How Many Symmetries Does Admit a Nonlinear Single-Input Control System Around an Equilibrium?

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How Many Symmetries Does Admit a Nonlinear Single-Input Control System Around an Equilibrium?

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Abstract

We describe all symmetries of a single-input nonlinear control system, that is not feedback linearizable and whose first order approximation is controllable, around an equilibrium point. For a system such that a feedback transformation, bringing it to the canonical form, is analytic we prove that the set of all local symmetries of the system is exhausted by exactly two 1-parameter families of symmetries, if the system is odd, and by exactly one 1-parameter family otherwise. We also prove that the form of the set of symmetries is completely described by the canonical form of the system: possessing a nonstationary symmetry, a 1-parameter family of symmetries, or being odd corresponds, respectively, to the fact that the drift vector field of the canonical form is periodic, does not depend on the first variable, or is odd. If the feedback transformation bringing the system to its canonical form is formal, we show an analogous result for an infinitesimal symmetry: its existence is equivalent to the fact that the drift vector field of the formal canonical form does not depend on the first variable. We illustrate our results by studying symmetries of the variable length pendulum.

Keywords: symmetries, nonstationary symmetries, infinitesimal symmetries, feedback transformations, odd systems.

1 Introduction

Recently there has been a growing interest in symmetries of nonlinear control systems. The structure of control systems possessing symmetries was analyzed e.g. by Grizzle and Marcus in [5] and Gardner, Shadwick, and Wilkens (for linearizable systems) in [3] and [4]. The role of symmetries in the optimal control problems has been studied, among others, by Jurdjevic [7] (for systems on Lie groups), van der Schaft [13], and Sussmann [14]. In [6], Jakubczyk gave a complete characterization of symmetries in terms of symbols of control systems.

In this paper we study symmetries of single-input nonlinear control affine systems whose linear approximation, at an equilibrium point p, is controllable. Recently, we proved in [11] that "almost any" single-input control system, which is truly nonlinear (that is non linearizable via feedback) does not admit any stationary symmetry, that is any symmetry preserving the equilibrium point p. "Almost any" refers to all systems away from a small class of odd systems which admit one nontrivial stationary symmetry, that is conjugated to minus identity by a diffeomorphism bringing the system to its canonical form (constructed in [15]). In the present paper, for the same class of systems, and around an equilibrium point p, we study nonstationary symmetries, that is, symmetries which do not preserve p. Our main result states that also for nonstationary symmetries a complete picture can be deduced from the canonical form. We prove that an analytic system, equivalent by an analytic feedback transformation to its canonical form, admits a nonstationary symmetry if and only if the drift vector field defining the canonical form is periodic with respect to the first variable and that the system admits a 1-parameter family of symmetries if and only if that drift vector field does not depend on the first variable. Moreover, we show that in the latter case the set of all symmetries is given either by exactly one 1-parameter family of symmetries (in the non odd case) or by exactly two 1-parameter families of symmetries (in the odd case). In the case when the feedback transformation, bringing the system to its canonical form, is given by a (not necessarily convergent) formal power series, we prove that an analogous result holds for a formal infinitesimal symmetry. In fact, its existence is equivalent to the fact that the drift of the formal canonical form does not depend on the first variable.

Let us consider the system

$$\Pi : \dot{x} = F(x, u),$$

where $x(\cdot) \in X$, a smooth n-dimensional manifold and

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$u(\cdot) \in U$, a smooth m-dimensional manifold. The map $F : X \times U \longrightarrow TX$ is assumed to be smooth and for any value $u \in U$ of the control parameter, F defines a smooth vector field F_u, where $F_u(\cdot) = F(\cdot, u)$.

Consider the field of velocities $\mathcal{F}$ associated to the system Π and defined as

$$\mathcal{F}(x) = \{F_u(x) \ : \ u \in U\} \subset T_xX.$$

We say that a diffeomorphism $\psi : X \longrightarrow X$ is a *symmetry* of Π if it preserves the field of velocities $\mathcal{F}$, that is,

$$\psi_*\mathcal{F} = \mathcal{F}.$$

Recall that for any vector field f on X and any diffeomorphism $y = \psi(x)$ of X, we put

$$(\psi_* f)(y) = \mathrm{D}\psi(\psi^{-1}(y)) \cdot f(\psi^{-1}(y)).$$

A *local symmetry* at $p \in X$ is a local diffeomorphism ψ of X_0 onto X_1, where X_0 and X_1 are respectively neighborhoods of p and $\psi(p)$, such that

$$(\psi_*\mathcal{F})(q) = \mathcal{F}(q)$$

for any $q \in X_1$.

A local symmetry ψ at p is called a *stationary symmetry* if $\psi(p) = p$ and a *nonstationary symmetry* if $\psi(p) \neq p$.

Let us consider a single-input control affine system

$$\Sigma : \ \dot{x} = f(x) + g(x)u,$$

where $x(\cdot) \in X$, $u(\cdot) \in U = \mathbb{R}$ and f and g are smooth vector fields on X. The field of velocities for the system Σ is the following field of affine lines

$$\mathcal{A}(x) = \{f(x) + ug(x) \ : \ u \in \mathbb{R}\} \subset T_xX.$$

A specification of the above definition says that a diffeomorphism $\psi : X \longrightarrow X$ is a symmetry of Σ if it preserves the affine line field $\mathcal{A}$ (in other words, the affine distribution $\mathcal{A}$ of rank 1), that is if

$$\psi_*\mathcal{A} = \mathcal{A}.$$

We will call $p \in X$ to be an *equilibrium point* of Σ if $0 \in \mathcal{A}(p)$. For any equilibrium point p there exists a unique $\tilde{u} \in \mathbb{R}$ such that $\tilde{f}(p) = 0$, where $\tilde{f}(p) = f(p) + \tilde{u}g(p)$. By the linear approximation of Σ at p we will mean the pair (F, G), where $F = \frac{\partial \tilde{f}}{\partial x}(p)$ and $G = g(p)$. It is easy to see that if the linear approximation of Σ at p is controllable then, in a neighborhood of p, the set E of equilibrium points is a smooth curve.

We will say that Σ is an *odd system* at $p \in X$ if it admits a stationary symmetry at p, denoted by ψ^-, such that

$$\frac{\partial \psi^-}{\partial x}(p) = -\mathrm{Id}. \qquad (1)$$

The paper is organized as follows. In Section 2 we recall a canonical form for single-input control systems (constructed in [15]) on which are based all our results. In Section 3 we state two theorems describing, respectively, local nonstationary symmetries and 1-parameter families of local symmetries of analytic control systems which are analytically equivalent to their canonical form. Section 4 deals with systems whose canonical form is formal. For such systems we describe formal infinitesimal symmetries. We illustrate our results by studying symmetries of the variable length pendulum in Section 5. A more detailed analysis of symmetries, together with proofs of all our results is given in [12].

2 Canonical form for single-input systems

In this section we will describe a canonical form for single-input nonlinear systems, that has been recently obtained by the authors [15], [16] as a completion of Kang normal form [8]. In Sections 3 and 4, we will show that symmetries take a very simple form when the system is brought to its canonical form.

Throughout the paper we will consider Σ around an equilibrium point $p = 0 \in X \subset \mathbb{R}^n$, where X is an open neighborhood of $0 \in \mathbb{R}^n$. All objects, that is, functions, maps, vector fields, control systems, etc., are considered in a neighborhood of $0 \in \mathbb{R}^n$ and assumed to be C^∞-smooth. Let h be a smooth $\mathbb{R}$-valued function. By

$$h(x) = h^{[0]}(x) + h^{[1]}(x) + h^{[2]}(x) + \cdots = \sum_{m=0}^{\infty} h^{[m]}(x)$$

we denote its Taylor series expansion at $0 \in \mathbb{R}^n$, where $h^{[m]}(x)$ stands for a homogeneous polynomial of degree m. Similarly, for a map ϕ of an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ (resp. for a vector field f on an open subset of $\mathbb{R}^n$) we will denote by $\phi^{[m]}$ (resp. by $f^{[m]}$) the term of degree m of its Taylor series expansion at $0 \in \mathbb{R}^n$, that is, each component $\phi_j^{[m]}$ of $\phi^{[m]}$ (resp. $f_j^{[m]}$ of $f^{[m]}$) is a homogeneous polynomial of degree m in x.

Together with Σ we will study also its Taylor series expansion, around $0 \in \mathbb{R}^n$, given by

$$\Sigma^\infty \ : \ \dot{\xi} = F\xi + Gu + \sum_{m=2}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}u), \qquad (2)$$

where $F = \frac{\partial f}{\partial \xi}(0)$ and $G = g(0)$. Throughout the paper the pair (F, G) is assumed to be controllable.

Let

$$\Gamma : \begin{array}{rcl} x & = & \phi(\xi) \\ u & = & \alpha(\xi) + \beta(\xi)v \end{array} \qquad (3)$$

be an invertible feedback transformation acting on Σ. Consider the Taylor series expansion of the transformation Γ given by

$$\Gamma^\infty : \begin{array}{rcl} x &=& T\xi + \sum_{m=2}^{\infty} \phi^{[m]}(\xi) \\ u &=& K\xi + Lv + \sum_{m=2}^{\infty} (\alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v). \end{array} \quad (4)$$

The transformation Γ^∞ can be viewed as a composition of the linear feedback $x = T\xi$, $u = K\xi + Lv$ and of *homogenous feedback transformations* Γ^m, for $m \geq 2$, defined by

$$\Gamma^m : \begin{array}{rcl} x &=& \xi + \phi^{[m]}(\xi) \\ u &=& v + \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v. \end{array}$$

The equivalence via Γ^∞, in those cases when we do not address the problem of convergence, will be called *formal feedback equivalence.*

Define $\bar{\xi}_i = (\xi_1, \ldots, \xi_i)$. Let m_0 denote the largest nonnegative integer such that for any $1 \leq k \leq n$, the distributions

$$\mathcal{D}^k = (g, ad_f g, \cdots, ad_f^{k-1} g)$$

have constant rank k and are involutive modulo terms of order $m_0 - 2$. It follows that the system Σ is feedback linearizable up to order $m_0 - 1$ (see [10]). We bring the homogeneous part of degree m_0 of the system into Kang normal form (see [8]) and without loss of generality we can assume that the system Σ^∞ takes the form

$$\Sigma^\infty : \begin{array}{rcl} \dot{\xi} &=& A\xi + Bu + \bar{f}^{[m_0]}(\xi) \\ && + \sum_{m=m_0+1}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}(\xi)u), \end{array} \quad (5)$$

where (A, B) is in Brunovský canonical form and

$$\bar{f}_j^{[m_0]}(\xi) = \begin{cases} \sum_{i=j+2}^{n} \xi_i^2 P_{j,i}^{[m_0-2]}(\bar{\xi}_i) & \text{if } 1 \leq j \leq n-2, \\ 0 & \text{if } n-1 \leq j \leq n, \end{cases} \quad (6)$$

that is the first nonvanishing homogeneous vector field $\bar{f}^{[m_0]}$ is in Kang normal form.

Let $(i_1, \ldots, i_{n-s})$, where $i_1 + \cdots + i_{n-s} = m_0$ and $i_{n-s} \geq 2$, be the largest, in the lexicographic ordering, $(n-s)$-tuple of nonnegative integers such that for some $1 \leq j \leq n-2$, we have

$$\frac{\partial^{m_0} \bar{f}_j^{[m_0]}}{\partial \xi_1^{i_1} \cdots \partial \xi_{n-s}^{i_{n-s}}} \neq 0.$$

Define

$$j^* = \sup \left\{ j = 1, \cdots, n-2 \ : \ \frac{\partial^{m_0} \bar{f}_j^{[m_0]}(\xi)}{\partial \xi_1^{i_1} \cdots \partial \xi_{n-s}^{i_{n-s}}} \neq 0 \right\}.$$

The following result was proved in [15].

Theorem 1 *The system Σ^∞, given by (5), is equivalent by a formal feedback Γ^∞ to a system of the form*

$$\Sigma_{CF}^\infty \ : \ \dot{x} = Ax + Bv + \sum_{m=m_0}^{\infty} \bar{f}^{[m]}(x), \quad (7)$$

where, for any $m \geq m_0$,

$$\bar{f}_j^{[m]}(x) = \begin{cases} \sum_{i=j+2}^{n} x_i^2 P_{j,i}^{[m-2]}(\bar{x}_i) & \text{if } 1 \leq j \leq n-2, \\ 0 & \text{if } n-1 \leq j \leq n, \end{cases} \quad (8)$$

additionally, we have

$$\frac{\partial^{m_0} \bar{f}_{j^*}^{[m_0]}}{\partial x_1^{i_1} \cdots \partial x_{n-s}^{i_{n-s}}} = \pm 1 \quad (9)$$

and, moreover, for any $m \geq m_0 + 1$,

$$\frac{\partial^{m_0} \bar{f}_{j^*}^{[m]}}{\partial x_1^{i_1} \cdots \partial x_{n-s}^{i_{n-s}}}(x_1, 0, \ldots, 0) = 0. \quad (10)$$

The form Σ_{CF}^∞ satisfying (8), (9),(10) will be called the *canonical form* of the system Σ^∞. The name is justified by the following result that we proved in [15].

Theorem 2 *Two systems Σ_1^∞ and Σ_2^∞ are formally feedback equivalent if and only if their canonical forms $\Sigma_{1,CF}^\infty$ and $\Sigma_{2,CF}^\infty$ coincide.*

3 Main result: analytic case

In this section we will state our main results describing all nonstationary symmetries of analytic systems. In general, we do not know whether a formal feedback transformation Γ^∞, bringing the system Σ to its canonical form Σ_{CF}^∞, converges, that is, whether the formal power series defining ϕ, α, and β, converge. If they do, we get a complete description of all local symmetries of analytic control systems around equilibria. Consider an analytic control system Σ on X. We will say that an analytic control system Σ_{CF}, defined on an open subset $X_{CF} \subset \mathbb{R}^n$, is the canonical form of Σ if there exists an analytic feedback transformation Γ, of the form (3), with $\phi(X) = X_{CF}$, which maps Σ into

$$\Sigma_{CF} \ : \ \dot{x} = Ax + \bar{f}(x) + Bv,$$

where

$$\bar{f}_j(x) = \begin{cases} \sum_{i=j+2}^{n} x_i^2 P_{j,i}(x_1, \cdots, x_i) & \text{if } 1 \leq j \leq n-2, \\ 0 & \text{if } n-1 \leq j \leq n, \end{cases}$$

and all $P_{j,i}$, and thus $\bar{f}$, are analytic functions on X_{CF} such that

$$\frac{\partial^{m_0} \bar{f}_{j^*}}{\partial x_1^{i_1} \cdots \partial x_{n-s}^{i_{n-s}}}(x_1, 0, \ldots, 0) = \pm 1.$$

The $(n-s)$-tuple $(i_1, \cdots, i_{n-s})$ and the index j^* are those defined in Section 2.

Let X be an open subset of $\mathbb{R}^n$ equipped with coordinates $(x_1, \ldots, x_n)^t$ such that $0 \in X$. We will say that a map f of X into $\mathbb{R}^n$ is periodic with respect to x_1 on X if there exists $c = (c_1, 0, \ldots, 0)^t$, $c_1 \neq 0$, such that $c \in X$ and $f(x) = f(x + c)$ for any $x, x + c \in X$.

Theorem 3 *Consider an analytic system Σ on an open subset $X \subset \mathbb{R}^n$. Assume that its linear approximation (F, G) at an equilibrium point $0 \in X$ is controllable and that Σ is not locally feedback linearizable at $0 \in X$. Suppose that Σ is equivalent via an analytic feedback transformation Γ, of the form (3), with $\phi(X) = X_{CF}$, to its canonical form Σ_{CF} defined on $X_{CF} \subset \mathbb{R}^n$. The system Σ admits a local nonstationary symmetry ψ, such that $\psi(0) = p \in X$, if and only if the drift $Ax + \bar{f}(x)$ of its canonical form Σ_{CF} is periodic with respect to x_1 on X_{CF}.*

Now we will describe systems that possess 1-parameter families of symmetries. We will say that ψ_{c_1}, where $c_1 \in (-\epsilon, \epsilon) \subset \mathbb{R}$, is a *nontrivial 1-parameter analytic family of local symmetries* if each ψ_{c_1} is a local analytic symmetry, $\psi_{c_1} \neq \psi_{c_2}$ if $c_1 \neq c_2$, and $\psi_{c_1}(x)$ is jointly analytic with respect to (x, c_1). Notice that, by taking ϵ sufficiently small, we can assume that all elements (with a possible exception for one symmetry) of a nontrivial analytic 1-parameter family of symmetries are nonstationary symmetries.

Theorem 4 *Consider an analytic system Σ on an open subset $Y \subset \mathbb{R}^n$. Assume that its linear approximation (F, G) at an equilibrium point $0 \in Y$ is controllable and that Σ is not locally feedback linearizable at $0 \in Y$. Suppose that Σ is equivalent via an analytic feedback transformation Γ, of the form (3), to its canonical form Σ_{CF}. If Σ is not odd (resp. odd) then there exists an open set $X \subset Y$ such that the following conditions are equivalent.*

(i) *Σ admits a nontrivial 1-parameter analytic family of local symmetries ψ_{c_1}, for $c_1 \in (-\delta, \delta)$, such that $\psi_{c_1}(0) \in X$.*

(ii) *Σ admits exactly one 1-parameter family of local nonstationary symmetries ψ_{c_1}, where $c_1 \in (-\epsilon, \epsilon)$ (resp. two 1-parameter families of local nonstationary symmetries ψ_{c_1} and $\psi_{c_1}^-$, where $c_1 \in (-\epsilon, \epsilon)$), which exhausts completely the set of all local symmetries of Σ, that is, for any local symmetry ψ of Σ, such that $\psi(0) \in X$, there exists a real c_1, satisfying $c_1 \in (-\epsilon, \epsilon)$ and such that on a neighborhood of $0 \in \mathbb{R}^n$ we have $\psi = \psi_{c_1}$ (resp. either $\psi = \psi_{c_1}$ or $\psi = \psi_{c_1}^-$).*

(iii) *The drift $Ax + \bar{f}$ of the canonical form Σ_{CF} of Σ satisfies $\bar{f}(x) = \bar{f}(x_2, \ldots, x_n)$.*

(iv) *For any $c_1 \in (-\epsilon, \epsilon)$, the translation $T_{c_1}(x) = (x_1 + c_1, x_2, \ldots, x_n)^t$ is a local symmetry (resp. the translation $T_{c_1}(x) = (x_1 + c_1, x_2, \ldots, x_n)^t$ and the composition of -*Id *with T_{c_1}, that is the map $T_{c_1}^-(x) = (-x_1 + c_1, -x_2, \ldots, -x_n)^t$ are local symmetries) of the canonical form Σ_{CF} on $X_{CF} = \phi(X)$, for some sufficiently small ϵ.*

We can summerize our results as follows. Consider the system Σ around an equilibrium point and bring it to its canonical form Σ_{CF} via an analytic feedback transformation Γ defined on X. Denote $X_{CF} = \phi(X)$, where ϕ is the diffeomorphism defining Γ. If the drift vector field $Ax + \bar{f}$ of Σ_{CF} satisfies $\bar{f}(x) = -\bar{f}(-x)$ then Σ_{CF} is odd and admits minus identity as the only nontrivial stationary symmetry. Otherwise Σ_{CF} does not admit any nontrivial stationary symmetry. If the drift $Ax + \bar{f}(x)$ is periodic, with respect to x_1, with a period c_1 such that $(c_1, 0, \ldots, 0)^t \in X_{CF}$, then Σ_{CF} admits the translation $T_{c_1}(x) = (x_1 + c_1, x_2, \ldots, x_n)^t$ as a local symmetry. Finally, if $Ax + \bar{f}(x)$ does not depend on x_1 then Σ_{CF} admits the 1-parameter family of translations $T_{c_1}(x) = (x_1 + c_1, x_2, \ldots, x_n)^t$, for $c_1 \in \mathbb{R}$, as local symmetries. This exhausts completely the set of all possible generators of symmetries of Σ_{CF}. Recall that ϕ denotes the analytic diffeomerphism defining Γ that brings Σ to its canonical form Σ_{CF}. The above analysis implies that, under the assumptions of Theorems 3 and 4, any local symmetry ψ of Σ is either of the form

$$\psi = \phi^{-1} \circ T_{c_1} \circ \phi$$

or of the form

$$\psi = \phi^{-1} \circ T_{c_1}^- \circ \phi$$

for some real c_1. Moreover, the latter case is possible only for odd systems.

4 Main result: formal case

We do not know whether, in general, the feedback transformation Γ^∞ bringing the system to its canonical form Σ_{CF}^∞ converges. If it does, Theorems 3 and 4 describe all local symmetries of Σ. If it does not, the canonical form Σ_{CF}^∞ is considered as a formal power series but even in this case it keeps, as we will show in the sequel, important information about symmetries.

We say that a vector field H on an open subset $X \subset \mathbb{R}^n$ is an *infinitesimal symmetry* of the system Σ if the (local) flow γ_H^t of H is a local symmetry of Σ, for any t for which it exists. Denote by $\mathcal{G}$ the distribution spanned by the vector field g.

Proposition 1 *A vector field H is an infinitesimal symmetry of Σ if and only if*

$$\begin{aligned}[H, f] &= 0 \bmod \mathcal{G}, \\ [H, g] &= 0 \bmod \mathcal{G}.\end{aligned}$$

This characterization of infinitesimal symmetries justifies the following notion. We say that a vector field formal series

$$H^{\infty}(\xi) = \sum_{m=0}^{\infty} h^{[m]}(\xi)$$

is a *formal infinitesimal symmetry* of the system

$$\Sigma^{\infty} \ : \ \dot{\xi} = f(\xi) + g(\xi)u = \sum_{m=1}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}u)$$

if

$$\begin{aligned}[H^{\infty}, f] &= 0 \bmod \mathcal{G}, \\ [H^{\infty}, g] &= 0 \bmod \mathcal{G}.\end{aligned}$$

Here, $[\cdot,\cdot]$ is understood as the Lie bracket of formal power series vector fields.

Theorem 5 *Consider the system Σ^{∞}. Assume that its linear approximation (F, G) is controllable and that Σ^{∞} is not feedback linearizable. The following conditions are equivalent.*

(i) *Σ^{∞} admits a formal infinitesimal symmetry.*
(ii) *The only formal infinitesimal symmetry of Σ^{∞} is $H^{\infty} = (\phi^{-1})_* \frac{\partial}{\partial x_1}$, where ϕ is the formal diffeomorphism defining a feedback transformation Γ^{∞} that brings Σ^{∞} into its canonical form Σ^{∞}_{CF}.*
(iii) *The canonical form Σ^{∞}_{CF} of Σ^{∞} satisfies $\bar{f}^{[m]}(x) = \bar{f}^{[m]}(x_2, \ldots, x_n)$, for any $m \geq m_0$, where the vector fields $\bar{f}^{[m]}$ are of the form (8), (9), and (10).*
(iv) *For any $c_1 \in \mathbb{R}$, the translation $T_{c_1}(x) = (x_1 + c_1, x_2, \ldots, x_n)^t$ is a symmetry of the canonical form Σ^{∞}_{CF}.*
(v) *The vector field $H^{\infty}_{CF} = \frac{\partial}{\partial x_1}$ is a formal infinitesimal symmetry of the canonical form Σ^{∞}_{CF}.*

This result, established in the formal category, provides the following necessary condition for the existence of analytic 1-parameter families of symmetries. Notice that below we do not assume that the feedback transformation Γ^{∞}, bringing Σ to its canonical form Σ^{∞}_{CF}, converges.

Proposition 2 *Consider an analytic system Σ and assume that its linear approximation is controllable and the system is not feedback linearizable. If Σ admits a nontrivial 1-parameter analytic family of local symmetries ψ_{c_1}, for $c_1 \in (-\epsilon, \epsilon)$, then the drift vector field of the canonical form Σ^{∞}_{CF} satisfies $\bar{f}^{[m]}(x) = \bar{f}^{[m]}(x_2, \ldots, x_n)$, for any $m \geq m_0$.*

We will end this section by giving a necessary condition for the existence of a family of local nonstationary symmetries which does not require to bring the system to its canonical form but only to normalize a finite number of terms.

Recall, see Section 2, that m_0 denotes the degree of the first nonlinearizable term. After having annihilated all terms $f^{[m]}$, for $2 \leq m \leq m_0 - 1$ we can thus bring the system Σ to the form

$$\tilde{\Sigma} \ : \ \dot{x} = Ax + Bu + \bar{f}^{[m_0]}(x) + O(x, v)^{m_0+1}, \quad (11)$$

where (A, B) is in Brunovský canonical form and the first nonlinearizable homogeneous vector field $\bar{f}^{[m_0]}$ is in Kang normal form (6).

Proposition 3 *Under the assumptions of Proposition 2, if $\bar{f}^{[m_0]}(x)$ in (11) depends on x_1 then Σ does not admit any nontrivial 1-parameter analytic family of local symmetries.*

We would like to emphasize that the above condition is checkable via an algebraic calculation. In [15] (see also [16]) we gave explicit polynomial transformations that bring a homogenous part of any degree of a system to Kang normal form [8]. Therefore a successive application of those polynomial transformations, of degree 2 up to m_0, brings Σ into $\tilde{\Sigma}$ for which we can apply Proposition 3. We will illustrate this result in Section 5 by describing all symmetries of the variable length pendulum.

5 Example

Example 1 Consider the variable length pendulum of Bressan and Rampazzo [1] (see also [2]). We denote by ξ_1 the length of the pendulum, by ξ_2 its velocity, by ξ_3 the angle with respect to the horizontal, and by ξ_4 the angular velocity. The control $u = \dot{\xi}_4 = \ddot{\xi}_3$ is the angular acceleration. The mass is normalized to 1. The equations are (compare [1] and [2])

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -g \sin \xi_3 + \xi_1 \xi_4^2 \\ \dot{\xi}_3 &= \xi_4 \\ \dot{\xi}_4 &= u,\end{aligned}$$

where g denotes the gravity.

Clearly, $\psi^{-}(\xi) = -\xi$ is a stationary symmetry of the system and therefore the variable length pendulum is an odd system. Since the vector field f satisfies $f(-\xi) = -f(\xi)$, the drift $\bar{f}(x)$ of its canonical form (see below) also satisfies $\bar{f}(-x) = -\bar{f}(x)$.

We would like to know whether the variable length pendulum admits nonstationary symmetries. The system is feedback linearizable modulo terms of degree three and the first nonlinearizable term is of degree $m_0 = 3$. The feedback transformation

$$\begin{aligned} x_1 &= \xi_1 \\ x_2 &= \xi_2 \\ x_3 &= -g(\xi_3 - \tfrac{\xi_3^3}{6}) \\ x_4 &= -g\xi_4(1 - \tfrac{\xi_3^2}{2}) \\ v &= g\xi_3\xi_4^2 - ug(1 - \tfrac{\xi_3^2}{2}) \end{aligned}$$

brings the system to the following Kang normal form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + \tfrac{1}{g^2}x_1x_4^2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u, \end{aligned}$$

modulo terms of order four. The vector field $\bar{f}^{[3]} = \frac{1}{g^2}x_1x_4^2\frac{\partial}{\partial x_2}$ depends on x_1 and thus, by Proposition 3, the variable length pendulum does not admit any nontrivial local 1-parameter group of symmetries.

In order to check whether this system admits a nonstationary symmetry we bring it into its canonical form. The analytic diffeomorphism $\tilde{x} = \phi(\xi)$, defined on $X = \mathbb{R} \times \mathbb{R} \times (-\pi, \pi) \times \mathbb{R}$, and given by (see [15])

$$\begin{aligned} \tilde{x}_1 &= \tfrac{1}{g}\xi_1 \\ \tilde{x}_2 &= \tfrac{1}{g}\xi_2 \\ \tilde{x}_3 &= -\sin\xi_3 \\ \tilde{x}_4 &= -\xi_4\cos\xi_3, \end{aligned}$$

and followed by a suitable feedback $u = \alpha(\xi) + \beta(\xi)v$, brings the system into the following canonical form Σ_{CF}

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= \tilde{x}_3 + \tfrac{\tilde{x}_1}{1-\tilde{x}_3^2}\tilde{x}_4^2 \\ \dot{\tilde{x}}_3 &= \tilde{x}_4 \\ \dot{\tilde{x}}_4 &= \tilde{v}. \end{aligned}$$

The vector field $\bar{f}(x) = \frac{\tilde{x}_1}{1-\tilde{x}_3^2}\tilde{x}_4^2\frac{\partial}{\partial \tilde{x}_2}$ is not periodic with respect to $\tilde{x}_1$ and thus, by Theorem 3, the system does not admit any nonstationary symmetry ψ such that $\psi(0) \in X = \mathbb{R} \times \mathbb{R} \times (-\pi, \pi) \times \mathbb{R}$. Hence the only local symmetries of the variable length pendulum in X are identity and minus identity. Notice that this example illustrates well a local character of presented results. Indeed, the variable length pendulum admits clearly the symmetry $\psi(\xi) = (\xi_1, \xi_2, \xi_3 + 2\pi, \xi_4)$, however it maps $0 \in \mathbb{R}^4$ outside the set $\mathbb{R} \times \mathbb{R} \times (-\pi, \pi) \times \mathbb{R}$ on which we can bring the system to its canonical form and therefore this symmetry cannot be detected by the canonical form.

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