Fall 2009

Introduction to Lie groups

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Lecture notes for a series of talks in Representation theory seminar Fall 2009.

Lecture 1

These lectures follow Fulton & Harris's *Representation Theory: A first course*, with some material taken from Spivak's *Calculus on Manifolds.*

Introduction

Definition (Lie Group): A Lie group is a C^∞ manifold with a group structure such that

$$(x, y) \to xy^{-1} \text{ is } C^\infty$$

A Lie group map is smooth group homomorphism(all homomorphism's smooth unless otherwise specified).

Some things we need from manifold theory

1) **The immersed v. closed issue.** It is possible to embed one manifold into another in such a way that the subspace topology from the big one does not coincide with the natural topology on the little one.

   A submanifold is said to be **closed** if manifold structure is the inherited one, otherwise, merely "immersed." "Immersed" is a weaker condition than closed, not a negation. That is,

$$\text{immersed } \not\Rightarrow \text{ not closed.}$$

2) Tangent space, to manifold M at point p (following Spivak *Calculus on Manifolds*).
   Let M be an n-dimensional manifold, $\subseteq \mathbb{R}^N$, $p \in M$ a point.

Let

$$f_1 : V_1 \subseteq \mathbb{R}^n \to U_1 \subseteq M$$
$$f_2 : V_2 \subseteq \mathbb{R}^n \to U_2 \subseteq M$$

be charts, $p \in U_1 \cap U_2$.

Say $f_1(x_1) = f_2(x_2) = p$. Then

$$\begin{aligned}(df_1)_{x_1} \cdot \mathbb{R}^n &= (df_2)_{x_1} \cdot \mathbb{R}^n \leq \mathbb{R}^N \\ &= \{v \subseteq \mathbb{R}^n \mid \exists \gamma : I \to M \text{ s.t. } \gamma(0) = p,\ \gamma'(0) = v\}.\end{aligned}$$

Define $T_pM := \big\{(p,v) \mid v \in (df)_{f^{-1}(p)}\mathbb{R}^n\big\}$. In addition, $(p,v_1)+(p,v_2) = (p, v_1+v_2)$.

$$\begin{aligned} f &: M_1 \to M_2 \\ f_* &: T_p(M_i) \to T_{f(p)}(M_2) \\ &\quad (p,v) \mapsto \big(f(p), df_p v\big) \end{aligned}$$

Tangent Bundle Let M be a manifold. The tangent bundle $\mathcal{T}M$ of M is a manifold, equipped with a map to M. As a set $\mathcal{T}M$ is simply the union of the tangent spaces $T_pM, p \in M$.

$$\begin{array}{c} \mathcal{T}M \\ \downarrow \\ M \end{array}$$

The manifold structure on $\mathcal{T}M$ is such that in sufficiently small open sets the projection $\mathcal{T}M \to M$ is like

$$\begin{array}{c} U \times \mathbb{R}^n \\ \downarrow \\ U \end{array}$$

Section: A section of the tangent bundle is a function $M \to \mathcal{T}M$ s.t. $f(x) \in T_xM\ \forall\, x$ (usually assumed C^∞ without mention).

Sections of the tangent bundle are often called *vector fields.*

Covering space map: A covering space map is a C^∞ map $\varphi : M \to N$ of manifolds such that

$$\forall x \in N \exists U \subset M \text{ s.t. } \varphi^{-1}(U) \cong U \times \{\text{a discrete set}\}.$$

Examples: $x^n : \mathbb{C} \to \mathbb{C}^x$, $n \neq 0$.

Key Theorem: Picard-Lindelöf

If M is a manifold, and v is a Lipschitz continuous vector field on M, (Lipschitz continuous is a weaker condition than C^1) then $\exists\ I \subseteq \mathbb{R}$ interval $\gamma : I \to M$ such that $0 \in I$, $\gamma(0) = p$, $\gamma(t) = v\big(\gamma(t)\big)\ \forall\ t \in I$.

Examples of Lie groups

1) $\mathbb{R}$,

2) $\mathbb{R}_+^\times \cong \mathbb{R}$ (isomorphism given by log)

3) $\mathbb{R}^\times \cong \mathbb{R} \times \{\pm 1\}$

4) One may construct the torus as $\mathbb{R}/\Lambda$, where Λ is a lattice. Doing so equips the torus with structure of a Lie group.

5) $GL_n\mathbb{R}$ = group of all invertible $n \times n$ matrices

6) $SL_n\mathbb{R}$ = group of all $n \times n$ matrices with $\det = 1$.

7) **Classical groups.** For any Q $n \times n$, one may consider

$$\{g \in GL_n\mathbb{R} \mid {}^T g Q_g = Q\}.$$

   a) if Q is **invertible** and **symmetric,** then the group obtained is called an **orthogonal** group, denoted $O(Q)$.

   b) if Q is **invertible** and **skew-symmetric,** then the group obtained is called a **symplectic** group, denoted $Sp(Q)$.

   One may also replace $\mathbb{R}$ by $\mathbb{C}$ in any of the above. In addition, one may consider

$$\{g \in GL_n\mathbb{C} \mid {}^T g Q \overline{g} = Q\}.$$

   c) if Q is invertible and symmetric, then this group is called a unitary group, denoted $U(Q)$.

Complex Lie Group Means as manifold, is complex manifold. I.e., charts come from open sets in $\mathbb{C}^n$ and maps $f \circ g^{-1}$ are holomorphic for two charts $f : U \to V, g : U' \to V'$. That is, they have a **complex derivative.**

($f : \mathbb{C}^n \to \mathbb{C}^m$ has a complex derivative means $\exists\ L : \mathbb{C}^n \to \mathbb{C}^m$ linear s.t.

$$\lim_{\|\overline{h}\| \to 0} \frac{1}{\|\overline{h}\|} \|f(x + \overline{h}) - f(\overline{x}) - L\overline{h}\| = 0.$$

This condition turns out to imply that, as a function of $2n$ real variables f is C^∞ with convergent power series expansion in some radius.)

Remark. $U(Q)$ is **not** a complex Lie group. $GL_n\mathbb{C}$, $SL_n\mathbb{C}$, orthogonal, symplectic groups are.

Isogenies and Isogeny Classes:
Let G be a connected Lie group with identity e. Let

$$\begin{array}{ccc} H & \leftarrow & \text{connected} \\ \text{pr} \downarrow & & \qquad \text{covering} \\ G & & \end{array}$$

be a covering space. Then for each $\tilde{e}$ in the preimage $pr^{-1}(e)$ or e, there exists unique C^∞ maps $\tilde{\mu} : H \times H \to H$ and $\tilde{\iota} : H \to H$ such that H can be made into a group with multiplication and inversion given by $\tilde{\mu}$ and $\tilde{\iota}$ respectively and pr becomes a homomorphism.

Next, let G be a Lie group and Γ a discrete subgroup of the center of G. then there is a unique C^∞ manifold structure on the quotient $G/Z(G)$ such that the natural projection $G \to G/Z(G)$ is C^∞.

Remark. $\Gamma \leq G$ discrete and normal, G connected $\Rightarrow \Gamma \leq Z(G)$.

Definition (Isogeny, Isogenous): Let G, H be Lie groups. An isogeny $\varphi : G \to H$ is a function which is both a Lie group map and a covering space map.

Two lie groups G_1, G_2 are said to be **isogenous** if there exists an isogeny $G_1 \to G_2$ or $\varphi : G_2 \to G_1$.

This is not an equivalence relation, but it generates an equivalence relation.

Examples of isogenies:

$$\begin{array}{l} z \longrightarrow z^n \\ \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \\ \quad \text{or} \\ S^1 \longrightarrow S^1 \quad \text{(treating } S^1 \text{ as the unit circle in } \mathbb{C}^\times \\ \mathbb{R} \longrightarrow S^1 \\ x \longrightarrow e^{2\pi i x} \\ \mathbb{R}^2 \longrightarrow \text{ torus, viewed as } \mathbb{R}^2/\Lambda. \end{array}$$

Define a homomorphism

$$SL_n(\mathbb{R}) \to GL_{n^2}(\mathbb{R})$$

as follows:

$$SL_n(\mathbb{R}) \text{ acts on } \mathrm{Mat}_{n\times n}(\mathbb{R}) \text{ by conjugation.}$$

$\mathrm{Mat}_{n\times n}(\mathbb{R})$ is an n^2-dimensional real vector space.

Write $Ad(g)X = gXg^{-1}$. Then for each g, the function $Ad(g)$ is a linear function of $X \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. Fix B an ordered basis of $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Then for each g, $\big[Ad(g)\big]_B$ (the matrix of the operator $Ad(g)$ with respect to the ordered basis B) is an element of $GL_{n^2}\mathbb{R}$. And the function $g \to \big[Ad(g)\big]_B$ is a homomorphism. The kernel of the function

$$g \to \big[Ad(g)\big]_B$$

is the center of $SL_n\mathbb{R}$ which is

$$\begin{cases} \{I_n\} & n \text{ odd} \\ \{\pm I_n\} & n \text{ even.} \end{cases}$$

Thus the function $g \to \big[Ad(g)\big]_B$ is an isogeny. The image is called $PSL_n\mathbb{R}$.

Proposition 1. *Let G be a connected Lie group, and $U \subseteq G$ an open set containing the identity element e. Then U generates G.*

Proof. Let $H \leq G$ be the subgroup generated by U. Clearly, $h \cdot U \in H$ for all $h \in H$. By continuity of multiplication $h \cdot U$ is an open neighborhood of h. It follows that h is open.

Let H^c denote the complement of H in G and $U' = \{u^{-1} \mid u \in U\}$. By continuity of the inversion map, U' is open. If $g \in H^c$ then $g \cdot U' \subset H^c$, for if $h \in g \cdot U' \cap H$ then $g \in h \cdot U \subset H$. It follows that H^c is also open, whence H is closed.

This proves that, in the general case, H is a union of connected components of G, and in the case G connected, $H = G$. □

Lecture 2

Definitions.

Let G be a Lie Group the Lie algebra of G is T_eG. It is denoted by the corresponding lower case gothic letter, in this case $\mathfrak{g}$.

Suppose that $M \subseteq \mathbb{R}^n$, is a manifold of dim $= k$ defined as the level set of an equation, i.e.

$$M = \{\underline{x} \in \mathbb{R}^n \mid \Phi(x) = \underline{0}\} \text{ some } \Phi : \mathbb{R}^n \to \mathbb{R}^{n-k}.$$

Then for $p \in M$, the subspace

$$\begin{aligned} V &= \left\{ \gamma'(0) \mid \begin{array}{c} \gamma : (a,b) \to M \\ \gamma(0) = p \end{array} \right\} \\ &= Im(df^{-1}_{f(p)}) \quad \begin{array}{c} \text{for any coordinate system} \\ (f : U \subseteq M \to V \subseteq \mathbb{R}^{n-k}). \end{array} \end{aligned}$$

introduced in the last lecture, such that

$$T_pM = \big\{(p,v) \mid v \in V\big\}$$

It can also be described as $\ker(d\Phi_p)$.

Example.

$$\begin{aligned} &G = GL_n(\mathbb{R}) \\ &\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R}) = \text{all } n \times n \text{ matrices.} \end{aligned}$$

Indeed, $X \to I + X$ is local coordinate at e.

$$\mathfrak{sl}_n\mathbb{R} = \{x \in \mathfrak{gl}_n(\mathbb{R}) \mid d\det_e = 0\}.$$

To prove this, it is necessary and sufficient to prove that

$$d\det_e = \text{trace}.$$

We give two proofs. First, take $A(t) = \big(a_{ij}(t)\big)$ are in $GL_n\mathbb{R}$.

$$\begin{aligned} \frac{d}{dt}\det A(t) &= \frac{d}{dt}\left(\sum_{\sigma \in s_n}\prod_{i=1}^{n} a_{i\sigma(i)}(t)\,\text{sgn}(\sigma)\right) \\ &= \sum_{\sigma \in s_n}\sum_{i=1}^{n}\left(a'_{i\sigma(i)}(t)\prod_{\substack{j=1\\ j\neq i}}^{n} a_{j\sigma(j)})(t)\right)\text{sgn}(\sigma). \end{aligned}$$

Now

$$(\det A)'(0) = \sum_{i=1}^{n} a'_{ii}(0) = \operatorname{tr}(A'(0))$$

(all other terms drop because

$$\#\{j \mid a_{j\sigma(j)}(0) = 0\} \geq 2 \quad \forall\, \sigma \neq id.)$$

Alternate Proof: Use $I + X$ coordinate.
$\det(I + X) = 1 + \operatorname{tr}(X) +$ higher order terms.

Classical groups For our next, example fix a matrix Q symmetric, invertible, $n \times n$, and consider $SO(Q) = \{g \in GL_n\mathbb{R} \mid {}^t g_g Q g = Q\}$.

This is defined by $\Phi(g) = 0$, where $\Phi(g) = {}^T g Q g - Q$.

Lemma 1.

$$d\Phi_e(X) = {}^T X Q + Q X.$$

Proof.

$$\frac{d}{dt} {}^T\gamma(t) Q \gamma(t) = {}^T\gamma'(t) Q \gamma(t) + {}^T\gamma(t) Q \gamma'(t)$$

at $t = 0$, get ${}^T\gamma'(0) Q + Q\gamma'(0)$. (Because $\gamma(0) = {}^T\gamma(0) = e$.) $\square$

Likewise, if $\Phi : GL(n, \mathbb{C}) \to \mathbb{C}$ is the function

$$\Phi(g) = ({}^T g Q \overline{g})$$

then

$$d\Phi_e(X) = {}^T X Q + Q\overline{X}.$$

$$\mathfrak{o}(q) = \mathfrak{so}(Q) = \{X \in \mathfrak{gl}_n\mathbb{R} \mid {}^T X Q = -QX\}.$$

If Q is skew symmetric, inverstible

$$\begin{aligned} Sp(Q) &= \{g \in GL_n\mathbb{R} \mid {}^T g Q g = Q\} \\ \mathfrak{sp}(Q) &= \{X \in GL_n\mathbb{R} \mid {}^T X Q = -QX \\ \mathfrak{u}(Q) &= \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid {}^T X Q = -Q\overline{X}\}. \end{aligned}$$

Now, a remarkable fact about Lie groups: if G is a connected Lie groups and

$$\rho : G \to H$$

is a C^∞ homomorphism, then $d\rho_e$ determines ρ.

Goals: Prove this and determine necessary and sufficient conditions on linear map $L : \mathfrak{g} \to \mathfrak{h}$ for existence of $\rho : G \to H$ with $d\rho_e = L$.

First: case $\dim G = 1$, $\mathfrak{g} \cong \mathbb{R}$, $L(t) = t \cdot X$ some $X \in \mathfrak{h}$. We must show $\forall X \in \mathfrak{h}$,

$$\#\{\varphi : \mathbb{R} \to H \text{ homomorphism} \mid \text{s.t. } \varphi'(0) = X\} \leq 1.$$

Proof. φ homomorphism

$$\begin{aligned} &\Rightarrow \varphi'(s) = (m_{\varphi(s)})_* \varphi'(0) = (m_{(s)})_* X \\ &\Rightarrow \varphi(s) \quad \text{follows the vector field } v_X \\ &\qquad\qquad \text{defined by } v_X(g) = (m_g)_* X. \end{aligned}$$

Uniqueness follows from Picard-Lindelöf.

On the other hand, let φ be integral curve vector field v_X passing through e. Fix s and consider

$$\alpha(t) = \varphi(s)\varphi(t)$$
$$\beta(t) = \varphi(s+t)$$

$$\begin{aligned} \alpha'(t) = (m_{\varphi(s)})_* \cdot \varphi'(t) &= (m_{\varphi(s)})_* (m_{\varphi(t)})_* \cdot X \\ &= (m_{\alpha(t)})_* \cdot X \\ \beta'(t) = (m_{\varphi(s+t)})_* X &= (m_{\beta(t)})_* \cdot X. \end{aligned}$$

Therefore α, β are two integral curves for v_X passing through $\varphi(s)$, <u>so they are equal</u>.

This proves $\varphi(s)\varphi(t) = \varphi(s+t)$ $\forall s, t$ sufficiently small. But a unique extension to homomorphism $\mathbb{R} \to H$ follows. $\square$

Idea: Suppose differential equations gives $\varphi : (-1, 1) \to H$ with the above property. For $t \in \mathbb{R}$ write $t = t_1 + \cdots + t_n$ s.t. $t_i \in (-1, 1)$ and define $\varphi(t) = \varphi(t_1) \cdots \varphi(t_n)$. Independence of choice?

$$t_1 + \cdots + t_n = t'_1 + \cdots + t'_m$$

Then these two partitions of the interval $[0, t]$ have a common refinement.

Observe: If G is a matrix group then φ_X is given explicitly by

$$\varphi_X(t) = I + t \cdot X + \frac{t^2}{2} \cdot X^2 + \frac{t^3}{6} X^3 + \cdots$$

Notably, $X \to \varphi_X(1)$ is the matrix exponential.

For general G we **define**

$$\exp(X) = \varphi_X(1).$$

$d\exp_0$ is a linear map $\mathfrak{g} \to \mathfrak{g}$, and in fact is the identity map. Indeed, to select an arc in $\mathfrak{g}$ with tangent vector X, use $\gamma(t) = tX$. Then

$$\begin{aligned} d\exp_0(X) &= \frac{d}{dt}\exp(tX)\,|_{t=0} \\ &= \varphi_X'(0) = X. \end{aligned}$$

Corollary 1. *Image of exp contains neighborhood of 1.*

Proposition 2. *G, H are Lie groups. $\rho : G \to H$, C^∞ homomorphism. Then the following diagram commutes*

$$\begin{array}{ccc} G & \xrightarrow{\rho} & H \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{g} & \xrightarrow[d\rho_e]{} & \mathfrak{h} \end{array}$$

That is,

$$\rho\big(\exp(X)\big) = \exp(d\rho_e X) \quad \forall\ X \in \mathfrak{g}.$$

Proof. Let

$$\begin{aligned} \varphi_1(t) &= \rho\big(\exp(tX)\big) = \rho \circ \varphi_X(t) \\ \varphi_1'(0) &= d\rho_e \varphi_X'(0) \\ &= d\rho_e \cdot X \end{aligned}$$

$$\therefore \varphi_1 = \varphi_{d\rho_e X} \quad \text{(uniqueness)}$$

$$\begin{aligned} \therefore \varphi_1(t) &= \exp\big(t(d\rho_e X)\big) \\ &= \exp(d\rho_e tX). \end{aligned}$$

□

Corollary 2. *If $\rho_e : G \to H$ and $\sigma : G \to H$ are both C^∞homomorphism, and G is* ***connected,*** *and $d\rho_e = d\sigma_e$, then $\rho = \sigma$.*

Proof.

$$\begin{aligned}
d\rho_e &= d\sigma_e \\
&\Rightarrow \exp(d\rho_e X) = \exp(d\sigma_e X) \quad \forall\ X \in \mathfrak{g} \\
&\Rightarrow \rho\big(\exp(X)\big) = \sigma\big(\exp(X)\big) \quad \forall\ X \in \mathfrak{g} \\
&\Rightarrow \rho(g) = \sigma(g) \quad \forall\ g \in \text{ image of } \exp .
\end{aligned}$$

But image of exp $\supseteq$ neighborhood of e. $\rho(g) = \sigma(g)\ \forall\ g \in$ neighborhood of e. Neighborhood of 1 generates G. □

Now, given G, H, and linear map $L : \mathfrak{g} \to H$, is it possible to tell by looking at L whether there exists ρ such that $d\rho_e = L$?

Lecture 3

Definitions.
G — Lie Group
$\mathfrak{g}$ — Lie algebra

The Lie Bracket

Assume first that $G \subseteq GL_n\mathbb{R}$. Consider $\Psi_g \in \mathrm{Aut}(G)$

$$\Psi_g(x) = gxg^{-1}.$$

The induced map $\mathfrak{g} \to \mathfrak{g}$ is also just $Ad(g) \cdot X = gXg^{-1}$. Now Ad is a *homomorphism* $G \to GL(\mathfrak{g})$. Its differential is a linear map

$$\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

denoted ad. Thus

$$\begin{gathered}
ad(X) : \mathfrak{g} \to \mathfrak{g} \\
ad(X)(Y) \in \mathfrak{g}.
\end{gathered}$$

The map ad is given explicitly by

$$\begin{aligned}
&\frac{d}{dt}\big(\gamma(t)X\gamma(t)^{-1}\big)\ |_{t=0} \\
&\quad = \gamma'(0)X\gamma(0)^{-1} + \gamma(0)X\left(\frac{d}{dt}\gamma(t)^{-1}\right)\ |_{t=0} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{d}{dt}\gamma(-t)\ |_{t=0} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -\gamma'(0) \\
&\gamma'(0)X - X\gamma'(0) \\
&[X, Y] = XY - YX.
\end{aligned}$$

Note: XY, YX need not be in $\mathfrak{g}$! $[X, Y] = ad(X)(Y)$ also makes sense if $G \not\subseteq GL_n\mathbb{R}$.

Let $\rho : G \to H$ be C^∞ homomorphism. Then for $X, Y \in G$ we have

$$\big[\rho_*(X), \rho_*(Y)\big] = \rho_*[X, Y].$$

Proof.

$$\begin{aligned}
\rho \circ \Psi_g &= \Psi_{\rho(g)\circ\rho} \\
\rho(ghg^{-1}) &= \rho(g)\rho(h)\rho(g)^{-1} \\
\rho_* \cdot Ad(g) &= Ad\big(\rho(g)\big) \circ \rho_* \\
\rho_* \cdot ad(X) &= ad(\rho_* X) \circ \rho_* \quad \forall\ X \\
\rho_*\big([X, Y]\big) &= [\rho_* X, \rho_* Y] \quad \forall\ X, Y,
\end{aligned}$$

as required. □

$[\quad,\quad]$ has the following properties:

1) It is bilinear.

2) $[X, Y] = -[Y, X]$

3) $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big]$ (Jacobi)

Definition. An abstract Lie algebra is a vector space $\mathfrak{g}$ equipped with a slow-symmetric bilinear map satisfying Jacobi identity.

Fundamental Theorems:

1) **Ado's Theorem:** Every Lie algebra is isogenous to a subalgebra of $\mathfrak{gl}_n$, for some n. (Justifies assumption we are always in $\mathfrak{gl}_n$.)

2) **Cambell-Hausdorff Formula:** For $U \subset \mathfrak{g}$ containing 0 and sufficiently small, $\exp : U \to G$ is one-to-one and we can define an inverse, denoted $\log$. Then $\log(\exp(X) \cdot \exp(Y))$ is given by a complicated formula involving the bracket $[\ ,\ ]$.

Corollary to Cambell-Hausdorff: If a subspce $\mathfrak{h}$ of $\mathfrak{gl}_n$ closed under $[\quad,\quad]$, then its image under exp is closed under multiplication.

Corollary to that: The map subgroups of $GL_n\mathbb{R} \to$ abstract Lie algebras is **surjective.**

Remark. The map $G \to \mathfrak{g}$ is far from injective. Indeed, $\mathfrak{g}_1 \cong \mathfrak{g}_2$ if G_1 is isogenous to G_2. This will turn out to be an "if and only if." We won't prove that now but note that if a map $\rho : G_1 \to G_2$ is given, and $d\rho_e$ is an isomorphism, then ρ is an isogeny.

Theorem 1. *G, H Lie groups. G simply connected,*

$$L : \mathfrak{g} \to \mathfrak{h} \textit{ linear.}$$

Then

$$\exists\ \rho : G \to H\ C^\infty \textit{ homomorphism with } \rho_* = L$$
$$\textit{iff } [LX, LY] = L[X, Y] \quad \forall\ X, Y \in \mathfrak{g}.$$

The "only if" part was shown earlier.

Proof. Put $K = G \times H$, $frakk = \mathfrak{g} \oplus \mathfrak{h}$.

$$\mathfrak{j} \subseteq \mathfrak{g} \oplus \mathfrak{h} = \mathfrak{k} = \big\{ \big(X, L(X)\big) \mid X \in \mathfrak{g} \big\}$$

(i.e., $\mathfrak{j}$ is the graph of the function L). The $\mathfrak{j}$ is closed under $[\quad,\quad]$, so

$$\exists\ J \leq G \times H \text{ with } T_e J = \mathfrak{j}.$$

(Remark: J is not necessarily a *closed* subgroup. See the closed v. immersed issue in Lecture 1.) Consider the map

$$\pi_1 : G \times H \to G$$

$$\text{projection } (g,h) \to g.$$

given by projection onto the first factor.

Restrict to a map $J \to G$.

The differential of this map is given by

$$\big(X, L(X)\big) \to X.$$

which is an isomorphism $\mathfrak{j} \to \mathfrak{g}$.

$$\therefore \pi \text{ induces anisogeny } J \to G.$$

G simply connected. $\therefore J \underline{\sim} G$.

Now $(g,h) \to h$ induces map $J \underline{\sim} G \to H$. $\square$

Ideal: $\mathfrak{h} \subset \mathfrak{g}$ is an ideal if

$$[X,Y] \in \mathfrak{h} \ \forall \ X \in \mathfrak{h}, \ Y \in \mathfrak{g},$$
$$(\Leftrightarrow H \trianglelefteq G).$$

$\mathfrak{g}$ is semisimple if it contains no nonzero solvable ideals.

Examples

$\mathfrak{sl}_n\mathbb{R}$, (can check)

$$\mathfrak{so}_n\mathbb{R}, \ \mathfrak{sp}_{2n}\mathbb{R},$$

not $\mathfrak{g}l_n\mathbb{R}$ because $GL_n\mathbb{R}$ has at least two normal subgroups: $SL_n\mathbb{R}$ and its center.

Reductive: $\mathfrak{g} = \mathfrak{g}_0 \oplus z$. $\mathfrak{g}_0$ semisimple, $z =$ center

Example. $\mathfrak{gl}_n\mathbb{R}$.

Definition: A **homomorphism** of Lie algebras is a linear map $\rho : \mathfrak{g} \to \mathfrak{h}$ ($\mathfrak{g}, \mathfrak{h}$, Lie algebras) such that

$$\rho\big([X,Y]\big) = \big[\rho(X), \rho(Y)\big].$$

A **representation** of Lie algebra $\mathfrak{g}$ on vector space V is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \operatorname{End}(V) \cong M_{\dim V \times \dim V}(\mathbb{R}).$$

So if ρ is a representation, then

$$\rho\big([X,Y]\big) = \rho(X)\circ\rho(Y) - \rho(Y)\rho(X).$$

Definition. Let V be a vector space and $T : V \to V$ an endomorphism. Then T is said to be **semisimple** if it is diagonalizable. i.e., if $\exists$ basis B of V s.t. v is an eigenvector of T $\forall$ $v \in B$. On the other hand, T is said to be **nilpotent** if $T^k = 0$, for some k.

Theorem 2. *Assume that V is a* **complex** *vector space. Then for all complex-linear endomorphisms T, $\exists!$ T_{ss} semisimple, and T_{nil} nilpotent s.t. $T = T_{ss} + T_{nil}$, $T_{ss}T_{nil} = T_{nil}T_{ss}$. Consequence of Jordan canonical form. These are called the semisimple and nilpotent parts of T.*

Over real vector spaces, the situation is more complex, for example the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is diagonalizable, but not over $\mathbb{R}$.

To sidestep this difficulty, we shall assume that we work with complex Lie algebras. Thus Lie group maps are henceforth assumed to be holomorphic (i.e., to have complex derivatives), and Lie algebra maps to be complex linear.

Theorem 3. *Let $\mathfrak{g}$ be a Lie algebra and take $X \in \mathfrak{g}$. Then $\exists$ X_{ss}, $X_{nil} \in \mathfrak{g}$ such that*

$$\begin{aligned} \rho(X)_{ss} &= \rho(X_{ss}) && \forall \textit{ representations} \\ \rho(X)_{nil} &= \rho(X)_{nil} && \rho \textit{ of } \mathfrak{g}. \end{aligned}$$

$\therefore$ *one may refer to an element of a Lie algebra as "semisimple" or "nilpotent" without ambiguity, and semisimple elements always act semisimply (i.e., by diagonalizable endomorphisms).*

Analysis of Semisimple Lie Algebras

Cartan Subalgebra: A Cartan subalgebra of a Lie algebra is defined as a maximal abelian subalgebra of semisimple elements.

Strategy: Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Consider $ad|_{\mathfrak{h}} : \mathfrak{h} \to End(\mathfrak{g})$.By the theorem above, each of the operators $ad(X)$, $X \in \mathfrak{h}$ is diagonalizable. Further, these operators commute. It follows that $\mathfrak{g}$ has a basis consisting of simultaneous eigenvectors for all of the

Decompose $\mathfrak{g}$ into eigenspaces for the adjoint action of $\mathfrak{h}$. Study the structure of the set of eigenvalues obtained.

Example. $\mathfrak{g} = \mathfrak{gl}_n\mathbb{C}$. Let $\mathfrak{h} = \{\text{diagonal elts}\}$

This is a maximal abelian subalgebra for if X is not diagonal then $\exists\ D$ diagonal with $DX - XD \neq 0$.

Now, for D diagonal,

$$(DX - XD)_{ij} = (d_i - d_j)X_{ij}$$

Example.

$$\begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix} \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} - \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (d_1 - d_2)s & (d_1 - d_3)t \\ (d_2 - d_1)u & 0 & (d_2 - d_3)w \\ (d_3 - d_1)x & (d_3 - d_2)y & 0 \end{pmatrix}.$$

Let E_{ij} denote the matrix with a 1 at the i, j entry and zeros elsewhere. Every such matrix is an eigenvector of $\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}$ for all d_1, d_2, d_3, and the eingevalue is $(d_i - d_j)$.

Now, what is meant by an "eigenvalue for $\mathfrak{h} \supset \mathfrak{g}$," in general?

Answer: a linear map $\mathfrak{h} \to \mathbb{C}$. The space of such functions is called $\mathfrak{h}^*$. In our example let $L_i = E_{ii}\ \ L_i^* : \mathfrak{h} \to \mathbb{R}$ be the map

$$(d_1 \cdots d_n) \to d_i\,.$$

Eigenvalues are $\{L_i^* - L_j^* \mid 1 \leq i, j \leq n\}$. Zero is one of them. The zero eigenspace for $\mathfrak{h}$ acting on $\mathfrak{g}$ is $\mathfrak{h}$ itself. The nonzero eigenvalues for $\mathfrak{h}$ acting on $\mathfrak{g}$ are called **roots.**

Key Fact: For each root α there is a 3-dimensional subalgebra $\mathfrak{s}_\alpha \subseteq \mathfrak{g}$ which is isomorphic to $\mathfrak{sl}_2\mathbb{C}$.

This, together with careful study of representations of $\mathfrak{sl}_2\mathbb{C}$ will take us a long way.

$\mathfrak{sl}_2\mathbb{C} = 2 \times 2$ traceless matrices (complex entries).

basis $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\mathfrak{h} = \mathrm{span}(H)$ is Cartan.

$$\begin{aligned} [H,X] &= 2X \qquad\qquad [H,Y] = -2Y \\ [X,Y] &= H. \end{aligned}$$

Returning to the general case for just a moment, if

$$\rho : \mathfrak{g} \to \mathrm{End}(V)$$

is a Lie algebra representation, then V decomposes into simultaneous eigenspaces for $\mathfrak{h}$ with eigenvalues $\lambda \in \mathfrak{h}^*$. These are called the **weights** of ρ.

Weights for $\mathfrak{sl}_2$.

$$\lambda(H) \to a \quad \text{some } a \in \mathbb{C}.$$

Example. $\mathfrak{sl}_2 \circlearrowright \mathbb{C}^2$ matrix multiplication.

$$\begin{array}{lcc} \text{weights:} & L_1, & -L_2 \\ & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \mathfrak{sl}_2 \circlearrowright \mathfrak{sl}_2 & [\quad , & \quad] \end{array}$$

$$\begin{aligned} X \cdot Y &= [X,Y] \\ [X_1,X_2] \cdot Y &= \big[[X_1,X_2],Y\big] \\ &= \big[X_1,[X_2,Y]\big] - \big[X_2,[X_1,Y]\big] \qquad \text{(Jacobi identity)} \\ &= X_1 \cdot X_2 \cdot Y - X_2 \cdot X_1 \cdot Y \end{aligned}$$

Eigenvectors $X,\ Y,\ H$

Eigenvalues $2,\ -2,\ 0.$

Let ρ be a representation of $\mathfrak{sl}_2$ on space v. Write $X.v$ for $\rho(X)(v)$. Then we have

$$\begin{aligned} X.Y.v - Y.X.v &= H.v \\ H.X.v &= 2.X.v + X.H.v \\ H.Y.v &= -2.Y.v + Y.H.v. \end{aligned}$$

Use this to study $\Lambda_V = \{a \in \mathbb{R} \mid a \text{ is a weight of } \rho\}$.

$$\begin{aligned} H.v = av \Rightarrow & H.X.v = (a+2)X.v \\ & H.Y.v \Rightarrow (a-2)Y.v \\ \Rightarrow & (a+2) \text{ is a weight or } X.v = 0 \\ & (a-2) \text{ is a weight or } Y.v = 0. \end{aligned}$$

If V is a finite dimensional, then Λ_V has maximal and minimal elements $a_{\max}, a_{\min}$.

Take $v_{\max} \in V_{a_{\max}}$, then

$$\begin{aligned} X.v_{\max} &= 0 \\ Y^n.v_{\max} &\in V_{a_{\max}-2n} \quad \text{each } n. \end{aligned}$$

Proposition 3. span$\big(\{Y^n.v_{\max} \mid n \geq 0\}\big)$ *is an invariant subspace. If V is irreducible, then it is all of V.*

Proof. Put $V_1 = \operatorname{span}\big(\{Y^n.v_{\max} \mid n \geq 0\}\big)$. We claim $Z.v_1 \in V_1$ for all $Z \in \mathfrak{sl}_2\mathbb{C},\ v_1 \in V_1$. It suffices to check basis elements.

$$Z = Y, H \text{ are obvious.}$$

$$\begin{aligned} X.v_{\max} &= 0 \\ X.Y.v_{\max} &= H.v_{\max} = a_{\max}v_{\max} \\ X.Y.Y.v_{\max} &= H.Y.v_{\max} + Y.X.Y.v_{\max} \\ &= (a_{\max} - 2)\cdot Y.v_{\max} + Y \cdot (a_{\max}v_{\max}) \\ &= (2a_{\max} - 2)\cdot Y.v_{\max} \\ X.Y^{k+1}.v_{\max} &= H.Y^k v_{\max} + Y.X.Y^k.v_{\max} \\ &= \sum_{i=0}^{k} (a_{\max} - 2i) Y^k v_{\max} \\ &= \big[(k + 1a_{\max} - k(k+1)\big] Y^k v_{\max}. \end{aligned}$$

□

Now, it follows from the proposition that if V is an irreducible representation of $\mathfrak{sl}_2\mathbb{C}$, then

$$\Lambda_V = \{a_{\max}, a_{\max} - 2, \ldots, a_{\max} - 2n = a_{\min}\} \quad \text{some } n$$

and, for this n

$$\begin{aligned} &Y^{n+1}.v_{\max} = 0, \text{ so } X.Y^{n+1}.v_{\max} \ = 0, \text{ but } Y^n.v_{\max} \neq 0 \\ \Rightarrow\ &\big((n+1)a_{\max} - n(n+1)\big) = 0. \end{aligned}$$

So $a_{\max} = n$. Thus

$$\Lambda_V = \{n, n-2, \ldots, n-2n = -n\} \quad \text{some } n.$$

□

Further, V_1, V_2 two representations with the same value of $a_{\max}$, then $\exists\ L : V_1 \to V_2$, vector space isomorphism s.t. $Z \cdot L(v_1) = L(Z.v_1)\ \forall\ v_1 \in V_1, Z \in \mathfrak{g}$. Such a map is called then an isomorphism of representations.