Normal Forms, Canonical Forms, and Invariants of Single Input Nonlinear Systems Under Feedback

Issa Amadou Tall
Southern Illinois University Carbondale, itall@math.siu.edu

Witold Respondek
INSA de Rouen

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Issa Amadou Tall[1] and Witold Respondek
Institut National des Sciences Appliquées de Rouen
Laboratoire de Mathématiques de l'INSA
76131 Mont Saint Aignan, France
{tall, wresp}@lmi.insa-rouen.fr

Abstract

We study the feedback group action on single-input nonlinear control systems. We follow an approach of Kang and Krener based on analysing, step by step, the action of homogeneous transformations on the homogeneous part of the system. We construct a dual normal form and dual invariants with respect to those obtained by Kang. We also propose a canonical form and show that two systems are equivalent via a formal feedback if and only if their canonical forms coincide. We give an explicit construction of transformations bringing the system to its normal, dual normal, and canonical form.

1 Introduction

The problem of transforming the nonlinear control single-input system

$$\Sigma \ : \ \dot{\xi} = f(\xi) + g(\xi)u$$

by a feedback transformation of the form

$$\Gamma \ : \ \begin{array}{rcl} x & = & \phi(\xi) \\ u & = & \alpha(\xi) + \beta(\xi)v \end{array}$$

to a simpler form has been extensively studied during the last twenty years. The transformation Γ brings Σ to the system

$$\tilde{\Sigma} \ : \ \dot{x} = \tilde{f}(x) + \tilde{g}(x)v,$$

whose dynamics are given by

$$\begin{array}{rcl} \tilde{f} & = & \phi_*(f + g\alpha) \\ \tilde{g} & = & \phi_*(g\beta), \end{array}$$

where for any vector field f and any diffeomorphism ϕ we denote

$$(\phi_* f)(x) = d\phi(\phi^{-1}(x)) \cdot f(\phi^{-1}(x)).$$

[1]On leave from the Dep. of Mathematics, University Cheikh Anta Diop, Dakar, Senegal, e-mail: iatall@smtp.refer.sn

A natural question to ask is whether we can take the system $\tilde{\Sigma}$ to be linear, i.e., whether we can linearize the system Σ via feedback. Necessary and sufficient geometric conditions for this to be the case have been given in [5] and [6]. Those conditions turn out to be, except for the planar case, restrictive and a natural problem which arises is to find normal forms for nonlinearizable systems. Although being natural, this problem is very involved and has been extensively studied during the last twenty years (see [2],[3],[7],[9],[10],[11],[12],[13] among others). In our paper we will follow a very fruitful approach proposed by Kang and Krener [11] and then followed by Kang [9],[10]. Their idea, which is closely related with classical Poincaré's technique for linearization of dynamical systems (see e.g. [1]), is to analyse the system Σ and the feedback transformation Γ step by step and, as a consequence, to produce a simpler equivalent system $\tilde{\Sigma}$ also step by step.

This method allowed Kang to produce a normal form for any single-input system with controllable linearization. The first goal of our paper is to propose a dual normal form. The second goal is to provide explicit transformations bringing the system to Kang normal form and to dual normal form. Neither Kang normal form nor dual normal form is unique: a given control can admit different Kang normal forms and different dual normal forms and therefore the third goal of the paper is to propose a canonical form.

The paper is organized as follows. In Section 2 we will introduce, following [11], homogeneous feedback transformations. We recall a normal form, obtained by Kang, and discuss invariants of homogeneous transformations, also obtained by him. We provide an explicit construction of transformations bringing the system to Kang normal form. In Section 3 we dualize the main results of Section 2: we give a dual normal form, explicitely construct transformations bringing the system to that form, and define dual invariants of homogeneous transformations. In Section 4 we construct our canonical form and prove that two control systems are fedback equivalent if and only if their canonical forms coincide. We illustrate our canonical form by analyzing different ball-and-beam systems with various values of the friction constant. Proofs of all results are given in [15] (see also [14]).

2 Normal form and m-invariants

All objects, i.e., functions, maps, vector fields, control systems, etc., are considered in a neighborhood of $0 \in \mathbb{R}^n$ and assumed to be C^∞-smooth. Let h be a smooth function. By

$$h(x) = h^{[0]}(x) + h^{[1]}(x) + h^{[2]}(x) + \cdots = \sum_{m=0}^{\infty} h^{[m]}(x)$$

we denote its Taylor series expansion around zero, where $h^{[m]}(x)$ stands for a homogeneous polynomial of degree m.

Similarly, for a map ϕ of an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ (resp. for a vector field f on an open subset of $\mathbb{R}^n$) we will denote by $\phi^{[m]}$ (resp. by $f^{[m]}$) the term of degree m of its Taylor expansion at zero, i.e., each component $\phi_j^{[m]}$ of $\phi^{[m]}$ (resp. $f_j^{[m]}$ of $f^{[m]}$) is a homogeneous polynomial of degree m in x. Denote also $\bar{x}_i = (x_1, \ldots, x_i)$.

Consider the Taylor series expansion of the system Σ given by

$$\Sigma^\infty \ : \ \dot{\xi} = F\xi + Gu + \sum_{m=2}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}u), \quad (2.1)$$

where $F = \frac{\partial f}{\partial \xi}(0)$ and $G = g(0)$. We will assume throughout the paper that $f(0) = 0$ and $g(0) \neq 0$.

Consider also the Taylor series expansion Γ^∞ of the feedback transformation Γ given by

$$\Gamma^\infty \ : \ \begin{array}{rcl} x &=& \phi(\xi) = T\xi + \sum_{m=2}^{\infty} \phi^{[m]}(\xi) \\ u &=& \alpha(\xi) + \beta(\xi)v \\ &=& K\xi + Lv + \sum_{m=2}^{\infty} (\alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v), \end{array} \quad (2.2)$$

where T is an invertible matrix and $L \neq 0$. Let us analyse the action of Γ^∞ on the system Σ^∞ step by step.

To start with, consider the linear system

$$\dot{\xi} = F\xi + Gu.$$

Throughout the paper we will assume that it is controllable. It can thus be transformed (see e.g. [8]) by a linear feedback transformation of the form

$$\Gamma^1 : \begin{array}{rcl} x &=& T\xi \\ u &=& K\xi + Lv \end{array}$$

to the Brunovský canonical form (A, B). Assuming that the linear part (F, G), of the system Σ^∞ given by (2.1), has been transformed to the Brunovský canonical form (A, B), we follow an idea of Kang and Krener [11], [9] and apply successively a series of transformations

$$\Gamma^m : \begin{array}{rcl} x &=& \xi + \phi^{[m]}(\xi) \\ u &=& v + \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v, \end{array} \quad (2.3)$$

for $m = 2, 3, \ldots$. A feedback transformation defined as an infinite series of successive compositions of Γ^m, $m = 1, 2, \ldots$, will also be denoted by Γ^∞ because, as a formal power series, it is of the form (2.2). We will not address the problem of convergence and we will call such a series of successive compositions a *formal feedback transformation.*

Observe that each transformation Γ^m, for $m \geq 2$, leaves invariant all homogeneous terms of degree smaller than m and we will call Γ^m a ***homogeneous** feedback transformation of order m.* We will study the action of Γ^m on the following system $\Sigma^{[m]}$

$$\dot{\xi} = A\xi + Bu + f^{[m]}(\xi) + g^{[m-1]}(\xi)u + O(\xi, u)^{m+1}. \quad (2.4)$$

The starting point is the following result, proved in [9]. Consider another system $\tilde{\Sigma}^{[m]}$ given by

$$\dot{x} = Ax + Bv + \tilde{f}^{[m]}(x) + \tilde{g}^{[m-1]}(x)v + O(x, v)^{m+1}. \quad (2.5)$$

Proposition 1 *The feedback transformation Γ^m, defined by (2.3), brings the system $\Sigma^{[m]}$, given by (2.4), to $\tilde{\Sigma}^{[m]}$, given by (2.5), if and only if the following relations hold for any $1 \leq j < n$:*

$$\left\{ \begin{array}{rcl} L_{A\xi}(\phi_j^{[m]}) - \phi_{j+1}^{[m]}(\xi) &=& \tilde{f}_j^{[m]}(\xi) - f_j^{[m]}(\xi) \\ L_B(\phi_j^{[m]}) &=& \tilde{g}_j^{[m-1]}(\xi) - g_j^{[m-1]}(\xi) \\ L_{A\xi}(\phi_n^{[m]}) + \alpha^{[m]}(\xi) &=& \tilde{f}_n^{[m]}(\xi) - f_n^{[m]}(\xi) \\ L_B(\phi_n^{[m]}) + \beta^{[m-1]}(\xi) &=& \tilde{g}_n^{[m-1]}(\xi) - g_n^{[m-1]}(\xi). \end{array} \right. \quad (2.6)$$

This proposition represents the essence of the method developped by Kang and Krener and used in our paper. The problem of studying the feedback equivalence of two systems Σ and $\tilde{\Sigma}$ requires, in general, solving a system of 1-st order partial differential equations. On the other hand, if we perform the analysis step by step, then the problem of establishing the feedback equivalence of two systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$ reduces to solving the algebraic system (2.6).

Recall the notation $\bar{x}_i = (x_1, \ldots, x_i)$. Using the above proposition, Kang [9] proved the following result.

Theorem 1 *The system $\Sigma^{[m]}$ can be transformed via a homogeneous feedback transformations Γ^m, into the following normal form $\Sigma_{NF}^{[m]}$:*

$$\begin{aligned}
\dot{x}_1 &= x_2 + \sum_{i=3}^{n} x_i^2 P_{1,i}^{[m-2]}(\bar{x}_i) + O(x,v)^{m+1} \\
&\vdots \\
\dot{x}_j &= x_{j+1} + \sum_{i=j+2}^{n} x_i^2 P_{j,i}^{[m-2]}(\bar{x}_i) + O(x,v)^{m+1} \\
&\vdots \\
\dot{x}_{n-2} &= x_{n-1} + x_n^2 P_{n-2,n}^{[m-2]}(\bar{x}_n) + O(x,v)^{m+1} \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= v,
\end{aligned} \tag{2.7}$$

where $P_{j,i}^{[m-2]}(\bar{x}_i) = P_{j,i}^{[m-2]}(x_1, \cdots, x_i)$ are homogeneous polynomials of degree $m-2$.

In order to construct invariants of homogeneous feedback transformations let us define

$$X_i^{m-1}(\xi) = ad^i_{A\xi + f^{[m]}(\xi)}(B + g^{[m-1]}(\xi))$$

and let $X_i^{[m-1]}$ be its homogeneous part of degree $m-1$.

Following Kang [9], we denote by $a^{[m]t,i}(\xi)$ the homogeneous part of degree $m-2$ of

$$CA^{t-1}\left[X_{i-1}^{m-1}, X_{i-2}^{m-1}\right]_{|W_{n-i+2}},$$

where $C = (1, 0, \cdots, 0)$ and the submanifolds W_i are defined as follows:

$$W_i = \{\xi \in R^n \mid \xi_{i+1} = \cdots = \xi_n = 0\}.$$

The functions $a^{[m]t,i}(\xi)$, for $2 \le i \le n-1$, $1 \le t \le n-i$, will be called *m-invariants* of $\Sigma^{[m]}$.

The following result of Kang [9] asserts that m-invariants $a^{[m]t,r}(\xi)$ are complete invariants of homogeneous feedback and, moreover, illustrates their meaning for the normal form $\Sigma_{NF}^{[m]}$. Consider two systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$. Let

$$\{\, a^{[m]t,i}(\xi) \;:\; 2 \le i \le n-1,\ 1 \le t \le n-i \,\} \text{ and}$$

$$\{\, \tilde{a}^{[m]t,i}(\xi) \;:\; 2 \le i \le n-1,\ 1 \le t \le n-i \,\}$$

denote, respectively, their m-inavariants.

Theorem 2 *The m-invariants have the following properties :*

(i) *Two systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$ are equivalent via a homogeneous feedback of order m, modulo higher order terms, if and only if*

$$a^{[m]t,i}(\xi) = \tilde{a}^{[m]t,i}(\xi)$$

for any $2 \le i \le n-1$ and any $1 \le t \le n-i$.

(ii) *The m-invariants $\bar{a}^{[m]t,i}$ of the system $\Sigma_{NF}^{[m]}$, defined by (2.7), are given by*

$$\bar{a}^{[m]t,i} = \frac{\partial^2}{\partial x_{n-i+2}^2} x_{n-i+2}^2 P_{t,n-i+2}^{[m-2]}(x_1, \ldots, x_{n-i+2})$$

for any $2 \le i \le n-1$ and any $1 \le t \le n-i$.

Our first aim is to find explicitely feedback transformations bringing the system $\Sigma^{[m]}$ to its normal form $\Sigma_{NF}^{[m]}$. Define the functions $\psi_{j,i}^{[m-1]}(\xi)$ by setting $\psi_{j,0}^{[m-1]}(\xi) = \psi_{1,1}^{[m-1]}(\xi) = 0$, and

$$\psi_{j,i}^{[m-1]}(\xi) = (-1)^{n-i+1} CA^{j-1} \times \left(ad_{A\xi}^{n-i} g^{[m-1]} + \sum_{t=1}^{n-i} (-1)^t ad_{A\xi}^{t-1} ad_{A^{n-i-t}B} f^{[m]} \right)$$

if $1 \le j < i \le n$ and

$$\psi_{j,i}^{[m-1]}(\xi) = L_{A^{n-i}B} f_{j-1}^{[m]}(\bar{\xi}_i) + L_{A\xi}(\psi_{j-1,i}^{[m-1]}(\bar{\xi}_i))_{|W_i} + \psi_{j-1,i-1}^{[m-1]}(\bar{\xi}_{i-1}) + \int_0^{\xi_i} L_{A^{n-i+1}B} \psi_{j-1,i}^{[m-1]}(\bar{\xi}_i) d\xi_i$$

if $1 \le i \le j$, where $\psi_{j,i}^{[m-1]}(\bar{\xi}_i)$ is the restriction of $\psi_{j,i}^{[m-1]}(\xi)$ to the submanifold W_i. Define

$$\begin{aligned}
\phi_j^{[m]}(\xi) &= \sum_{i=1}^{n} \int_0^{\xi_i} \psi_{j,i}^{[m-1]}(\bar{\xi}_i) d\xi_i \quad \text{if } 1 \le j \le n-1, \\
\phi_n^{[m]}(\xi) &= f_{n-1}^{[m]}(\xi) + L_{A\xi}\phi_{n-1}^{[m]}(\xi), \\
\alpha^{[m]}(\xi) &= -\left(f_n^{[m]}(\xi) + L_{A\xi}\phi_n^{[m]}(\xi)\right), \\
\beta^{[m-1]}(\xi) &= -\left(g_n^{[m-1]}(\xi) + L_B\phi_n^{[m]}(\xi)\right).
\end{aligned} \tag{2.8}$$

We have the following result.

Theorem 3 *The feedback transformation*

$$\Gamma^m \;:\; \begin{aligned} x &= \xi + \phi^{[m]}(\xi) \\ u &= v + \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v, \end{aligned}$$

where $(\phi^{[m]}, \alpha^{[m]}, \beta^{[m-1]})$ are defined by (2.8), brings the system $\Sigma^{[m]}$ to a normal form $\Sigma_{NF}^{[m]}$ given by (2.7).

3 Dual normal form and dual m-invariants

In the normal form $\Sigma_{NF}^{[m]}$ given by (2.7), all the components of the control vector field $g^{[m-1]}$ are annihilated and all non removable nonlinearities are grouped in $f^{[m]}$. Kang and Krener in their pioneering paper [11] have shown that it is possible to transform, via a transformation Γ^2 of order 2, the second order system

$$\Sigma^{[2]} \ : \dot{\xi} = A\xi + Bu + f^{[2]}(\xi) + g^{[1]}(\xi)u + O(\xi, u)^3$$

to a dual normal form. In that form the components of the drift $f^{[2]}$ are annihilated while all non removable nonlinearities are, this time, present in $g^{[1]}$. The aim of this Section is to propose, for an arbitrary m, a dual normal form for the system $\Sigma^{[m]}$. Our dual normal form, on the one hand, generalizes that given in [11] for second order terms and, on the other hand, dualizes, the normal form $\Sigma_{NF}^{[m]}$. The structure of this Section will follow that of Section 2: we will give the dual normal form, then we define and study dual m-invariants, and, finally, we give an explicit construction of transformations bringing the system to its dual normal form. Our first result asserts that we can always bring $\Sigma^{[m]}$ to a dual normal form.

Theorem 4 *The system $\Sigma^{[m]}$ can be transformed via a homogeneous feedback transformations Γ^m, into the dual normal form $\Xi_{NF}^{[m]}$ given by:*

$$\begin{array}{rcl}
\dot{x}_1 &=& x_2 \\
\dot{x}_2 &=& x_3 + vQ_{2,n}^{[m-2]}(\bar{x}_n)x_n + O(x,v)^{m+1} \\
&\vdots& \\
\dot{x}_j &=& x_{j+1} + \sum\limits_{i=n-j+2}^{n} vQ_{j,i}^{[m-2]}(\bar{x}_i)x_i + O(x,v)^{m+1} \\
&\vdots& \\
\dot{x}_{n-1} &=& x_n + \sum\limits_{i=3}^{n} vQ_{j,i}^{[m-2]}(\bar{x}_i)x_i + O(x,v)^{m+1} \\
\dot{x}_n &=& v\,,
\end{array} \tag{3.1}$$

where $Q_{j,i}^{[m-2]}(\bar{x}_i) = Q_{j,i}^{[m-2]}(x_1, \cdots, x_i)$ are homogeneous polynomials of degree $m-2$.

Now we will define dual invariants. To start with, recall that the homogeneous vector field $X_i^{[m-1]}$ is defined by taking the homogeneous part of degree $m-1$ of the vector field

$$X_i^{m-1} = ad^i_{A\xi + f^{[m]}}(B + g^{[m-1]}).$$

Consider the system $\Sigma^{[m]}$ and for any j, such that $2 \leq j \leq n-1$, define the functions $b_j^{[m-1]}$ by setting

$$\begin{array}{rcl}
b_j^{[m-1]} &=& g_j^{[m-1]} + \sum\limits_{k=0}^{j-2} L_B L_{A\xi}^{j-k-2}(f_{k+1}^{[m]}) \\
&+& \sum\limits_{i=1}^{n} (-1)^{n-i+1} L_B L_{A\xi}^{j-1} \int_0^{\xi_i} CX_{n-i}^{[m-1]}(\bar{\xi}_i) d\xi_i.
\end{array}$$

The functions $b_j^{[m-1]}$ will be called *dual m-invariants* of the system $\Sigma^{[m]}$. Consider two systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$ of the form (2.4) and (2.5), respectively. Let

$$\{\, b_j^{[m-1]}(\xi) \ : \ 2 \leq j \leq n-1 \,\} \text{ and}$$

$$\{\, \tilde{b}_j^{[m-1]}(\xi) \ : \ 2 \leq j \leq n-1 \,\}$$

denote, respectively, their dual m-inavariants. The following result gives a dual of Theorem 2.

Theorem 5 *The dual m-invariants have the following properties:*

(i) *Two systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$ are equivalent via a homogeneous feedback of order m, modulo higher order terms, if and only if, for any $2 \leq j \leq n-1$,*

$$b_j^{[m-1]}(\xi) = \tilde{b}_j^{[m-1]}(\xi).$$

(ii) *The dual m-invariants of the system $\Xi_{NF}^{[m]}$, defined by (3.1), are given, for any $2 \leq j \leq n-1$, by*

$$\tilde{b}_j^{[m-1]} = \sum_{i=n-j+2}^{n} Q_{j,i}^{[m-2]}(x_1, \ldots, x_i)x_i.$$

This result asserts therefore that the dual m-invariants, similarly like m-invariants, form a set of complete invariants of the homogeneous feedback transformation. Notice however that the same information is encoded in both sets of invariants in different ways.

Now define the following functions

$$\begin{array}{rcl}
\phi_1^{[m]}(\xi) &=& \sum_{i=1}^{n} (-1)^{n-i+1} \int_0^{\xi_i} CX_{n-i}^{[m-1]}(\bar{\xi}_i) d\xi_i \\
\phi_{j+1}^{[m]}(\xi) &=& f_j^{[m]} + L_{A\xi}(\phi_j^{[m]}) \\
\alpha^{[m]} &=& -\left(f_n^{[m]} + L_{A\xi}(\phi_n^{[m]})\right) \\
\beta^{[m-1]} &=& -\left(g_n^{[m-1]} + L_B(\phi_n^{[m]})\right).
\end{array} \tag{3.2}$$

Theorem 6 *The feedback transformation*

$$\Gamma^m \ : \ \begin{array}{rcl} x &=& \xi + \phi^{[m]}(\xi) \\ u &=& v + \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v, \end{array}$$

where $(\phi^{[m]}, \alpha^{[m]}, \beta^{[m-1]})$ are defined by (3.2), brings the system $\Sigma^{[m]}$ to a dual normal form $\Xi_{NF}^{[m]}$ given by (3.1).

4 Canonical form

Consider the system Σ of the form (2.1). Apply successively to it a series of transformations Γ^m, $m = 1, 2, \ldots,$

such that each Γ^m brings $\Sigma^{[m]}$ to its normal form $\Sigma_{NF}^{[m]}$; for instance we can take a series of transformations defined by (2.8). In a dual way, apply successively to Σ a series of transformations Γ^m, $m = 1, 2, \ldots$, such that each Γ^m brings $\Sigma^{[m]}$ to its dual normal form $\Xi_{NF}^{[m]}$; for instance we can take a series of transformations defined by (3.2). Successive repeating of, respectively, Theorem 1 and Theorem 4 gives the following normal forms.

Theorem 7 *Consider the system Σ given by* (2.1).

(i) *There exists a formal feedback transformation Γ^∞ bringing the system Σ to a normal form Σ_{NF}^∞ given by*

$$\left\{\begin{array}{rcl} \dot{x}_j &=& x_{j+1} + \sum_{i=j+2}^{n} x_i^2 P_{j,i}(\bar{x}_i), \quad 1 \le j \le n-2 \\ \dot{x}_{n-1} &=& x_n \\ \dot{x}_n &=& v, \end{array}\right.$$

where $P_{j.i}(\bar{x}_i) = P_{j.i}(x_1, \cdots, x_i)$ are formal power series.

(ii) *There exists a formal feedback transformation Γ^∞ bringing the system Σ to a normal form Ξ_{NF}^∞ given by*

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_j &=& x_{j+1} + \sum_{i=n-j+2}^{n} vQ_{j,i}(\bar{x}_i)x_i, \quad 2 \le j \le n-1 \\ \dot{x}_n &=& v, \end{array}\right.$$

where $Q_{j,i}(\bar{x}_i) = Q_{j,i}(x_1, \cdots, x_i)$ are formal power series.

A natural and fundamental question which arises is whether the system Σ^∞ can admit two different normal forms, that is, whether the normal forms given by Theorem 7 are in fact canonical forms. It turns out that a given system can admit different normal forms, see [9], and the aim of this Section is to construct a canonical form for Σ^∞.

Consider the system Σ^∞ of the form (2.1). Let the first homogeneous term of Σ^∞, which cannot be annihilated by a feedback transformation, be of order m_0. As proved by Krener [12], the order m_0 is given by the largest integer $j+1$ such that all distributions $\mathcal{D}^k = \mathrm{span}\{g, \ldots, ad_f^{k-1} g\}$, for $1 \le k \le n-1$, are involutive modulo terms of order $j-1$. We can thus, due to Theorems 1 and 2, assume that, after applying a suitable feedback, Σ^∞ takes the form

$$\dot{\xi} = A\xi + Bu + \bar{f}^{[m_0]}(\xi) + \sum_{m=m_0+1}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}(\xi)u),$$

where (A, B) is in Brunovský canonical form and the first nonvanishing homogeneous vector field $\bar{f}^{[m_0]}$ is of the form

$$\bar{f}_j^{[m_0]}(\xi) = \left\{\begin{array}{ll} \sum_{i=j+2}^{n} \xi_i^2 P_{j,i}^{[m_0-2]}(\bar{\xi}_i), & 1 \le j \le n-2 \\ 0, & n-1 \le j \le n. \end{array}\right.$$

Let $(i_1, \ldots, i_{n-s})$, where $i_1 + \cdots + i_{n-s} = m_0$, be the largest, in the lexicographic ordering, $(n-s)$-tuple of nonnegative integers such that, for some $1 \le j \le n-2$, we have

$$\frac{\partial^{m_0} \bar{f}_j^{[m_0]}(\xi)}{\partial \xi_1^{i_1} \cdots \partial \xi_{n-s}^{i_{n-s}}} \neq 0.$$

Define

$$j^* = \sup\left\{ j = 1, \cdots, n-2, \;\middle|\; \frac{\partial^{m_0} \bar{f}_j^{[m_0]}(\xi)}{\partial \xi_1^{i_1} \cdots \partial \xi_{n-s}^{i_{n-s}}} \neq 0 \right\}.$$

We have the following result.

Theorem 8 *The system Σ^∞ given by* (2.1) *is equivalent by a formal feedback Γ^∞ to a system of the form*

$$\Sigma_{CF}^\infty \; : \; \dot{x} = Ax + Bv + \sum_{m=m_0}^{\infty} \bar{f}^{[m]}(x), \qquad (4.1)$$

where, for any $m \ge m_0$,

$$\bar{f}_j^{[m]}(x) = \left\{\begin{array}{ll} \sum_{i=j+2}^{n} x_i^2 P_{j,i}^{[m-2]}(\bar{x}_i), & 1 \le j \le n-2 \\ 0, & n-1 \le j \le n; \end{array}\right. \qquad (4.2)$$

additionally, we have

$$\frac{\partial^{m_0} \bar{f}_{j^*}^{[m_0]}(x)}{\partial x_1^{i_1} \cdots \partial x_{n-s}^{i_{n-s}}} = \pm 1 \qquad (4.3)$$

and, moreover, for any $m \ge m_0 + 1$,

$$\frac{\partial^{m_0} \bar{f}_{j^*}^{[m]}(x)}{\partial x_1^{i_1} \cdots \partial x_{n-s}^{i_{n-s}}}(x_1, 0, \ldots, 0) = 0. \qquad (4.4)$$

The form Σ_{CF}^∞ satisfying (4.2), (4.3) and (4.4) will be called the *canonical form* of Σ^∞. The name is justified by the following result.

Theorem 9 *Two systems Σ_1^∞ and Σ_2^∞ are formally feedback equivalent if and only if their canonical forms $\Sigma_{1,CF}^\infty$ and $\Sigma_{2,CF}^\infty$ coincide.*

Kang [9], generalizing [11], proved that any system Σ^∞ can be brought by a formal feedback to the normal form (4.1), for which (4.2) is satisfied. He also observed that his normal forms are not unique. Our results, Theorems 8 and 9, complete his study. We show that for each degree m of homogenity we can use a 1-dimensional subgroup of feedback transformations which preserves the "triangular" structure of (4.2) and at the same time allows us to normalize one term. The form of (4.3) and (4.4) is a result of this normalization.

Example We consider the well known ball-and-beam example [4], whose Lagrange equations are given by

$$\begin{array}{rcl}(\frac{J_b}{R^2}+1)\ddot{r}+G\sin\theta+\beta\dot{r}-r\dot{\theta}^2 & = & 0\\ (r^2+J_b)\ddot{\theta}+2r\dot{r}\dot{\theta}+Gr\cos\theta & = & \tau,\end{array}$$

where we take the mass of the ball equal to one and the momentum of the beam equal to zero. Let J_b denote the momentum of the ball, r its position, τ the torque applied to the beam, θ its angle with respect to the horizontal and $\beta \geq 0$ the viscous friction constant. We set $k_0 = \frac{1}{G^2}$ and $\tau = 2r\dot{r}\dot{\theta} + Gr\cos\theta + (r^2+J)u$, where u denotes the control variable. In the coordinates system $(\xi_1, \xi_2, \xi_3, \xi_4) = (r, \dot{r}, \theta, \dot{\theta})$, we obtain the following equations

$$\begin{array}{rcl}\dot{\xi}_1 & = & \xi_2\\ \dot{\xi}_2 & = & -\beta\xi_2 - G\sin\xi_3 + \xi_1\xi_4^2\\ \dot{\xi}_3 & = & \xi_4\\ \dot{\xi}_4 & = & u.\end{array} \qquad (4.5)$$

The coordinates change

$$\begin{array}{rcl}y_1 & = & \xi_1\\ y_2 & = & \xi_2\\ y_3 & = & -\beta\xi_2 - G\sin\xi_3\\ y_4 & = & \beta(\beta\xi_2 + G\sin\xi_3) - (G\cos\xi_3)\xi_4,\end{array}$$

together with a feedback $u = \alpha(\xi) + \beta(\xi)w$, takes the system (4.5) into the following one

$$\begin{array}{rcl}\dot{y}_1 & = & y_2\\ \dot{y}_2 & = & y_3 + y_1 Q(y_2, y_3)(y_4^2 + \beta y_3)^2\\ \dot{y}_3 & = & y_4 - \beta y_1 Q(y_2, y_3)(y_4^2 + \beta y_3)^2\\ \dot{y}_4 & = & w,\end{array}$$

where Q is an analytic function satisfying $Q(0) = k_0$. Applying a suitable feedback transformation (see [14] and [15] for details), we show that the ball-and-beam system is feedback equivalent to the following canonical form

$$\begin{array}{rcl}\dot{x}_1 & = & x_2 + \beta(x_1x_3^2 + x_3^2P_{1,3}(\bar{x}_3) + x_4^2P_{1,4}(\bar{x}_4))\\ \dot{x}_2 & = & x_3 + x_1x_4^2 + x_4^2P_{2,4}(\bar{x}_4)\\ \dot{x}_3 & = & x_4\\ \dot{x}_4 & = & v,\end{array}$$

where $P_{1,3}$, $P_{1,4}$ and $P_{2,4}$ are formal power series whose 1-jets vanish at zero and $P_{2,4}(\bar{x}_4) = x_2R_1(\bar{x}_2) + x_3R_2(\bar{x}_3) + x_4R_3(\bar{x}_4)$. If $\beta = 0$ then $P_{2,4}(\bar{x}_4) = x_1P(x_3)$. As a conclusion, two ball-and-beam systems are feedback equivalent if and only if they have the same friction constant β.

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