 1)$ with $a \in D_L\langle 1, -z\rangle$. Then $q(x, (u, v)) = (\rho, 1)$ unless $a \in D_L\langle 1, -u\rangle$. Thus $D_L\langle 1, -z\rangle \subset D_L\langle 1, -u\rangle$ for all $(u, v) \in H$, that is, $D_L\langle 1, -z\rangle \subset C_L(B)$. Then $C_L(B) = D_L\langle 1, -z\rangle$ and so $B = \{1, z\}$. But $C_L(B) \subset B$ so that $D_L\langle 1, -z\rangle \subset \{1, z\}$ and $|G(L)| = 4$, a case we are excluding.

Thus there does exist an $h_1 \in H$ and an $x \in D_R\langle 1, -h\rangle$ such that $q(x, h_1) = (\rho, 1)$. Then in S we have $q(\alpha x, h_1) = (\rho, 1)$ and so $Q_S(h) \subset Q_S(\alpha x)$. We obtain:

$$|Q_S(\alpha x) \cap Q_S(h)| = 2.$$

On the other hand:

$$D_S\langle 1, -\alpha x\rangle \cap D_S\langle 1, -h\rangle = \{1, -\alpha x\}\,(D_R\langle 1, -x\rangle \cap H) \cap \{1, \alpha\}D_R\langle 1, -h\rangle.$$

Here $D_R\langle 1, -x\rangle \cap H \subset D_R\langle 1, -h\rangle$ as $h \in C_R(H)$. Also, by construction, x is an element of $D_R\langle 1, -h\rangle$ and $h \in C_R(H) \subset D_R\langle 1, -h\rangle$. So $-1, x \in D_R\langle 1, -h\rangle$ and $-\alpha x \in \alpha D_R\langle 1, -h\rangle$. Thus $D_S\langle 1, -\alpha x\rangle \subset D_S\langle 1, -h\rangle$. By [4,5.2]:

$$|Q_S(\alpha x) \cap Q_S(h)| = \frac{|D_S\langle 1, -\alpha xh\rangle|}{|D_S\langle 1, -\alpha x\rangle \cap D_S\langle 1, -h\rangle|} = \frac{2|D_R\langle 1, -xh\rangle \cap H|}{2|D_R\langle 1, -x\rangle \cap H|},$$

and $D_R\langle 1, -xh\rangle \cap H = D_R\langle 1, -x\rangle \cap H$ as $h \in C_R(H)$ implies $H \subset D_R\langle 1, -h\rangle$. So $|Q_S(\alpha x) \cap Q_S(h)| = 1$, a contradiction. Hence $\pi_1(H) = G(L)$. $\square$

Corollary 2.3. *Suppose R is of local type with $|G(R)| \neq 4$. If S is an H-extension of R then either:*

(a) *$H = 1$ and $S = R[E_1]$, with E_1 generated by α, or*
(b) *$H = G(R)$ and $S = D_1 \sqcap R$, with D_1 generated by α.*

Proof. If $|G(R)| = 2$ then it is clear that $H = 1$ or $H = G(R)$. If $|G(R)| \geq 8$ the take $R_2 = 1$ in (2.2) to get $H = 1$ or $G(R)$. Now apply (1.1). □

3. Products: General Lemmas.

We start with a lemma that may be of general interest.

Lemma 3.1. *Let R be a Witt ring of elementary type. Let K be a proper subgroup of $G(R)$ and let $y \in G(R)$. If*

$$G(R) = \bigcup_{k \in K} D\langle 1, -yk\rangle,$$

then $y \in rad(R) \cdot K$.

Proof. We first prove the result for non-degenerate R where we must show $y \in K$. It suffices to prove this for subgroups K of index two. Namely, if K_0 is any subgroup satisfying the hypothesis let $\mathcal{A}$ denote the set of subgroups K of index two that contain K_0. Then for any $K \in \mathcal{A}$:

$$G(R) = \bigcup_{k \in K_0} D\langle 1, -yk\rangle \subset \bigcup_{k \in K} D\langle 1, -yk\rangle.$$

Assuming the result holds for subgroups of index two, we obtain $y \in K$. Then $y \in \bigcap_{K \in \mathcal{A}} K = K_0$, as desired.

So suppose $[G(R) : K] = 2$. We work by induction on $|G(R)|$. We need to prove that $y \in K$ when R is of local type, a group ring or a product. First suppose R is of local type. Then $K = D\langle 1, -a\rangle$, for some $a \in G(R)$. We have:

$$G(R) = \bigcup_{k \in D\langle 1, -a\rangle} D\langle 1, -yk\rangle = D\langle 1, -y, ay\rangle.$$

Multiplying by $-a$ gives $G(R) = D\langle\langle -a, -y\rangle\rangle'$, the pure part of the Pfister form $\langle\langle -a, -y\rangle\rangle$. In particular, $-1 \in D\langle\langle -a, -y\rangle\rangle'$ so $\langle\langle -a, -y\rangle\rangle = 0$ and $y \in D\langle 1, -a\rangle = K$.

Next let $R = R_0[E_1]$, with $E_1 = \{1, t\}$ and $|G(R_0)| \geq 2$ (if $G(R_0) = 1$ then R is degenerate). Suppose $y \notin K$. We claim $G(R_0) \subset K$. Choose any $g \in G(R_0)$. Then there exist $k_1, k_2 \in K$ with $-gt \in D\langle 1, -yk_1\rangle$ and $-t \in D\langle 1, -yk_2\rangle$. We get $gt = yk_1$ and $t = yk_2$ since $y \notin K$. Hence $g = k_1k_2 \in K$. This proves the claim. Both $G(R_0)$ and K have index two so $K = G(R_0)$. From $t = yk_2$ we have $y \in tG(R_0)$. Pick $g \in G(R_0)^{\cdot}$. Then $g \in D\langle 1, -yk_3\rangle$ for some $k_3 \in K$. But $yk_3 \in yK = tG(R_0)$, so this is impossible. The contradiction implies $y \in K$.

Lastly, say $R = R_1 \sqcap R_2$. Write $y = (y_1, y_2)$. Now $K \cap (G(R_1) \times 1)$ is a subgroup of index at most two in $G(R_1) \times 1$. Let K_1 be its projection into $G(R_1)$. Similarly, let K_2

be the projection of $K \cap (1 \times G(R_2))$ into $G(R_2)$. Then $[G(R_i) : K_i] \leq 2$, for $i = 1, 2$. If $K_2 = G(R_2)$ then:

$$\bigcup_{k \in K} D_R\langle 1, -yk\rangle = \bigcup_{k_1 \in K_1} D\langle 1, -y_1k_1\rangle \times G(R_2)$$

so that $G(R_1) = \bigcup_{K_1} D\langle 1, -y_1k_1\rangle$. By induction then $y_1 \in K_1$ and hence $y = (y_1, y_2) \in K_1 \times G(R_2) = K$. In the same way, if $K_1 = G(R_1)$ then $y \in K$ as desired. So we may assume that $[G(R_i) : K_i] = 2$ for $i = 1, 2$. Write $K = \{1, \gamma\}K_1 \times K_2$. We have:

$$G(R) = \bigcup_{k_1 \in K_1, k_2 \in K_2} \left(D\langle 1, -y_1k_1\rangle \times D\langle 1, -y_2k_2\rangle \cup D\langle 1, -y_1\gamma_1k_1\rangle \times D\langle 1, -y_2\gamma_2k_2\rangle\right),$$

where $\gamma = (\gamma_1, \gamma_2)$.

Suppose $y_1 \in K_1$. If $G(R_1) = \cup D\langle 1, -y_1\gamma_1k_1\rangle$ then by induction we have $y_1\gamma_1 \in K_1$ and so $\gamma_1 \in K_1$. Then $K = K_1 \times G(R_2)$ and $K_2 = G(R_2)$ a case we have already covered. We may thus assume there exists a $g_1 \in G(R_1) \setminus \cup D\langle 1, -y_1\gamma_1k_1\rangle$. Then $g_1 \times G(R_2) \subset \cup(D\langle 1, -y_1k_1\rangle \times D\langle 1, -y_2k_2\rangle)$ and so $G(R_2) = \cup D\langle 1, -y_2k_2\rangle$. By induction $y_2 \in K_2$ and $y = (y_1, y_2) \in K_1 \times K_2 \subset K$, and we are done.

We may thus assume $y_1 \notin K_1$. Similarly, $y_2 \notin K_2$. Pick, for $i = 1, 2$, a $g_i \in G(R_i) \setminus \cup D\langle 1, -y_ik_i\rangle$, which is possible by induction. Then $g_1 \times G(R_2) \subset \cup(D\langle 1, -y_1\gamma_1k_1\rangle \times D\langle 1, -y_2\gamma_2k_2\rangle)$ and so $G(R_2) = \cup D\langle 1, -y_2\gamma_2k_2\rangle$. By induction once more, we have $y_2\gamma_2 \in K_2$. Similarly, $y_1\gamma_1 \in K_1$. Then $y \in (\gamma_1, \gamma_2)(K_1 \times K_2) \subset K$, as desired. This proves the result for non-degenerate R.

Now suppose R is degenerate. Write $R = D \sqcap R_2$, with $\mathrm{rad}(R) = G(D) \times 1$ and R_2 non-degenerate. Let π_2 be the projection of $G(R)$ onto $G(R_2)$. Set $K_2 = \pi_2(K)$ and write $y = (y_1, y_2)$, with $y_1 \in G(D)$ and $y_2 \in G(R_2)$. Our assumption is:

$$\begin{aligned} G(R) = G(D) \times G(R_2) &= \bigcup_{(k_1,k_2) \in K} D\langle (1, 1), -(y_1k_1, y_2k_2)\rangle \\ &= G(D) \times \Big(\bigcup_{k_2 \in K_2} D_{R_2}\langle 1, -y_2k_2\rangle\Big). \end{aligned}$$

From the non-degenerate case we get $y_2 \in K_2 = \pi_2(K)$. Thus there exists a $d \in G(D)$ such that $(d, y_2) \in K$. Hence $y = (y_1, y_2) = (dy_1, 1)(d, y_2) \in \mathrm{rad}(R) \cdot K$. $\square$

Our key reduction lemma follows.

Lemma 3.2. *Let $R = R_1 \sqcap R_2$ and suppose S is an H-extension of R. If $H = H_1 \times G(R_2)$ then there exists a Witt ring T that is an H_1-extension of R_1 such that $S \cong T \sqcap R_2$.*

Proof. We first construct T. Let $G(T)$ be a group containing $G(R_1)$ as a subgroup of index 2; write $G(T) = \{1, \beta\}G(R_1)$. Let $\varphi : G(T) \to \{1, \alpha\}(G(R_1) \times 1)$ be the isomorphism sending $g_1 \mapsto (g_1, 1)$ and $\beta g_1 \mapsto \alpha(g_1, 1)$, where $g_1 \in G(R_1)$. For $z \in G(T)$ define:

$$D_T\langle 1, -z\rangle = \varphi^{-1}(D_S\langle 1, -\varphi(z)\rangle \cap \{1, \alpha\}(G(R_1) \times 1)).$$

We check that T is an H_1-extension of R_1. If $z \in G(R_1) \setminus H_1$ then:

$$\begin{aligned} D_T\langle 1, -z\rangle &= \varphi^{-1}((D_{R_1}\langle 1, -z\rangle \times G(R_2)) \cap \{1, \alpha\}(G(R_1) \times 1)) \\ &= \varphi^{-1}(D_{R_1}\langle 1, -z\rangle \times 1) \\ &= D_{R_1}\langle 1, -z\rangle. \end{aligned}$$

If $z \in H_1$ then:

$$\begin{aligned} D_T\langle 1, -z\rangle &= \varphi^{-1}(\{1, \alpha\}(D_{R_1}\langle 1, -z\rangle \times G(R_2)) \cap \{1, \alpha\}(G(R_1) \times 1)) \\ &= \varphi^{-1}(\{1, \alpha\}(D_{R_1}\langle 1, -z\rangle \times 1)) \\ &= \{1, \beta\}D_{R_1}\langle 1, -z\rangle. \end{aligned}$$

Lastly, if $z \in G(R_1)$ then:

$$D_T\langle 1, -\beta z\rangle = \varphi^{-1}(D_S\langle 1, -\alpha(z, 1)\rangle \cap \{1, \alpha\}(G(R_1) \times 1)).$$

Now:

$$\begin{aligned} D_S\langle 1, -\alpha(z, 1)\rangle &= \{1, -\alpha(z, 1)\}\left((D_{R_1}\langle 1, -z\rangle \times G(R_2)) \cap (H_1 \times G(R_2))\right) \\ &= \{1, -\alpha(z, 1)\}\left((D_{R_1}\langle 1, -z\rangle \cap H_1) \times G(R_2)\right) \end{aligned}$$

Since $(1, -1) \in D_{R_1}\langle 1, -z\rangle \cap H_1 \times G(R_2)$ we have:

$$D_S\langle 1, -\alpha(z, 1)\rangle = \{1, \alpha(-z, 1)\}\left((D_{R_1}\langle 1, -z\rangle \cap H_1) \times G(R_2)\right).$$

Hence:

$$\begin{aligned} D_T\langle 1, -\beta z\rangle &= \varphi^{-1}(\{1, \alpha(-z, 1)\}(D_{R_1}\langle 1, -z\rangle \cap H_1) \times G(R_2)) \cap \{1, \alpha(-z, 1)\}(G(R_1) \times 1)) \\ &= \varphi^{-1}(\{1, \alpha(-z, 1)\}(D_{R_1}\langle 1, -z\rangle \cap H_1) \times 1) \\ &= \{1, -\beta z\}(D_{R_1}\langle 1, -z\rangle \cap H_1). \end{aligned}$$

We begin the verification that $(G(T), D_T)$ is linked, so that T is indeed a Witt ring. Let $t = \alpha^{\epsilon_1}(u, v) \in G(S)$ and let $\beta^{\epsilon_2}x, \beta^{\epsilon_3}y \in G(T)$ with each $\epsilon_i = 0$ or 1.

Claim. If $t \in \varphi(\beta^{\epsilon_3}y)D_S\langle 1, -\varphi(\beta^{\epsilon_2}x)\rangle$ then $\beta^{\epsilon_1}u \in \beta^{\epsilon_3}yD_T\langle 1, -\beta^{\epsilon_2}x\rangle$.

We first assume that $\epsilon_1 = 0$. We have four cases:

Case 1: $\epsilon_2 = 0, \epsilon_3 = 0$. Here $(uy, v) \in D_S\langle 1, -(x, 1)\rangle$, hence $uy \in D_{R_1}\langle 1, -x\rangle \subset D_T\langle 1, -x\rangle$.

Case 2: $\epsilon_2 = 0, \epsilon_3 = 1$. Here $\alpha(uy, v) \in D_S\langle 1, -(x, 1)\rangle$. We must have that $x \in H_1$ so $\alpha(uy, v) \in \{1, \alpha\}(D_{R_1}\langle 1, -x\rangle \times G(R_2))$. Then $uy \in D_{R_1}\langle 1, -x\rangle$. Thus $\beta uy \in D_T\langle 1, -x\rangle = \{1, \beta\}D_{R_1}\langle 1, -x\rangle$.

Case 3: $\epsilon_2 = 1, \epsilon_3 = 0$. Here:

$$\begin{aligned} (uy, v) &\in \{1, -\alpha(x, 1)\}((D_{R_1}\langle 1, -x\rangle \times G(R_2)) \cap H) \\ &= \{1, -\alpha(x, 1)\}((D_{R_1}\langle 1, -x\rangle \cap H_1) \times G(R_2)) \end{aligned}$$

Thus $uy \in D_{R_1}\langle 1, -x\rangle \cap H_1$. We obtain $uy \in D_T\langle 1, -\beta x\rangle = \{1, -\beta x\}(D_{R_1}\langle 1, -x\rangle \cap H_1)$.

Case 4: $\epsilon_2 = 1, \epsilon_3 = 1$. Here $\alpha(uy, v) \in \{1, -\alpha(x, 1)\}((D_{R_1}\langle 1, -x\rangle \cap H_1) \times G(R_2))$ so that $(-xuy, v) \in (D_{R_1}\langle 1, -x\rangle \cap H_1) \times G(R_2)$. Thus $-xuy \in D_{R_1}\langle 1, -x\rangle \cap H_1$ and $\beta uy \in D_T\langle 1, -\beta x\rangle = \{1, -\beta x\}(D_{R_1}\langle 1, -x\rangle \cap H_1)$.

The four cases with $\epsilon_1 = 1$ are identical to the four above cases. For example, if $\epsilon_1 = 1, \epsilon_2 = 0, \epsilon_3 = 0$ then we have $\alpha(uy, v) \in D_S\langle 1, -(x, 1)\rangle$, which is *Case 2* above. Thus the **Claim** is proven.

We can now check linkage in T. Let $x, y, z, w \in G(T)$ and suppose:

$$xD_T\langle 1, -y\rangle \cap D_T\langle 1, -yz\rangle \cap wD_T\langle 1, -z\rangle \neq \emptyset.$$

Apply φ to get:

$$\varphi(x)D_S\langle 1, -\varphi(y)\rangle \cap D_S\langle 1, -\varphi(yz)\rangle \cap \varphi(w)D_S\langle 1, -\varphi(z)\rangle \neq \emptyset.$$

By linkage on S, there exists a $t \in G(S)$ in:

$$\varphi(y)D_S\langle 1, -\varphi(x)\rangle \cap D_S\langle 1, -\varphi(xw)\rangle \cap \varphi(z)D_S\langle 1, -\varphi(w)\rangle.$$

Now apply the **Claim** :

$$yD_T\langle 1, -x\rangle \cap D_T\langle 1, -xw\rangle \cap zD_T\langle 1, -w\rangle \neq \emptyset,$$

as desired.

Lastly, set $W = T \sqcap R_2$. Then $G(W) = (\{1, \beta\}G(R_1)) \times G(R_2)$. Set $\gamma = (\beta, 1)$ so that $G(W) = \{1, \gamma\}(G(R_1) \times G(R_2)) = \{1, \gamma\}G(R)$. We will show W is an H-extension of R, via γ, and hence that $S \cong W$.

First let $h = (h_1, g_2)) \in H$, where $h_1 \in H_1 \subset G(R_1)$ and $g_2 \in G(R_2)$. Then:

$$\begin{aligned} D_W\langle 1, -h\rangle &= D_T\langle 1, -h_1\rangle \times D_{R_2}\langle 1, -g_2\rangle \\ &= (\{1, \beta\}D_{R_1}\langle 1, -h_1\rangle) \times D_{R_2}\langle 1, -g_2\rangle \\ &= \{1, \gamma\}D_R\langle 1, -h\rangle. \end{aligned}$$

Next let $g = (g_1, g_2) \in G(R) \setminus H$, with $g_1 \in G(R_1) \setminus H_1$ and $g_2 \in G(R_2)$. Then:

$$\begin{aligned} D_W\langle 1, -g\rangle &= D_T\langle 1, -g_1\rangle \times D_{R_2}\langle 1, -g_2\rangle \\ &= D_{R_1}\langle 1, -g_1\rangle \times D_{R_2}\langle 1, -g_2\rangle = D_R\langle 1, -g\rangle. \end{aligned}$$

Lastly, let $g = (g_1, g_2) \in G(R)$, with $g_1 \in G(R_1)$ and $g_2 \in G(R_2)$. Then:

$$\begin{aligned} D_W\langle 1, -\gamma g\rangle &= D_W\langle 1, -(\beta g_1, g_2)\rangle \\ &= D_{R_1}\langle 1, -\beta g_1\rangle \times D_{R_2}\langle 1, -g_2\rangle \\ &= \big(\{1, -\beta g_1\}(D_{R_1}\langle 1, -g_1\rangle \cap H_1)\big) \times D_{R_2}\langle 1, -g_2\rangle \\ &= (D_{R_1}\langle 1, -g_1\rangle \cap H_1) \times D_{R_2}\langle 1, -g_2\rangle \\ &\cup \gamma\big(-g_1(D_{R_1}\langle 1, -g_1\rangle \cap H_1) \times -g_2 D_{R_2}\langle 1, -g_2\rangle\big) \\ &= \{1, -\gamma(g_1, g_2)\}(D_R\langle 1, -(g_1, g_2)\rangle \cap H) \\ &= \{1, -\gamma g\}(D_R\langle 1, -g\rangle \cap H). \end{aligned}$$

Thus $W = T \sqcap R_2$ is an H-extension of R and so is isomorphic to S. $\square$

Our last general lemma is the most technical, but it also does most of the work.

Lemma 3.3. *Let $u \in G(R)$ and $h \in H$. Then:*

$$D_R\langle 1, uh, -h\rangle \setminus uH \subset \bigcup_{t \in h(D\langle 1,-u\rangle \cap H)} D_R\langle 1, -t\rangle.$$

Proof. Let $w \in D_R\langle 1, uh, -h\rangle \setminus uH$. Then $w \in D_R\langle 1, -hv\rangle$ for some $v \in D_R\langle 1, -u\rangle$ and $uw \notin H$. We have:

$$q(\alpha u, h) = q(u, h) = q(u, vh) = q(uw, vh).$$

Thus, by linkage, there exists a $t \in G(S)$ such that:

$$q(\alpha u, h) = q(\alpha u, t) = q(uw, t) = q(uw, vh).$$

The third equality gives $vht \in D_S\langle 1, -uw\rangle$. Since $uw \notin H$ this implies $t \in G(R)$. Then the first two equalities give:

$$\begin{aligned} ht &\in D_S\langle 1, -\alpha u\rangle \cap G(R) = D_R\langle 1, -u\rangle \cap H, \\ t &\in D_S\langle 1, -\alpha w\rangle \cap G(R) = D_R\langle 1, -w\rangle \cap H. \end{aligned}$$

Hence $w \in D_R\langle 1, -t\rangle$ where $t \in h(D_R\langle 1, -u\rangle \cap H)$. $\square$

4. Products: Degenerate Witt rings.

If R is a degenerate Witt ring then $R = D \sqcap R_2$, for some Witt ring R_2 and where $G(D) = \{1, d\}$, with $D_D\langle 1, 1\rangle = D_D\langle 1, d\rangle = \{1, d\}$. We will often use the fact that if $(u, v) \in G(R)$ then $D_R\langle 1, -(u, v)\rangle = D_R\langle 1, -(du, v)\rangle$.

Lemma 4.1. *Suppose $R = D \sqcap R_2$ is degenerate. Let π_1 be the projection of $G(R)$ onto $G(D)$. Let S be an H-extension of R. Then either $G(D) \times 1 \subset H$ or S is isomorphic to an H_0-extension of R, for some subgroup $H_0 \subset G(R)$ with $\pi_1(H_0) = 1$.*

Proof. Suppose $(d, 1) \notin H$ and that $\pi_1(H) \neq 1$, so that $(d, y) \in H$ for some $y \in G(R_2)$. Let G_2 be the subgroup of $G(R_2)$ such that $1 \times G_2 = H \cap (1 \times G(R_2))$. Then $H = (1 \times G_2) \cup (d \times yG_2)$. Set $H_0 = 1 \times \{1, y\}G_2$, and note that $\pi_1(H_0) = 1$.

Let $\beta^2 = 1$ and set $G(S_0) = \{1, \beta\}G(R_2)$. Define S_0-value set s so that S_0 is an H_0-extension of R. We wish to show $S \cong S_0$. Extend G_2 to a subgroup K of index two in $G(R_2)$ that does not contain y. Define $\varphi : G(S) \to G(S_0)$ by $\alpha \mapsto \beta$ and for $(u, v) \in G(R)$:

$$\varphi(u, v) = \begin{cases} (u, v), & \text{if } v \in K \\ (du, v), & \text{if } v \notin K. \end{cases}$$

It is quickly checked that φ is an isomorphism. We will show:

$$\varphi(D_S\langle 1, -s\rangle) = D_{S_0}\langle 1, -\varphi(s)\rangle, \tag{4.2}$$

for all $s \in G(S)$. This shows both that S_0 is a Witt ring and that $S \cong S_0$.

Claim. If $(u,v) \in G(R)$ then $\varphi(D_R\langle 1, -(u,v)\rangle) = D_R\langle 1, -(u,v)\rangle$.

$$\begin{aligned} D_R\langle 1, -(u,v)\rangle &= \{1,d\} \times D_{R_2}\langle 1, -v\rangle \\ &= 1 \times (D_{R_2}\langle 1, -v\rangle \cap K) \cup d \times (D_{R_2}\langle 1, -v\rangle \cap K) \\ &\cup 1 \times (D_{R_2}\langle 1, -v\rangle \cap yK) \cup d \times (D_{R_2}\langle 1, -v\rangle \cap yK). \end{aligned}$$

Thus:

$$\begin{aligned} \varphi(D_R\langle 1, -(u,v)\rangle &= 1 \times (D_{R_2}\langle 1, -v\rangle \cap K) \cup d \times (D_{R_2}\langle 1, -v\rangle \cap K) \\ &\cup d \times (D_{R_2}\langle 1, -v\rangle \cap yK) \cup 1 \times (D_{R_2}\langle 1, -v\rangle \cap yK) \\ &= D_R\langle 1, -(u,v)\rangle, \end{aligned}$$

proving the **Claim**.

We now check (4.2) in various cases. First suppose $s = (u,v) \in G(R)$, with $v \in K$. Then $s \in H$ iff $u = 1$ and $v \in G_2$. We have $\varphi(s) = s$ and $\varphi(s) \in H_0$ iff $u = 1$ and $v \in G_2$ iff $s \in H$. $D_S\langle 1, -s\rangle = \{1, \alpha\}D_R\langle 1, -s\rangle$ or $D_R\langle 1, -s\rangle$ depending on whether or not $s \in H$. So by the **Claim**, $\varphi(D_S\langle 1, -s\rangle) = \{1, \beta\}D_R\langle 1, -s\rangle$ or $D_R\langle 1, -s\rangle$ depending on whether or not $\varphi(s) \in H_0$. Thus $\varphi(D_S\langle 1, -s\rangle) = D_{S_0}\langle 1, -\varphi(s)\rangle$.

Next suppose that $s = (u,v) \in G(R)$ with $v \in yK$. Then $s \in H$ iff $u = d$ and $v \in yG_2$. We have $\varphi(s) = (du, v)$ so that $\varphi(s) \in H_0$ iff $u = d$ and $v \in yG_2$ iff $s \in H$. Again using the **Claim**:

$$D_S\langle 1, -s\rangle = \begin{cases} \{1, \alpha\}D_R\langle 1, -s\rangle, & \text{if } s \in H \\ D_R\langle 1, -s\rangle, & \text{if } s \notin H. \end{cases}$$

Thus:

$$\varphi(D_S\langle 1, -s\rangle) = \begin{cases} \{1, \beta\}D_R\langle 1, -s\rangle, & \text{if } \varphi(s) \in H_0 \\ D_R\langle 1, -s\rangle & \text{if } \varphi(s) \notin H_0. \end{cases}$$

Now $D_R\langle 1, -s\rangle = D_R\langle 1, -(u,v)\rangle = D_R\langle 1, -(du,v)\rangle = D_R\langle 1, -\varphi(s)\rangle$. Hence we have as desired that $\varphi(D_S\langle 1, -s\rangle) = D_{S_0}\langle 1, -\varphi(s)\rangle$.

Now suppose $s = \alpha(u,v) \in \alpha G(R)$. Then:

$$\begin{aligned} D_S\langle 1, -s\rangle &= \{1, -\alpha(u,v)\}(D_R\langle 1, -(u,v)\rangle \cap H) \\ &= \{1, -\alpha(u,v)\}[(\{1,d\} \times D_{R_2}\langle 1, -v\rangle) \cap (1 \times G_2 \cup d \times yG_2)] \\ &= \{1, -\alpha(u,v)\}[1 \times (D_{R_2}\langle 1, -v\rangle \cap G_2) \cup d \times (D_{R_2}\langle 1, -v\rangle \cap yG_2)] \end{aligned}$$

Now:

$$\begin{aligned} &\varphi((1 \times (D_{R_2}\langle 1, -v\rangle \cap G_2)) \cup (d \times (D_{R_2}\langle 1, -v\rangle \cap yG_2))) \\ &\qquad = 1 \times (D_{R_2}\langle 1, -v\rangle \cap G_2) \cup 1 \times (D_{R_2}\langle 1, -v\rangle \cap yG_2) \\ &\qquad = 1 \times (D_{R_2}\langle 1, -v\rangle \cap \{1, y\}G_2) \\ &\qquad = D_R\langle 1, -(u,v)\rangle \cap H_0. \end{aligned}$$

Thus if $v \in K$ then:

$$\begin{aligned}\varphi(D_S\langle 1,-s\rangle) =& \varphi(D_S\langle 1,-\alpha(u,v)\rangle \\ &= \{1,-\beta(u,v)\}(D_R\langle 1,-(u,v)\rangle \cap H_0) = D_{S_0}\langle 1,-\varphi(s)\rangle,\end{aligned}$$

verifying (4.2) in this case.

Lastly, if $v \in yK$ then:

$$\begin{aligned}\varphi(D_S\langle 1,-s\rangle) &= \varphi(D_S\langle 1,-\alpha(u,v)\rangle \\ &= \{1,-\beta(du,v)\}(D_R\langle 1,-(u,v)\rangle \cap H_0) \\ &= \{1,-\beta(du,v)\}(D_R\langle 1,-(du,v)\rangle \cap H_0) \\ &= D_{S_0}\langle 1,-\varphi(s)\rangle.\end{aligned}$$

Thus (4.2) holds in all cases. □

5. Products: Local type factors.

Lemma (3.3) looks simpler when one factor has local type.

Lemma 5.1. *Suppose $R = L \sqcap R_2$, with L of local type. Suppose S is an H-extension of R. Let $h = (h_1, h_2) \in H$ and $u = (u_1, u_2) \in G(R)$ such that $u_1 \notin D_L\langle 1,-h_1\rangle$ while $u_2 \in D_{R_2}\langle 1,-h_2\rangle$. Then:*

$$G(R) = uH \cup (u_1 \times G(R_2)) \cup \bigcup_{t \in h(D_R\langle 1,-u\rangle \cap H)} D_R\langle 1,-t\rangle.$$

Proof. Write $Q(L) = \{1,\rho\}$. Then $\langle\langle -u,-h\rangle\rangle = (\rho, 1)$. Hence $-D_R\langle\langle -u,-h\rangle\rangle' = \{(x,y) \in G(R) : x \neq 1\}$. Now $-u \cdot \langle -u,-h,uh\rangle \simeq \langle 1,uh,-h\rangle$. Thus $D_R\langle 1,uh,-h\rangle = \{(x,y) \in G(R) : x \neq u_1\}$. Apply (3.3). □

Lemma 5.2. *Suppose $R = L \sqcap R_2$, with L of local type. Let π_1 be the projection of $G(R)$ onto $G(L)$. Let S be an H-extension of R and suppose that $\pi_1(H) = G(L)$. Let $u = (u_1, u_2) \in G(R)$.*

(a) *If $|G(L)| \geq 4$ then $\pi_1(D_R\langle 1,-u\rangle \cap H) = D_L\langle 1,-u_1\rangle$.*
(b) *If $L = \mathbb{Z}$ and $\pi_1(D_R\langle 1,-u\rangle \cap H) \neq D_L\langle 1,-u_1\rangle$ for some u, then $H = 1 \times H_2 \cup -1 \times -H_2$, where $1 \times H_2 = H \cap (1 \times G(R_2))$ and H_2 is an ordering on R_2.*

Proof. Suppose that $\pi_1(D_R\langle 1,-u\rangle \cap H) = K < D_L\langle 1,-u_1\rangle$.
Claim. If $v \in G(L) \setminus K$ then $(v,-1) \in H$.

Since $\pi_1(H) = G(L)$ there exists a $w \in G(R_2)$ such that $(v,w) = h \in H$. Now:

$$\begin{aligned}D_R\langle 1,uh,-h\rangle &= -u(D_L\langle\langle -u,-v\rangle\rangle' \times D_{R_2}\langle\langle -u_2,-w\rangle\rangle') \\ &\supset u_1G(L)^{\cdot} \times u_2T,\end{aligned}$$

where $T = -D_{R_2}\langle\langle -u_2,-w\rangle\rangle'$. Also:

$$\bigcup_{t \in h(D_R\langle 1,-u\rangle \cap H)} D_R\langle 1,-t\rangle \subset \left(\bigcup_{k \in K} D_L\langle 1,-vk\rangle\right) \times G(R_2).$$

Now $v \notin K$ implies $G(L) \neq \cup D_L\langle 1, -vk\rangle$ by (3.1). Choose a $g \in G(L) \setminus \cup D_L\langle 1, -vk\rangle$.

We check that we may assume $g \neq u_1$. If $v \in D_L\langle 1, -u_1\rangle$ then we have $u_1 D_L\langle 1, -v\rangle = D_L\langle 1, -v\rangle \subset \cup D_L\langle 1, -vk\rangle$ and so no $g \in G(L) \setminus \cup D_L\langle 1, -vk\rangle$ is equal to u_1. If instead $v \notin D_L\langle 1, -u_1\rangle$ then, as $|K| < |D_L\langle 1, -u_1\rangle|$, there exists a $w \neq u_1$ such that $K \subset D_L\langle 1, -u_1\rangle \cap D_L\langle 1, -w\rangle$. If $v \notin D_L\langle 1, -w\rangle$ then w is not in any $D_L\langle 1, -vk\rangle$, for $k \in K$, and we may take $g = w$. If $v \in D_L\langle 1, -w\rangle$ then $v \notin D_L\langle 1, -u_1 w\rangle$ and we may take $g = u_1 w$.

We thus have $g \in u_1 G(L)^{\cdot} \setminus \cup D_L\langle 1, -vk\rangle$. So $g \times u_2 T \subset uH$ by (3.3). Hence $u_1 g \times T \subset H$. Then $(u_1 g, u_2), (u_1 g, -u_2 w) \in H$ and so $(1, -w) \in H$. We obtain that $(v, -1) = (v, w)(1, -w) \in H$ and the **Claim** is proven.

Now suppose $u_1 \neq 1$. Let $x \in G(L)^{\cdot}$. Since $|K| < |D_L\langle 1, -u_1\rangle|$ we have $|K| \leq \frac{1}{4}|G(L)|$. So we can choose $v \in G(L) \setminus \{1, x\}K$. Then $(v, -1)$ and $(vx, -1)$ are in H by the **Claim**. Hence $(x, 1) \in H$. This shows that $G(L) \times 1 \subset H$. But then $D_L\langle 1, -u_1\rangle \times 1 \subset D_R\langle 1, -u\rangle \cap H$ and $\pi_1(D_R\langle 1, -u\rangle \cap H) = D_L\langle 1, -u_1\rangle$, as desired.

Next suppose $u_1 = 1$ and $|G(L)| \geq 4$. We show $\pi_1(D_R\langle 1, -u\rangle \cap H) = G(L)$. Pick any $a \in G(L)$ and pick a $b \in G(L)^{\cdot}$ such that $a \in D_L\langle 1, -b\rangle$. This is possible since $|G(L)| \geq 4$ implies there are at least two b's such that $a \in D_L\langle 1, -b\rangle$. So there is such a b not equal to 1. Then, by the above paragraph, $\pi_1(D_R\langle 1, -(b, u_2)\rangle \cap H) = D_L\langle 1, -b\rangle$ contains a. Thus there exists a $k \in D_{R_2}\langle 1, -u_2\rangle$ such that $(a, k) \in H$. Thus $(a, k) \in D_R\langle 1, -(u_1, u_2)\rangle$, as $u_1 = 1$, and so $a \in \pi_1(D_R\langle 1, -u\rangle \cap H)$.

Lastly, suppose $u_1 = 1$ and $L = \mathbb{Z}$. Here $K = \{1\}$ and $v = -1$. The **Claim** shows that $(-1, -1) \in H$. Now $H_2 = H \cap (1 \times G(R_2))$ has index 2 in H since $1 \times G(R_2)$ has index 2 in $G(R)$ and $H \not\subset 1 \times G(R_2)$. Hence $H = 1 \times H_2 \cup -1 \times -H_2$. The last paragraph of the proof of the **Claim** gives $u_1 g \times T \subset H$, where $g \neq u_1$. Thus $g = -1$ and after multiplying by $-1 \in H$ we get:

$$1 \times D_{R_2}\langle\langle -u_2, -w\rangle\rangle' \subset H. \tag{5.3}$$

This holds for all $w \in G(R_2)$ such that $(-1, w) \in H$, that is, for all $w \in -H_2$. Thus for any $h_2 \in H_2$:

$$\begin{aligned} -u_2 D_{R_2}\langle 1, h_2\rangle &\subset D_{R_2}\langle -u_2, h_2, -u_2 h_2\rangle \subset H_2 \\ D_{R_2}\langle 1, h_2\rangle &\subset H_2. \end{aligned}$$

Thus H_2 is a preoredering. Also (5.3) holds for any $u_2 \in G(R_2)$ with $\pi_1(D_R\langle 1, -(1, u_2)\rangle) = 1$. That is, $D_R\langle 1, -(1, u_2)\rangle \cap (-1 \times -H_2) = \emptyset$ or equivalently, u_2 is not in $D_{R_2}\langle 1, h_2\rangle$ for any $h_2 \in H_2$. For such a u_2, (5.3) implies $-u_2 \in H_2$. Hence:

$$G(R_2) = -H_2 \cup \bigcup_{h_2 \in H_2} D_{R_2}\langle 1, h_2\rangle.$$

But H_2 is a preordering so that $\cup D_{R_2}\langle 1, h_2\rangle \subset H_2$. Thus $G(R_2) = -H_2 \cup H_2$, H_2 has index 2 in $G(R_2)$ and so H_2 is an ordering. $\square$

Notation. Suppose $R = R_1 \sqcap R_2$ and that H is a subgroup of $G(R)$. For $x \in G(R_1)$ set $F(x) = \{y \in G(R_2) : (x, y) \in H\}$.

Lemma 5.4. *Let $R = R_1 \sqcap R_2$ and let S be an H-extension of R. Let π_1 be the projection of $G(R)$ onto $G(R_1)$ and suppose $\pi_1(H) = G(R_1)$. Then for all $x \in G(R_1)$:*

(a) *$F(x)$ is non-empty.*
(b) *$F(1)$ is a subgroup of $G(R_1)$.*
(c) *$F(x)$ is a coset of $F(1)$.*

Proof. No $F(x)$ is empty since $\pi_1(H) = G(R_1)$. Clearly $F(1)$ is a subgroup. Fix $y_0 \in F(x)$. If $y \in F(1)$ then $(1, y), (x, y_0) \in H$ implies $(x, yy_0) \in H$ and so $yy_0 \in F(x)$. This says $y_0F(1) \subset F(x)$.

Now let $y \in F(x)$. Then $(x, y_0), (x, y) \in H$ so that $(1, yy_0) \in H$. Hence $yy_0 \in F(1)$ and we have the reverse inclusion $F(x) \subset y_0F(1)$. □

Lemma 5.5. *Let $R = R_1 \sqcap R_2$ be of elementary type. Let S be an H-extension of R. Let π_1 be the projection of $G(R)$ onto $G(R_1)$. Suppose the following:*

(1) *$\pi_1(H) = G(R_1)$.*
(2) *$F(a) \cap \mathrm{rad}(R_2) \subset \{1\}$, for all $a \in G(R_1)$.*
(3) *For all $u = (u_1, u_2) \in G(R)$ we have $\pi_1(D_R\langle 1, -u\rangle \cap H) = D_{R_1}\langle 1, -u_1\rangle$.*

Then $H = G(R_1) \times H_2$, for some subgroup $H_2 \subset G(R_2)$.

Proof. Let $a \in G(R_1)$. We will first show that:

$$G(R_2) = \bigcup_{k \in F(a)} D_{R_2}\langle 1, -k\rangle. \tag{5.6}$$

Pick any $b \in D_{R_1}\langle 1, -a\rangle$ and any $g \in G(R_2)$. Then $a \in D_{R_1}\langle 1, -b\rangle = \pi_1(D_R\langle 1, -(b, g)\rangle \cap H)$ by assumption (3). Hence there exists a $k \in D_{R_2}\langle 1, -g\rangle$ with $(a, k) \in H$. That is, $g \in D_{R_2}\langle 1, -k\rangle$ for some $k \in F(a)$, proving (5.6).

Write $F(a) = yF(1)$ as in (5.4). Then (5.6) becomes:

$$G(R_2) = \bigcup_{k \in F(1)} D_{R_2}\langle 1, -yk\rangle.$$

Thus $y \in \mathrm{rad}(R_2) \cdot F(1)$ by (3.1). That is, there exists a $d \in \mathrm{rad}(R_2)$ such that $d \in yF(1) = F(a)$. By assumption (2) then $d = 1$. Hence $y \in F(1)$ and so $F(a) = F(1)$. By assumption (1) we have $(a, m) \in H$ for some $m \in G(R_2)$. Then $m \in F(a) = F(1)$ so that $(1, m) \in H$ also. So $(a, 1) = (a, m)(1, m) \in H$. Hence $G(R_1) \times 1 \subset H$ and $H = G(R_1) \times F(1)$. □

We first complete the case of a local factor L with $|G(L)| \geq 8$.

Corollary 5.7. *Let $R = L \sqcap R_2$, with R_2 of elementary type, L of local type and $|G(L)| \geq 8$. Let S be an H-extension of R. Suppose $F(a) \cap \mathrm{rad}(R_2) \subset \{1\}$ for all $a \in G(L)$. Then either $H = 1 \times H_2$ or $H = G(L) \times H_2$ for some subgroup $H_2 \subset G(R_2)$.*

Proof. Again let π_1 denote the projection of $G(R)$ onto $G(L)$. We know that $\pi_1(H) = 1$ or $G(L)$, by (2.2). If $\pi_1(H) = 1$ then clearly $H = 1 \times H_2$ for some subgroup H_2. So suppose that $\pi_1(H) = G(L)$, the first hypothesis of (5.5). We are assuming the second hypothesis as well. And (5.2) shows the third hypothesis of (5.5) holds. Hence $H = G(L) \times H_2$, for some subgroup H_2. □

The argument for $R = \mathbb{Z} \sqcap R_2$ is different.

Lemma 5.8. *Let R be a real Witt ring of elementary type. Let $P \subset G(R)$ be an ordering. Suppose that for all $x \in P$ that:*

$$P = \bigcup_{k \in D\langle 1, -x\rangle \cap P} D\langle 1, xk\rangle.$$

Then $R = \mathbb{Z} \sqcap R_2$, for some Witt ring R_2.

Proof. Suppose $\mathbb{Z}$ is not a factor of R. Then R has a group ring factor that is real. Thus $R = R_0[E_1] \sqcap R_2$, for some Witt rings R_0, R_2, and we may assume $P = P_0\{1, t\} \times G(R_2)$, where $P_0 \subset G(R_0)$ is an ordering on R_0 and $E_1 = \{1, t\}$. Then take $x = (t, 1)$. We have that $D\langle 1, -x\rangle = \{1, -t\} \times G(R_2)$ and $D\langle 1, -x\rangle \cap P = 1 \times G(R_2)$. Thus:

$$\begin{aligned} P = \bigcup_{k \in D\langle 1, -x\rangle \cap P} D\langle 1, xk\rangle &= \bigcup_{g_2 \in G(R_2)} D\langle (1, 1), (t, g_2)\rangle \\ &= \{1, t\} \times G(R_2) \end{aligned}$$

Hence $P_0 = 1$ and $R_0 = \mathbb{Z}$, giving a contradiction. □

Lemma 5.9. *Let $R = \mathbb{Z} \sqcap R_2$ and suppose S is an H-extension of R. Then one of the following occurs.*

(a) *$H = 1 \times H_2$ for some subgroup $H_2 \subset G(R_2)$.*
(b) *$R = \mathbb{Z} \sqcap R_3$, for some Witt ring R_3, and $\{\pm 1\} \times 1 \subset H$.*
(c) *$R = \mathbb{Z} \sqcap \mathbb{Z} \sqcap R_3$, for some Witt ring R_3, and $(1, 1) \times G(R_3) \subset H$.*

Proof. Again let π_1 be the projection of $G(R)$ onto $G(\mathbb{Z}) = \{\pm 1\}$. If $\pi_1(H) = 1$ the we are in case (a). Thus we may assume that $\pi_1(H) = G(\mathbb{Z})$. If for every $u \in G(R)$ we have that $\pi_1(D_R\langle 1, -u\rangle \cap H) = D\langle 1, -\pi_1(u)\rangle$ then (5.5) implies we are in case (b). So suppose this fails for some $u \in G(R)$. Then by (5.2) $H = 1 \times H_2 \cup (-1 \times -H_2)$, for some ordering H_2 of $G(R_2)$. We will first show that for every $h_2 \in H_2$ that:

$$H_2 = \bigcup_{k \in D_{R_2}\langle 1, -h_2\rangle \cap H_2} D_{R_2}\langle 1, h_2 k\rangle.$$

Consider $\varphi = \langle (1, 1), (1, h_2), (1, -1)\rangle \in S$. We compute its value set two ways. First:

$$D_S\langle (1, 1), (1, h_2), (1, -1)\rangle = \bigcup_{\beta \in D_S\langle (1,1), (1, -h_2)\rangle} D_S\langle (1, 1), \beta(1, h_2)\rangle.$$

Now $(-1, h_2) \notin H$ so $D_S\langle (1, 1), (1, -h_2)\rangle = 1 \times D_{R_2}\langle 1, -h_2\rangle$. For φ to represent an element of $\alpha(1 \times G(R_2))$ we must have $\beta \in -H = H$. That is, $\beta = (1, \beta_2)$ with $\beta_2 \in H_2$. Thus:

$$D_S(\varphi) \cap \alpha(1 \times G(R_2)) = \alpha \cdot \bigcup_{\beta_2 \in D_{R_2}\langle 1, -h_2\rangle \cap H_2} (1 \times D_{R_2}\langle 1, \beta_2 h_2\rangle).$$

Next:

$$D_S(\varphi) = (1, h_2) \cdot \bigcup_{\gamma \in D_S\langle(1,1),(1,-1)\rangle} D_S\langle(1,1), \gamma(1,h_2)\rangle.$$

For any $x \in H_2$ take $\gamma = (1, x) \in D_S\langle(1,1),(1,-1)\rangle$. Then since $(-1, -xh_2) \in H$:

$$\begin{aligned}\alpha(1,x) &\in (1,h_2)D_S\langle(1,1), -(-1,-xh_2)\rangle \\ &= (1,h_2)\cdot\{1,\alpha\}(1 \times D_{R_2}\langle 1, xh_2\rangle)\end{aligned}$$

Thus $D_S(\varphi) \cap \alpha(1 \times G(R_2)) = \alpha(1 \times H_2)$. The two computations of $D_S(\varphi)$ thus yield $H_2 = \cup D_{R_2}\langle 1, \beta h_2\rangle$, over $\beta \in D_{R_2}\langle 1, -h_2\rangle \cap H_2$.

We may now apply (5.8) to obtain $R_2 = \mathbb{Z} \sqcap R_3$, for some Witt ring R_3. Let $H_3 \subset G(R_3)$ be the subgroup such that $H_2 \cap (1 \times G(R_3)) = 1 \times H_3$. We note that both H_2 and $1 \times G(R_3)$ have index two in $G(R_2)$.

If $H_2 = 1 \times G(R_3)$ then $(1,1) \times G(R_3) \subset H$ and we are in case (c). So suppose $H_2 \neq 1 \times G(R_3)$. Then $1 \times H_3$ has index two in H_2 and H_3 has index two in $G(R_3)$. Write $H_2 = 1 \times H_3 \cup (-1 \times zH_3)$, for some $z \in G(R_3)$. Then:

$$H = [(1,1) \times H_3] \cup [(1,-1) \times zH_3] \cup [(-1,-1) \times -H_3] \cup [(-1,1) \times -zH_3]. \tag{5.10}$$

Now $[G(R_3) : H_3] = 2$ implies at least one of the cosets $zH_3, -H_3, -zH_3$ equals H_3. Say $zH_3 = H_3$. Then the second term of (5.10) shows $(1,1,1), (1,-1,1) \in H$. Set R_4 equal to the product of the first copy of $\mathbb{Z}$ and R_3. Then $R = \mathbb{Z} \sqcap R_4$ and $\{\pm 1\} \times 1 \subset H$. We are thus in case (b). Next say $-H_3 = H_3$. Then $(1,-1) \in 1 \times H_3 \subset H_2$. Since H_2 is an ordering we have $D_{R_2}\langle(1,1),(1,-1)\rangle \subset H_2$. But the $1 \times G(R_3) \subset H_2$, a case we have already considered. Lastly, suppose $-zH_3 = H_3$. Then the fourth term of (5.10) shows $(1,1,1), (-1,1,1) \in H$. This is case (b) again. $\square$

6. Products: Group ring factors.

Lemma 6.1. *Let $R = R_1 \sqcap R_2$, with $R_1 = R_0[E_1]$ and E_1 generated by t. Let S be an H-extension of R. Let π_1 be the projection of $G(R)$ onto $G(R_1)$ and suppose $\pi_1(H) \not\subset G(R_0)$. Then either $\pi_1(H) = G(R_1)$ or $1 \times G(R_2) \subset H$.*

Proof. From $\pi_1(H) \not\subset G(R_0)$ we may assume $h = (t, g_2) \in H$, for some $g_2 \in G(R_2)$. Suppose $\pi_1(H) \neq G(R_1)$, Choose $-g_1 \in G(R_1) \setminus \pi_1(H)$. Then $-g_1 t \notin \pi_1(H)$. Set $u = (g_1 t, 1)$ and note that $\pi_1(D_R\langle 1, -u\rangle \cap H) = 1$. Now:

$$\begin{aligned}D_R\langle 1, uh, -h\rangle &= D_{R_1}\langle 1, g_1, -t\rangle \times D_{R_2}\langle 1, g_2, -g_2\rangle \\ &\supset g_1 \times G(R_2).\end{aligned}$$

Also:

$$\bigcup_{k \in h(D_R\langle 1,-u\rangle \cap H)} D_R\langle 1, -k\rangle \subset D_{R_1}\langle 1, -t\rangle \times G(R_2) = \{1, -t\} \times G(R_2).$$

Hence by (3.3), $g_1 \times G(R_2) \subset uH$. Multiplying by u gives $t \times G(R_2) \subset H$. Thus $1 \times G(R_2) \subset H$. $\square$

Lemma 6.2. *Let $R = R_1 \sqcap R_2$, with $R_1 = R_0[E_1]$ and E_1 generated by t. Let S be an H-extension of R. Let π_1 be the projection of $G(R)$ onto $G(R_1)$ and suppose $\pi_1(H) = G(R_1)$. If $u_1 \in G(R_0)^{\cdot}$ and $u = (u_1, u_2)$ then $\pi_1(D_R\langle 1, -u\rangle \cap H) = D_{R_1}\langle 1, -u_1\rangle$.*

Proof. Set $K = \pi_1(D_R\langle 1, -u\rangle \cap H)$ and suppose $K < D_{R_1}\langle 1, -u_1\rangle$. Let $g \in G(R_0)$. Then $(gt, g_2) \in H$ for some $g_2 \in G(R_2)$, since $\pi_1(H) = G(R_1)$. Now $D_R\langle 1, uh, -h\rangle$ contains $-gtD_{R_1}\langle 1, -u_1\rangle \times u_2T$, where $T = -D_{R_2}\langle\langle -u_2, -g_2\rangle\rangle'$. Also:

$$\begin{aligned}\bigcup_{w \in h(D_R\langle 1,-u\rangle \cap H)} D_R\langle 1, -w\rangle &\subset \bigcup_{k \in K} D_{R_1}\langle 1, -kgt\rangle \times G(R_2)\\ &= (\{1\} \cup -gtK) \times G(R_2).\end{aligned}$$

Hence by (3.3), if $y \in D_{R_1}\langle 1, -u_1\rangle \setminus K$ then:

$$\begin{aligned}-gty \times u_2T &\subset uH\\ -gtu_1y \times T &\subset H.\end{aligned}$$

Now u_2 and $-u_2g_2$ are in T so $(-gtu_1y, u_2)$ and $(-gtu_1y, -u_2g_2)$ are in H. Thus $(1, -g_2) \in H$ and as result $(gt, -1) \in H$.

This holds for all $g \in G(R_0)$ so we have that $tG(R_0) \times -1 \subset H$. Thus $G(R_0) \times 1 \subset H$. But the $D_{R_1}\langle 1, -u_1\rangle \times 1 \subset D_R\langle 1, -u\rangle \cap H$ and $\pi_1(D_R\langle 1, -u\rangle \cap H) = D_{R_1}\langle 1, -u_1\rangle$, a contradiction. □

Lemma 6.3. *Let $R = R_1 \sqcap R_2$, with $R_1 = R_0[E_1]$ non-degenerate and E_1 generated by t. Let S be an H-extension of R. Let π_1 be the projection of $G(R)$ onto $G(R_1)$ and suppose $\pi_1(H) = G(R_1)$. Then $G(R_1) \times 1 \subset H$.*

Proof. We will first show $F(g) = F(1)$ for all $g \in G(R_0)$. Let $g \in G(R_0)^{\cdot}$. Pick $u_1 \in G(R_0)^{\cdot}$ such that $g \in D_{R_1}\langle 1, -u_1\rangle$. For all $u_2 \in G(R_2)$, since $g \in D_{R_1}\langle 1, -u_1\rangle = \pi_1(D_R\langle 1, -(u_1, u_2)\rangle \cap H)$, there exists a $k \in G(R_2)$ with $(g, k) \in H$ and $k \in D_{R_2}\langle 1, -u_2\rangle$. That is, $G(R_2) = \cup_{k \in F(g)} D_{R_2}\langle 1, -k\rangle$. By (3.1) and (5.4), $F(g) = F(1)$.

We next show $G(R_0) \times 1 \subset H$. Continue to let $g \in G(R_0)^{\cdot}$. Now, as $g \in \pi_1(H) = G(R_1)$, we have $(g, m) \in H$ for some $m \in G(R_2)$. Then $m \in F(g) = F(1)$ so $(1, m) \in H$ and hence $(g, 1) = (g, m)(1, m) \in H$. This shows $G(R_0) \subset H$.

We will be done if we show $F(t) = F(1)$. Then, if $(t, k) \in H$ we get $(1, k)$ and hence $(t, 1)$ are in H. Apply the previous paragraph to get $G(R_1) \times 1 = \{1, t\}G(R_0) \times 1 \subset H$.

So suppose $F(t) \neq F(1)$. We have by (3.1):

$$G(R_2) \neq \bigcup_{k \in F(t)} D_{R_2}\langle 1, -k\rangle.$$

Pick $u_2 \in G(R_2) \setminus \cup D_{R_2}\langle 1, -k\rangle$. Set $u = (-t, u_2)$. Then, as there is no k with $(t, k) \in H$ and $k \in D_{R_2}\langle 1, -k\rangle$, we have $\pi_1(D_R\langle 1, -u\rangle \cap H) = 1$. Pick any $g \in G(R_0)^{\cdot}$ (we note that $|G(R_0)| > 1$, else R_1 is degenerate). Pick any $g_2 \in F(g)$ and set $h = (g, g_2) \in H$. Now:

$$\begin{aligned}D_R\langle 1, uh, -h\rangle &= D_{R_1}\langle 1, -gt, -g\rangle \times D_{R_2}\langle 1, u_2g_2, -g_2\rangle\\ &\supset -gt \times D_{R_2}\langle 1, u_2g_2, -g_2\rangle.\end{aligned}$$

Also:

$$\bigcup_{w\in h(D_R\langle 1,-u\rangle\cap H)} D_R\langle 1,-w\rangle \subset D_{R_1}\langle 1,-g\rangle \times G(R_2).$$

Hence, by (3.3),

$$-gt \times D_{R_2}\langle 1, u_2g_2, -g_2\rangle \subset uH$$
$$-t \times D_{R_2}\langle g_2, u_2, -1\rangle \subset H.$$

In particular, $(-t, g_2) \in H$. Since $(-1, 1) \in G(R_0) \times 1 \subset H$, we get $(t, g_2) \in H$. But then $g_2 \in F(t) \cap F(g)$, which equals $F(t) \cap F(1)$ by previous work. $F(t)$ is a coset of $F(1)$, by (5.4), so in fact $F(t) = F(1)$ as desired. □

7. The Main Theorem.

Theorem 7.1. *Let R be a Witt ring of elementary type. If S is an H-extension of R, for some subgroup $H \subset G(R)$, then S is also of elementary type.*

Proof. We argue by induction on $|G(R)|$. If $|G(R)| \leq 2$ then either $H = 1$ or $H = G(R)$ and we are done by (1.1). Suppose $|G(R)| > 2$. If $R = A \sqcap B$ then π_A will denote the projection of $G(R)$ onto $G(A)$.

Now suppose R is degenerate. Write $R = D_k \sqcap R_2$, where R_2 is non-degenerate and $G(D_k) \times 1 = \mathrm{rad}(R)$. If some $d \in \mathrm{rad}(R) \cap H$ then write $R = D_1 \sqcap R_3$, where d generates D_1. We have $G(D_1) \times 1 \subset H$ and so $H = G(D_1) \times H_3$, for some subgroup $H_3 \subset G(R_3)$. Then (3.2) implies $S = D_1 \sqcap S_3$, where S_3 is an H_3-extension of R_3. By induction, S_3 is of elementary type and so S is also. We may thus assume $\mathrm{rad}(R)\cap H = 1$. If $\pi_{D_k}(H)\cap G(D_k) \neq 1$ then by (4.1) we may replace H, without affecting S, by another subgroup H_0 such that $\pi_{D_k}(H_0)\cap G(D_k) = 1$. We assume this has already been done so that $\pi_{D_k}(H)\cap G(D_k) = 1$. We note this also holds trivially if R is non-degenerate and $k = 0$.

Next suppose R has a local type factor L with $|G(L)| \geq 8$. Write $R = L \sqcap R_4$. D_k is a factor of R_4. We check the hypothesis of (5.7). Let $a \in G(L)$ and suppose $x \in F(a)\cap\mathrm{rad}(R_4)$. This means $(a, x) \in H$ and so $x \in \pi_{D_k}(H) = 1$. Thus $F(a)\cap\mathrm{rad}(R_4) \subset \{1\}$, as desired. Apply (5.6) to get that either $G(L) \times 1 \subset H$ or $\pi_L(H) = 1$. In the first case, $H = G(L) \times H_4$, for some subgroup H_4 of $G(R_4)$. Then (3.2) implies $S = L \sqcap S_4$, where S_4 is an H_4-extension of R_4. By induction, S_4 is of elementary type and so S is also. We may thus assume we are in the second case: $\pi_L(H) = 1$.

Now suppose the local factor is $L_1 = \mathbb{Z}$. There are three cases according to (5.9). In case (b) we can write $R = \mathbb{Z}\sqcap R_5$, with $\{\pm 1\}\times 1 \subset H$. Then $H = G(\mathbb{Z})\times H_5$, for some subgroup $H_5 \subset G(R_5)$. Applying (3.2) again gives $S = \mathbb{Z} \sqcap S_5$, where S_5 is an H_5-extension of R_5. Induction again shows S is of elementary type. In case (c) we can write $R = \mathbb{Z} \sqcap \mathbb{Z} \sqcap R_6$, with $(1, 1) \times G(R_6) \subset H$. Once again (3.2) yields $S = S_0 \sqcap R_6$, where S_0 is an H_0-extension of $\mathbb{Z} \sqcap \mathbb{Z}$, for some subgroup H_0. Since $\mathbb{Z} \sqcap \mathbb{Z} \cong \mathbb{Z}[E_1]$, (1.5) shows S_0 is of elementary type (in (1.5)(b) we have $R_0 = \mathbb{Z}$ so that its extension is of elementary type as $|G(R_0)| = 2$). Thus S is also of elementary type. We may thus assume we are in case (a) of (5.9), namely, that $\pi_{L_1}(H) = 1$. This is the same conclusion as when the local factor has at least 8 square classes.

The only local type factors we have omitted are those with 4 square classes and these Witt rings are group rings. We are thus in the following position: $R = Y \sqcap W_1 \sqcap \ldots \sqcap W_n$, where Y is a product of D_k with $k \geq 0$, and local type rings L with $|G(L)| \neq 4$ and each W_i is a non-degenerate group ring. (It is possible that $Y = 1$.) We also have $\pi_Y(H) = 1$, so that if $n = 0$ then $H = 1$ and we are done by (1.1). So suppose $n \geq 1$. Write $W_i = V_i[E_1]$. We first suppose that $\pi_{W_i}(H) \not\subset G(V_i)$ for some i. Write $R = W_i \sqcap R_7$. There are two possibilities according to (6.1).

The first possibility is that $1 \times G(R_7) \subset H$. Write $H = H_0 \times G(R_7)$, for some subgroup H_0 of $G(W_i)$. Then (3.2) gives that $S = S_0 \sqcap R_7$, for S_0, some H_0-extension of W_i. If $G(R_7) \neq 1$ then we are done by induction. We drop the subscript i and suppose then that $R = W = W_0[E_n]$, where W_0 is basic and $n \geq 1$. In cases (a),(c),(d) we have S is of elementary type. In case (b) $S = S_0[E_n]$, where S_0 is an H-extension of W_0, and so agsin S is of elementary type by induction.

The second possibility in (6.1) is that $\pi_{W_i}(H) = G(W_i)$. Then by (6.3) we have $H = G(W_i) \times H_7$, where H_7 is a subgroup of R_7. Apply (3.2) once again to get that $S = W_i \sqcap S_7$, where S_7 is an H_7-extension of R_7. Induction gives that S is of elementary type. This completes the result when (6.1) applies, that is, when $\pi_{W_i}(H) \not\subset G(V_i)$, for some i.

We may thus assume we have $R = Y \sqcap W_1 \sqcap \ldots \sqcap W_n$, with $\pi_Y(H) = 1$ and every $\pi_{W_i}(H) \subset V_i$. Choose $t_i \notin V_i$, for each i and set $g = (1, t_1, \ldots, t_n) \in G(R)$. Then:

$$g \notin \pm \bigcup_{h \in H^{\cdot}} D_R\langle 1, -h\rangle,$$

as for any $h \in H^{\cdot}$ has a coordinate in $G(V_i)^{\cdot}$. By (1.2), $g \notin B(R)$. Thus R is itself a group ring, a case we covered two paragraphs ago. $\square$

The previous sections can be used to determine the possible H-extensions of a given ring R. As an example, consider $R = (D_1 \sqcap L_3)[E_2]$. Let d generate D_1, $-1, a, b$ generate L_3 and s, t generate E_2. Thus $G(R) = \mathrm{gp}(d, -1, a, b, s, t)$, where $\mathrm{gp}(A)$ denotes the group generated by A. $G(R)$ has 2825 subgroups, 47 of which will yield H-extensions. Up to isomorphism, there are exactly 8 H-extensions of R. Below we list the 8 extensions S along with one choice for the corresponding subgroup H.

1. $(D_1 \sqcap L_3)[E_3]$	1
2. $(D_1 \sqcap (D_1 \sqcap L_3)[E_1])[E_1]$	$\mathrm{gp}(d, -1, a, b, s)$
3. $D_1 \sqcap (D_1 \sqcap L_3)[E_2]$	$G(R)$
4. $(\mathbb{Z} \sqcap (D_1 \sqcap L_3)[E_1])[E_1]$	$\mathrm{gp}(d, a, b, s)$
5. $\mathbb{Z} \sqcap (D_1 \sqcap L_3)[E_2]$	$\mathrm{gp}(d, a, b, s, t)$
6. $(D_1 \sqcap L_3[E_1])[E_2]$	$\mathrm{gp}(d)$
7. $(D_2 \sqcap L_3)[E_2]$	$\mathrm{gp}(d, -1, a, b)$
8. $(D_1[E_1] \sqcap L_3)[E_2]$	$\mathrm{gp}(-1, a, b)$

We give a brief sketch of how this list was derived. Begin by running through the cases of (1.5). In (a) $H = 1$ and S is (1) by (1.1). In (c), $G(R_0)$ is a proper subgroup of H,

so S is (2) or (3), depending on whether or not $H = G(R)$. In (d) $G(R_0) \subset \pm H, -1 \notin H$ and $H \not\subset G(R_0)$. Thus H looks like a subgroup K of index 2 in $G(R_0)$ that does not contain -1, together with one or more elements from $\{t, s, ts\}$. S is (4) if $|H \cap E_2| = 2$ and (5) if $|H \cap E_2| = 4$. In (1.5)(b) $S = S_0[E_2]$, where S_0 is an H-extension of R_0. Now $R_0 = D_1 \sqcap L_3$. By (4.1) we can assume that either $H = G(D_1) \times H_2$, or that $H = 1 \times H_2$, for some $H_2 \subset G(L_3)$. Now $H_2 = 1$ or $H_2 = G(L_3)$ by (2.3) and (5.7). Since we have already done the case $H = 1$ this gives three choices: $G(D_1) \times 1, G(D_1) \times G(L_3)$ and $1 \times G(L_3)$. The corresponding S is (6), (7) and (8), respectively.

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