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CYCLIC PERMUTATIONS IN DOUBLY-TRANSITIVE GROUPS

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INTRODUCTION

Let Ω be a finite set of size n. A **cyclic permutation** on Ω is a permutation whose cycle decomposition is one cycle of length n. This paper classifies all finite doubly-transitive permutation groups which contain a cyclic permutation. The classification appears in Table 1.

We use (G, Ω) for a finite doubly-transitive permutation group G acting on a finite set Ω. For other notation and definitions see the self-contained article Cameron [1].

CLASSIFICATION

(G, Ω) has a unique minimal normal subgroup $N = soc(G)$, which is either elementary abelian or simple.

In the first case suppose (G, Ω) has an elementary abelian regular normal subgroup N of size p^d, where $d \geq 1$. Let $g \in G$ be a cyclic permutation, it has order p^d. Now $G \leq AGL(d, p) \leq GL(d + 1, p)$. By considering the JCF of g we have $p^{d-1} + 1 \leq d + 1$, so $d = 1$ or $p = d = 2$. So G contains no cyclic permutations unless $d = 1$ or $p = d = 2$. See Table 1, $d = 1$ corresponds to row a and $p = d = 2$ to row b.

In the second case, when N is simple, N is known because of the classification of the finite simple groups. Cameron [1] tabulates all simple groups, N, which occur as socles of finite doubly-transitive groups.

We have $N \leq G \leq Aut(N)$. For each row of the table in [1] we will check such G for cyclic permutations:

$\mathbf{N = A_n}$: Clearly A_n contains a cyclic permutation if and only if n is odd. When $n \geq 5$ and n is odd, then $Aut(A_n) \cong S_n$. Hence $G \cong A_n$ or S_n, see rows c and d of Table 1.

$\mathbf{N = PSL(d, q)}$: Here Zsigmondy's theorem may be used. If $G = PSL(2, 8)$ there is nothing to prove. Consider $GL(1, q^d) \lhd \Gamma L(1, q^d) \leq \Gamma L(d, q)$. Except for the case that $d = 2$ and q is a Mersenne prime, let p be a primitive prime divisor of $q^d - 1$ and let P be a Sylow p-subgroup of $GL(1, q^d)$. We may check that $\Gamma L(1, q^d) = N_{\Gamma L(d,q)}(P)$ and $GL(1, q^d) = C_{\Gamma L(1,q^d)}(P)$. Now p does not divide $q - 1$, so any cyclic permutation must be the image in $P\Gamma L(d, q)$ of a cyclic subgroup of $\Gamma L(d, q)$ containing P or a conjugate, and so must be a conjugate of the image of $GL(1, q^d)$. Hence such a cyclic permutation must lie in $PGL(d, q)$. Finally, if $d = 2$ and q is a Mersenne prime, a similar argument can be made with a subgroup P of order 4. Hence, for every $d \geq 2$ and prime power q, a group G for which $PSL(d, q) \leq G \leq P\Gamma L(d, q)$ contains a cyclic permutation if and only if $PGL(d, q) \leq G$. See row e of Table 1. Thus, we have decided which subgroups of $P\Gamma L(d, q)$ have cyclic permutations, see p.179 of Feit [3].

$\mathbf{N = PSU(3, q)}$: Here we use Liebeck, Praeger, and Saxl [4] which lists all maximal factorizations of all finite simple groups and their automorphism groups. Let $g \in G$ be a cyclic permutation. In this case N is already doubly-transitive and so we need only consider $G = N\langle g\rangle$. If M is any maximal subgroup of G containing g, then $G = MG_\alpha$ is a maximal factorization and appears in these lists.

From the lists on p.13 of [4] only $G = PSU(3, q)$ for $q = 3$, 5, and 8 has a maximal factorization. In the first two cases the group A does not contain an element of order $q^3 + 1$, so we may exclude them. In the final case, since $G = N\langle g\rangle$, so G/N is cyclic, and then this case is out by their remark. Hence, $PSU(3, q)$ contains no cyclic permutations.

$\mathbf{N =\ ^2B_2(q)}$ and $\mathbf{^2G_2(q)}$: The lists also take care of these two groups.

$\mathbf{N = PSp(2d, 2)}$: Here both permutation representations have even degree, hence a cyclic permutation is an odd permutation, but N is complete.

For the remaining cases we refer to the "Atlas of Finite Groups" by Conway, Curtis, Norton, Parker, and Wilson [2]. The only groups which contain cyclic permutations are those with prime degree, see the last three rows of Table 1. (See also p.179 of Feit [3].)

This completes the examination of the Table in [1]. For every finite doubly-transitive group G we have determined whether or not it contains a cyclic permutation, those which do are listed in Table 1.

TABLE 1

G	n	N
a) $AGL(1,p), p$ any prime	p	C_p
b) S_4	4	$C_2 \times C_2$
c) $S_n, n \geq 5$	n	A_n
d) A_n, n odd and ≥ 5	n	A_n
e) Any G with $PGL(d,q) \leq G \leq P\Gamma L(d,q)$	$(q^d - 1)/(q-1)$	$PSL(d,q)$
$(d,q) \neq (2,2)$, $(2,3)$, or $(2,4)$		
f) $PSL(2,11)$	11	$PSL(2,11)$
g) M_{11}	11	M_{11}
h) M_{23}	23	M_{23}

REMARKS

(i) The groups S_2 and S_3 occur in row a as $AGL(1,2)$ and $AGL(1,3)$ respectively.

(ii) Groups in rows a and b have an elementary abelian socle, groups in rows $c-h$ a non-abelian simple socle.

(iii) No two groups from Table 1 are isomorphic except S_5 from row c and $PGL(2,5)$ from row e, these two groups have inequivalent representations being of degrees 5 and 6 respectively.

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