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Weighted Canonical Forms of Nonlinear Single-Input Control Systems with Noncontrollable Linearization

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Weighted Canonical Forms of Nonlinear Single-Input Control Systems with Noncontrollable Linearization

Issa A. Tall and Witold Respondek

Abstract— **We propose a weighted canonical form for single-input systems with noncontrollable first order approximation under the action of formal feedback transformations. This weighted canonical form is based on associating different weights to the linearly controllable and linearly noncontrollable parts of the system. We prove that two systems are formally feedback equivalent if and only if their weighted canonical forms coincide up to a diffeomorphism whose restriction to the linearly controllable part is identity.**

INTRODUTION

The feedback classification of nonlinear control single-input systems of the form

$$\Sigma : \dot{x} = f(x) + g(x)u$$

under the action of feedback transformations of the form

$$\Gamma : \begin{array}{l} z = \phi(x) \\ u = \alpha(x) + \beta(x)v \end{array}$$

has been extensively studied during the past years. Normal forms for such systems have been computed [4], [5], [6], [10], [12] using a fruitful approach proposed by Kang and Krener, which generalizes to control systems a method developed by Poincaré for dynamical systems (see, e.g., [1]). This method is based on analyzing the action of the homogeneous components of the feedback group on the homogeneous components, of the same degree, of the system.

The problem of obtaining canonical forms is more complicated because it involves analyzing the action of homogenous components of lower degree of the feedback group on the homogenous components of higher degree of the system. Recently canonical forms for single-input systems, with controllable linearization, have been obtained by the authors [10], [13] who proved that two systems are feedback equivalent if and only if their canonical forms coincide. Construction of those canonical forms has led to a complete description of symmetries of single-input control systems with controllable linearization. Those symmetries have been fully described by the authors [8], [9] using the canonical form: possessing a stationary symmetry, a non stationary symmetry, a one 1-parameter family of symmetries or two 1-parameter families of symmetries corresponds, respectively, to the fact that the drift of the canonical form is odd, is periodic with respect to the first variable, does not depend on the first variable or is odd and does not depend on the first variable.

The aim of the present paper is to construct a canonical form for single-input systems with uncontrollable linearization. We recall that normal forms for such systems have been already obtained [11], [12], [5], [7] and so constructing canonical forms has been a challenging problem. Analyzing the proposed canonical forms should allow to describe symmetries and feedback invariants of single-input control systems with noncontrollable linearization.

I. NORMAL FORMS

All objects, that is, functions, maps, vector fields, control systems, etc., are considered in a neighborhood of $0 \in \mathbb{R}^n$ and assumed to be C^∞-smooth. Consider the system

$$\Sigma : \dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R},$$

where $f(0) = 0$ and $g(0) \neq 0$ and let

$$\Lambda : \dot{x} = Fx + Gu$$

be its linearization around the equilibrium point $0 \in \mathbb{R}^n$. We assume this linearization to be noncontrollable, that is

$$\operatorname{rank}[\, G\ FG\ \cdots\ F^{n-1}G\,] = n - r,$$

for some positive integer r. Applying a linear feedback transformation we can always assume that the linear part (F, G) of the system is in Jordan-Brunoský canonical form

$$(A, B) = (\begin{pmatrix} A^1 & 0 \\ 0 & A^2 \end{pmatrix}, \begin{pmatrix} 0 \\ B^2 \end{pmatrix})$$

that is, the uncontrollable part, of dimension r, is defined by the matrix A^1 in the Jordan form and the controllable part, of dimension $n-r$, is defined by the pair (A^2, B^2) in the Brunovský form.

We will be using the same notation $\mathcal{S}_r(\mathbb{R}, 0)$ for the space $C^\infty(\mathbb{R}^r, 0)$ of smooth functions defined locally at $0 \in \mathbb{R}^r$ as well as for the space $\mathbb{R}[[x_1, \ldots, x_r]]$ of formal power series in $x_1, \ldots, x_r$ with real coefficients. For a smooth $\mathbb{R}$-valued function h, defined in a neighborhood of $0 \times 0 \in \mathbb{R}^r \times \mathbb{R}^{n-r}$, we denote by

$$h(x) = h^{[0]}(x) + h^{[1]}(x) + h^{[2]}(x) + \cdots = \sum_{m=0}^{\infty} h^{[m]}(x)$$

its Taylor series expansion at $0 \times 0 \in \mathbb{R}^r \times \mathbb{R}^{n-r}$, where $h^{[m]}(x)$ stands for a homogeneous polynomial of degree m in the variables $x_{r+1}, \cdots, x_n$ whose coefficients are in $\mathcal{S}_r(\mathbb{R}, 0)$.

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Following [12], we will use different weights corresponding to the uncontrollable and controllable parts:

$$f^{\langle m\rangle} = \left(f_1^{[m-1]}, \cdots, f_r^{[m-1]}, f_{r+1}^{[m]}, \cdots, f_n^{[m]}\right)^T$$
$$g^{\langle m\rangle} = \left(g_1^{[m-1]}, \cdots, g_r^{[m-1]}, g_{r+1}^{[m]}, \cdots, g_n^{[m]}\right)^T$$
$$\phi^{\langle m\rangle} = \left(\phi_1^{[m-1]}, \cdots, \phi_r^{[m-1]}, \phi_{r+1}^{[m]}, \cdots, \phi_n^{[m]}\right)^T,$$

where, for any $1 \leq j \leq r$, we set $f_j^{[-1]}(x) = g_j^{[-1]}(x) = \phi_j^{[-1]}(x) = 0$, and $h^{\langle m\rangle}(x) = h^{[m]}(x)$ for a homogeneous polynomial. We will consider the action of the Taylor series expansion Γ^∞ of the feedback transformation Γ given by

$$\Gamma^\infty : \begin{array}{l} z = Tx + \sum\limits_{m=0}^{\infty} \phi^{\langle m\rangle}(x) \\ u = Kx + Lv + \sum\limits_{m=0}^{\infty} (\alpha^{\langle m\rangle}(x) + \beta^{\langle m-1\rangle}(x)v), \end{array} \tag{I.1}$$

on the Taylor series expansion of the system Σ given by

$$\Sigma^\infty : \dot{x} = Fx + Gu + \sum_{m=0}^{\infty} \left(f^{\langle m\rangle}(x) + g^{\langle m-1\rangle}(x)u\right). \tag{I.2}$$

After having transformed (F, G) into its Jordan-Brunovský form we then study the action of the weighted homogeneous feedback

$$\Gamma^{\langle m\rangle} : \begin{array}{l} z = x + \phi^{\langle m\rangle}(x) \\ u = v + \alpha^{\langle m\rangle}(x) + \beta^{\langle m-1\rangle}(x)v \end{array}$$

on the weighted homogeneous system

$$\Sigma^{\langle m\rangle} : \dot{x} = Ax + Bu + f^{\langle 1\rangle}(x) + f^{\langle m\rangle}(x) + g^{\langle m-1\rangle}(x)u,$$

where the last $n-r$ components of the vector field $f^{\langle 1\rangle}$ are equal to zero (which can always be achieved by a feedback transformation).

Denote $\bar{z}_i = (z_1, \ldots, z_i)$. We proved the following result in [12].

Theorem I.1 *For any $m \geq 2$, there exists a weighted feedback transformation $\Gamma^{\langle m\rangle}$, that transforms the weighted homogeneous system $\Sigma^{\langle m\rangle}$ into its weighted homogeneous normal form*

$$\Sigma_{NF}^{\langle m\rangle} : \dot{z} = Az + Bv + \bar{f}^{\langle 1\rangle}(z) + \bar{f}^{\langle m\rangle}(z),$$

with $\bar{f}^{\langle 1\rangle}(z) = f^{\langle 1\rangle}(z)$ and the components of the vector field $\bar{f}^{\langle m\rangle}(z)$ satisfy

$$\bar{f}_j^{\langle m\rangle}(z) = \begin{cases} z_{r+1}^{m-1} S_{j,m}(\bar{z}_r) + \sum\limits_{i=r+2}^{n} z_i^2 Q_{j,i}^{\langle m-3\rangle}(\bar{z}_i) \\ \text{if } 1 \leq j \leq r, \\ \sum\limits_{i=j+2}^{n} z_i^2 P_{j,i}^{\langle m-2\rangle}(\bar{z}_i), \text{ if } r+1 \leq j \leq n-2, \\ 0, \qquad\qquad \text{if } n-1 \leq j \leq n, \end{cases} \tag{I.3}$$

where $S_{j,m}(\bar{z}_r)$ are C^∞-functions of the variables $z_1, \cdots, z_r$, the functions $P_{j,i}^{\langle m-2\rangle}$ and $Q_{j,i}^{\langle m-3\rangle}$ are homogeneous polynomials, respectively of degrees $m-2$ and $m-3$, of the variables $z_{r+1}, \cdots, z_i$, with coefficients in $\mathcal{S}_r(\mathbb{R}, 0)$.

Denote by $\lambda = \{\lambda_1, \ldots, \lambda_r\}$ the spectrum of A^1, that is of the uncontrollable linear part of the system (I.2). We say that an eigenvalue λ_j is *resonant* if there is a r-tuple $(\alpha_1, \ldots, \alpha_r)$ of positive integers such that

$$\alpha_1 + \cdots + \alpha_r \geq 2 \quad \text{and} \quad \lambda_j = \alpha_1\lambda_1 + \cdots + \alpha_r\lambda_r. \tag{I.4}$$

The set $\mathcal{R}_j$ of all r-tuples $\alpha = (\alpha_1, \ldots, \alpha_r)$ satisfying (I.4) is called the *resonant set* associated to the eigenvalue λ_j.

A normalization of the vector field $f^{\langle 1\rangle}(x)$ followed by a successive repeating of Theorem I.1, for $m = 2, 3, \cdots$, yield the following result, see [12]:

Theorem I.2 *There exists a formal feedback transformation Γ^∞ of the form (I.1), which brings the system Σ^∞, given by (I.2), into its normal form*

$$\Sigma_{NF}^{\infty} : \dot{z} = Az + Bv + \bar{f}^{\langle 1\rangle}(z) + \bar{f}(z),$$

where the components $\bar{f}_j(z)$ of $\bar{f}(z)$ satisfy

$$\bar{f}_j(z) = \begin{cases} z_{r+1} S_j(\bar{z}_{r+1}) + \sum\limits_{i=r+2}^{n} z_i^2 Q_{j,i}(\bar{z}_i) \\ \text{if } 1 \leq j \leq r, \\ \sum\limits_{i=j+2}^{n} z_i^2 P_{j,i}(\bar{z}_i), \qquad \text{if } r+1 \leq j \leq n-2, \\ 0, \qquad\qquad\qquad \text{if } n-1 \leq j \leq n, \end{cases}$$

and (if the eigenvalues of A^1 are distinct) the components $\bar{f}_j^{\langle 1\rangle}(z)$ of $\bar{f}^{\langle 1\rangle}(z)$ satisfy

$$\bar{f}_j^{\langle 1\rangle}(z) = \begin{cases} \sum\limits_{\alpha \in \mathcal{R}_j} \gamma_{j,\alpha} z_1^{\alpha_1} \cdots z_r^{\alpha_r}, & \text{if } 1 \leq j \leq r \\ 0, & \text{if } r+1 \leq j \leq n. \end{cases} \tag{I.5}$$

Above, $P_{j,i}$, $Q_{j,i}$ and S_j are formal power series of the indicated variables, and $\gamma_{j,\alpha} \in \mathbb{R}$.

II. Canonical forms

The objective of this section is to produce a canonical form for systems under consideration.

Consider the system Σ^∞ of the form (I.2) and assume that its linear part (F, G) has been already brought to the Brunovský-Jordan canonical form (A, B). Let the first weighted homogeneous term of Σ^∞ which cannot be annihilated by a feedback transformation be of degree $\langle m_0\rangle$, $m_0 \geq 2$. This means we can assume (see Theorem I.1) that, after applying a suitable feedback, Σ^∞ takes the form

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu + \bar{f}^{\langle 1\rangle}(x) + \bar{f}^{\langle m_0\rangle}(x) \\ & + & \sum\limits_{m=m_0+1}^{\infty} \left(f^{\langle m\rangle}(x) + g^{\langle m-1\rangle}(x)u\right), \end{array} \tag{II.1}$$

where the components of the first non vanishing homogeneous vector field $\bar{f}^{\langle m_0\rangle}$ are of the form (I.3) for $m = m_0$.

Let s be the smallest nonnegative integer such that

$$L_{A^sB}\bar{f}_j^{\langle m_0\rangle} = \frac{\partial \bar{f}_j^{\langle m_0\rangle}}{\partial x_{n-s}} \neq 0 \tag{II.2}$$

for some $1 \leq j \leq n-2$. This implies that

$$ad_{A^{k-1}B}\bar{f}^{\langle m_0\rangle} = 0 \tag{II.3}$$

for any $1 \leq k \leq s$.

We define j_* to be the smallest integer $1 \leq j \leq n-2$ such that (II.2) is satisfied. Thus, for any $1 \leq j \leq j_* - 1$, we have

$$L_{A^sB}\bar{f}_j^{\langle m_0\rangle} = \frac{\partial \bar{f}_j^{\langle m_0\rangle}}{\partial x_{n-s}} = 0. \tag{II.4}$$

Let $(i_{r+1}, \cdots, i_{n-s})$, where $i_{r+1} + \cdots + i_{n-s} = \langle m_0\rangle$ and $i_{n-s} \geq 2$, be the smallest, in the lexicographic ordering, $(n-s)$-tuple of nonnegative integers such that

$$\frac{\partial^{\langle m_0\rangle}\bar{f}_{j_*}^{\langle m_0\rangle}}{\partial x_{r+1}^{i_{r+1}}\cdots\partial x_{n-s}^{i_{n-s}}} = \theta_{j_*}(\bar{x}_r) \neq 0. \tag{II.5}$$

By $i_{r+1} + \cdots + i_{n-s} = \langle m_0\rangle$ we mean that $i_{r+1} + \cdots + i_{n-s} = m_0 - 1$ if $1 \leq j_* \leq r$ and $i_{r+1} + \cdots + i_{n-s} = m_0$ if $r+1 \leq j_* \leq n-2$.

For simplicity we will assume that $\theta_{j_*}(0) \neq 0$. We have the following result.

Theorem II.1 *The system Σ^∞, given by (I.2), is equivalent by a formal feedback Γ^∞, given by (I.1), to a system of the form*

$$\Sigma_{CF}^\infty : \dot{z} = Az + Bv + \bar{f}^{\langle 1\rangle}(z) + \sum_{m=m_0}^{\infty} \bar{f}^{\langle m\rangle}(z),$$

where, for any $m \geq m_0$, the components of $\bar{f}^{\langle m\rangle}(z)$ are given by (I.3) and those of $\bar{f}^{\langle 1\rangle}(z)$ by (I.5); additionally, we have

$$\frac{\partial^{\langle m_0\rangle}\bar{f}_{j_*}^{\langle m_0\rangle}}{\partial z_{r+1}^{i_{r+1}}\cdots\partial z_{n-s}^{i_{n-s}}} = \pm 1 \tag{II.6}$$

and, moreover, for any $m \geq m_0 + 1$,

$$\frac{\partial^{\langle m_0\rangle}\bar{f}_{j_*}^{\langle m\rangle}}{\partial z_{r+1}^{i_{r+1}}\cdots\partial z_{n-s}^{i_{n-s}}}(\bar{z}_r, z_{r+1}, 0, \ldots, 0) = 0. \tag{II.7}$$

The form Σ_{CF}^∞ satisfying (I.3), (I.5), (II.6) and (II.7) will be called the *weighted canonical form* of Σ^∞. The following definition is crucial for an interpretation of the weighted canonical form.

Definition II.2 (i) Given a system Σ^∞ whose linear part is in Jordan-Brunovský canonical form, we will say that an invertible change of coordinates $z = \phi(x)$ is a *diffeomorphism of the uncontrollable part* if

$$\phi_j(x) = \phi_j(x_1, \cdots, x_r), \quad \text{for } 1 \leq j \leq r$$
$$\phi_j(x) = kx_j, \quad k \in \mathbb{R}, \qquad \text{for } r+1 \leq j \leq n.$$

(ii) We will say that two systems

$$\Sigma : \dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R} \ \text{ and}$$
$$\tilde{\Sigma} : \dot{z} = \tilde{f}(z) + \tilde{g}(z)v, \quad z \in \mathbb{R}^n,\ v \in \mathbb{R}$$

such that the linearizations of both are in the Jordan-Brunovsky canonical forms, *coincide on controllable parts*, if there exists a formal diffeomorphism of the uncontrollable parts transforming Σ into $\tilde{\Sigma}$.

Of course, we should speak about linearly controllable and linearly uncontrollable parts but we skip the word "linearly" by abuse of language. The name of the weighted canonical form is justified by the following result:

Theorem II.3 *Two systems Σ_1^∞ and Σ_2^∞ are formally feedback equivalent if and only if their weighted canonical forms $\Sigma_{1,CF}^\infty$ and $\Sigma_{2,CF}^\infty$ coincide on controllable parts.*

III. Proofs

In this section we will prove our main results, which are Theorems II.1 and II.3.

A Proof of Theorem II.1

The proof of Theorem II.1 consists of three steps. In the first step, we will normalize the vector field $f^{\langle 1\rangle}$. In the second step we will show that the component $\bar{f}_{j_*}^{\langle m_0\rangle}$ of the first non vanishing weighted homogeneous term can be normalized. Finally, we will prove, by an induction argument, that the terms of degree $\langle m_0 + l - 1\rangle$ can be put into their canonical form.

It is a well known result of Poincaré (see, e.g., [1]) that if all eigenvalues are distinct, then by a formal diffeomorphism of the uncontrollable part we can get rid of all nonresonant terms and bring $\dot{x}_j = \lambda_j x_j + f_j^{\langle 1\rangle}(x)$ into $\dot{z}_j = \lambda_j z_j + \bar{f}_j^{\langle 1\rangle}(z)$, for $1 \leq j \leq r$, where $\bar{f}_j^{\langle 1\rangle}(z)$ is of the form (I.5).

To perform the second step of the proof of the theorem, we need to show that the coefficient $\theta_{j_*}(\bar{x}_r)$ of the homogeneous term $x_{r+1}^{i_{r+1}}\cdots x_{n-s}^{i_{n-s}}$ of $\bar{f}_{j_*}^{\langle m_0\rangle}(x)$ can be normalized to either 1 or -1.

To see this, consider the weighted homogeneous system

$$\Sigma^{\langle m_0\rangle} : \dot{x} = Ax + Bu + \bar{f}^{\langle 1\rangle}(x) + \bar{f}^{\langle m_0\rangle}(x)$$

and apply a weighted linear feedback defined by:

$$\begin{aligned} z_j &= x_j && 1 \leq j \leq r, \\ z_{r+1} &= \epsilon(\bar{x}_r)x_{r+1}, \\ z_{j+1} &= L_{Ax+\bar{f}^{\langle 1\rangle}(x)}^{j-r}(\epsilon(\bar{x}_r)x_{r+1}), && r < j < n, \end{aligned} \tag{III.1}$$

with $(\epsilon(\bar{x}_r))^{m_0-1} = \pm\theta_{j_*}(\bar{x}_r)$, completed by

$$v = \alpha^{\langle l+1\rangle}(x) + \beta^{\langle l\rangle}(x)u = -L_{Ax+\bar{f}^{\langle 1\rangle}(x)}^{n-r}(\epsilon(\bar{x}_r)x_{r+1}).$$

Notice that for any $r+1 \leq j \leq n$, we have

$$z_j = \eta_{j,r+1}(\bar{x}_r)x_{r+1} + \cdots + \eta_{j,j}(\bar{x}_r)x_j,$$

where $\eta_{j,j}(\bar{x}_r) = \epsilon(\bar{x}_r)$ with $\epsilon(0) \neq 0$. It thus follows that the inverse of the transformation (III.1) is such that

$$x_j = \sigma_{j,r+1}(\bar{z}_r)z_{r+1} + \cdots + \sigma_{j,j}(\bar{z}_r)z_j,$$

for any $r+1 \leq j \leq n$, and $\sigma_{j,j} = 1/\eta_{j,j}$.

Using the fact that the transformation (III.1) and its inverse are triangular, we can show (see [14] for details) that, by applying a weighted homogeneous feedback of degree $\langle m_0 \rangle$, we take the system $\tilde{\Sigma}^{\langle m_0 \rangle}$ into its normal form where the condition (II.6) is satisfied.

In order to normalize $f_{j_*}^{\langle m \rangle}$, for $m \geq m_0 + 1$, we will need the following Lemma whose proof is straightforward and follows from the condition (II.3). Define the flag of involutive distributions $\mathcal{D}^1 \subset \cdots \subset \mathcal{D}^{s+1}$ as following

$$\mathcal{D}^k = \text{span}\left\{ \frac{\partial}{\partial z_{n-k+1}}, \cdots, \frac{\partial}{\partial z_n} \right\}$$

for any $1 \leq k \leq s+1$.

Lemma III.1 *Let $1 \leq k \leq s$. For any vector field $H \in \mathcal{D}^k$ we have*

$$\left[\bar{f}^{\langle m_0 \rangle}(z), H(z) \right] \in \mathcal{D}^k.$$

Moreover,

$$ad^{s-k+1}_{Az+f^{\langle 1 \rangle}(z)} \left[\bar{f}^{\langle m_0 \rangle}(z), H(z) \right] \in \mathcal{D}^{s+1}.$$

Let us suppose that the system (II.1)-(I.3) is of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + \bar{f}^{\langle 1 \rangle}(x) + \sum_{m=m_0}^{m_0+l-1} \bar{f}^{\langle m \rangle}(x) \\ &+ \sum_{m=m_0+l}^{\infty} \left(f^{\langle m \rangle}(x) + g^{\langle m-1 \rangle}(x)u \right), \end{aligned} \tag{III.2}$$

where the vector fields $\bar{f}^{\langle m \rangle}(x)$, for $m_0 \leq m \leq m_0 + l - 1$,satisfy the conditions (I.3), (II.6), and (II.7).

Consider the feedback transformation

$$\Gamma^{\langle l+1 \rangle} : \begin{array}{l} z = x + \phi^{\langle l+1 \rangle}(x) \\ u = v + \alpha^{\langle l+1 \rangle}(x) + \beta^{\langle l \rangle}(x)v, \end{array} \tag{III.3}$$

where the components $\phi_j^{\langle l+1 \rangle}(x)$ of $\phi^{\langle l+1 \rangle}(x)$ are defined as follows:

$$\begin{aligned} &\phi_j^{\langle l+1 \rangle}(x) = 0 && 1 \leq j \leq r, \\ &\phi_{r+1}^{\langle l+1 \rangle}(x) = \mu(\bar{x}_r)x_{r+1}^{l+1}, \\ &\phi_{j+1}^{\langle l+1 \rangle}(x) = L_{Ax+\bar{f}^{\langle 1 \rangle}(x)}\phi_j^{\langle l+1 \rangle}(x), && r < j < n, \end{aligned} \tag{III.4}$$

completed by the feedback

$$v = \alpha^{\langle l+1 \rangle}(x) + \beta^{\langle l \rangle}(x)u = -L_{Ax+\bar{f}^{\langle 1 \rangle}(x)}\phi_n^{\langle l+1 \rangle}(x).$$

The importance of this transformation is that it leaves invariant all terms of degree less than $\langle m_0 + l - 1 \rangle$ a nd takes the system (III.2) into the form

$$\begin{aligned} \dot{z} &= Az + Bv + \bar{f}^{\langle 1 \rangle}(z) + \sum_{m=m_0}^{m_0+l-1} \bar{f}^{\langle m \rangle}(z) \\ &+ \sum_{m=m_0+l}^{\infty} \left(\tilde{f}^{\langle m \rangle}(z) + \tilde{g}^{\langle m-1 \rangle}(z)v \right), \end{aligned} \tag{III.5}$$

where

$$\begin{array}{rcl} \tilde{f}^{\langle m_0+l \rangle}(z) & = & f^{\langle m_0+l \rangle}(z) + \left[\bar{f}^{\langle m_0 \rangle}(z), \phi^{\langle l+1 \rangle}(z) \right] \\ \tilde{g}^{\langle m_0+l-1 \rangle}(z) & = & g^{\langle m_0+l-1 \rangle}(z). \end{array}$$

Denote by $a^{\langle m_0+l \rangle j,i+2}$, $\hat{a}^{\langle m_0+l \rangle j,i+2}$, and $\tilde{a}^{\langle m_0+l \rangle j,i+2}$ the weighted homogeneous invariants (see [12]) associated, respectively, to the weighted homogeneous systems

$$\begin{array}{rcl} \Sigma^{\langle m_0+l \rangle} : \dot{z} & = & Az + Bu + f^{\langle 1 \rangle}(z) \\ & + & f^{\langle m_0+l \rangle}(z) + g^{\langle m_0+l-1 \rangle}(z)u, \\ \hat{\Sigma}^{\langle m_0+l \rangle} : \dot{z} & = & Az + Bu + f^{\langle 1 \rangle}(z) \\ & + & \hat{f}^{\langle m_0+l \rangle}(z) + \hat{g}^{\langle m_0+l-1 \rangle}(z)u, \end{array}$$

and

$$\begin{array}{rcl} \tilde{\Sigma}^{\langle m_0+l \rangle} : \dot{z} & = & Az + Bu + f^{\langle 1 \rangle}(z) \\ & + & \tilde{f}^{\langle m_0+l \rangle}(z) + \tilde{g}^{\langle m_0+l-1 \rangle}(z)u, \end{array}$$

where

$$\hat{f}^{\langle m_0+l \rangle}(z) = \left[\bar{f}^{\langle m_0 \rangle}(z), \phi^{\langle l+1 \rangle}(z) \right] \text{ and } \hat{g}^{\langle m_0+l-1 \rangle}(z) = 0.$$

It follows that

$$\tilde{a}^{\langle m_0+l \rangle j,i+2} = a^{\langle m_0+l \rangle j,i+2} + \hat{a}^{\langle m_0+l \rangle j,i+2} \tag{III.6}$$

for all $(j,i) \in \Delta_r$, where we define the subset $\Delta_r = \Delta_r^1 \cup \Delta_r^2 \subset \mathbb{N} \times \mathbb{N}$ by

$$\Delta_r^1 = \{ (j,i) : 1 \leq j \leq r \text{ and } 0 \leq i \leq n-r-1 \},$$

$$\Delta_r^2 = \{ (j,i) : r < j \leq n-2 \text{ and } 0 \leq i \leq n-j-2 \}.$$

By a tedious calculation (see [14] for details) we can prove that by an appropriate choice of feedback transformation (III.3)-(III.4), i.e., that of $\mu(\bar{x}_r)$, we can have

$$\tilde{a}^{\langle m_0+l \rangle j_*,s+2} = a^{\langle m_0+l \rangle j_*,s+2} + \hat{a}^{\langle m_0+l \rangle j_*,s+2} = 0,$$

where $(j_*, s) \in \Delta_r$ is given by (II.5).

Applying a normalizing weighted homogeneous transformation of degree $\langle m_0 + l \rangle$, we thus take the system (III.5) into the form

$$\begin{aligned} \dot{z} &= Az + Bv + \bar{f}^{\langle 1 \rangle}(z) + \sum_{m=m_0}^{m_0+l} \bar{f}^{\langle m \rangle}(z) \\ &+ \sum_{m=m_0+l+1}^{\infty} \left(\tilde{f}^{\langle m \rangle}(z) + \tilde{g}^{\langle m-1 \rangle}(z)v \right), \end{aligned} \tag{III.7}$$

where for any $m_0 \leq m \leq m_0 + l$, the components of the vector field $\bar{f}^{\langle m \rangle}(z)$ are given by (I.3), (I.5), (II.6) and (II.7).

This completes the proof of Theorem II.1. □

B. Proof of Theorem II.3

Let us consider two systems Σ_1^∞ and Σ_2^∞ and let

$$\Sigma_{1,CF}^\infty : \dot{x} = Ax + Bu + \bar{f}^{\langle 1 \rangle}(x) + \sum_{m=m_{0,1}}^{\infty} \bar{f}^{\langle m \rangle}(x) \text{ and}$$

$$\Sigma_{2,CF}^\infty : \dot{z} = Az + Bv + \tilde{f}^{\langle 1 \rangle}(z) + \sum_{m=m_{0,2}}^{\infty} \tilde{f}^{\langle m \rangle}(z)$$

denote respectively their weighted canonical forms, where $m_{0,1}$ and $m_{0,2}$ denote the degrees of the first non linearizable homogeneous parts. It is obvious that Σ_1^∞ and Σ_2^∞ are feedback equivalent if their canonical forms $\Sigma_{1,CF}^\infty$ and $\Sigma_{2,CF}^\infty$ coincide on controllable parts. To prove the converse, we assume that the systems Σ_1^∞ and Σ_2^∞ are formal feedback equivalent while their weighted canonical forms fail to coincide on controllable parts. Since Σ_1^∞ and Σ_2^∞ are feedback equivalent, so are their weighted canonical forms $\Sigma_{1,CF}^\infty$ and $\Sigma_{2,CF}^\infty$. It means that there exists a transformation Γ^∞ which brings $\Sigma_{1,CF}^\infty$ into $\Sigma_{2,CF}^\infty$. First remark that, from the definition of the integer m_0, we necessarily have $m_{0,1} = m_{0,2}$. Then, Theorem 2 of [12], and the fact that the components $\bar{f}_{j_*}^{\langle m_0\rangle}$ and $\tilde{f}_{j_*}^{\langle m_0\rangle}$ are normalized (see (II.6)), ensure that $\bar{f}^{\langle m_0\rangle} = \tilde{f}^{\langle m_0\rangle}$.

Let l be the largest integer such that for any $i \leq l$, we have $\bar{f}^{\langle m_0+i-1\rangle} = \tilde{f}^{\langle m_0+i-1\rangle}$. This means that the transformation Γ^∞ leaves invariant all terms of degree smaller than $m_0 + l$ of the system $\Sigma_{1,CF}^\infty$. The form of the transformation follows then from the following lemma.

Lemma III.2 *A transformation Γ^∞ leaves invariant all terms of degree smaller than $\langle m_0 + l\rangle$ of the system $\Sigma_{1,CF}^\infty$ if and only if Γ^∞ is of the form*

$$\Gamma^\infty : \begin{array}{l} z = Tx + \sum\limits_{m=l+1}^{\infty} \phi^{\langle m\rangle}(x) \\ u = kv + \sum\limits_{m=l+1}^{\infty} \left(\alpha^{\langle m\rangle}(x) + \beta^{\langle m-1\rangle}(x)v\right), \end{array} \tag{III.8}$$

where $k \in \mathbb{R}$, T is an invertible matrix preserving the Jordan Brunovský form, and for any m such that $l+1 \leq m \leq m_0 + l - 1$, the triplet $(\phi^{\langle m\rangle}, \alpha^{\langle m\rangle}, \beta^{\langle m-1\rangle})$ is given by

$$\begin{array}{ll} \phi_j^{\langle m\rangle}(x) = 0 & 1 \leq j \leq r, \\ \phi_{r+1}^{\langle m\rangle}(x) = \mu_m(\bar{x}_r)x_{r+1}^m, & \\ \phi_{j+1}^{\langle m\rangle}(x) = L_{Ax+\bar{f}^{\langle 1\rangle}(x)}\phi_j^{\langle m\rangle}(x), & r < j < n, \end{array} \tag{III.9}$$

and

$$\alpha^{\langle m\rangle}(x) + \beta^{\langle m-1\rangle}(x)u = -L_{Ax+\bar{f}^{\langle 1\rangle}(x)}\phi_n^{\langle m\rangle}(x).$$

The transformation above is defined modulo a composition with a diffeomorphism of the uncontrollable part given by Definition II.2.

The proof of this lemma is identical to that given in [12] and will be omitted for space reasons.

Since the transformation Γ^∞ brings $\Sigma_{1,CF}^\infty$ into $\Sigma_{2,CF}^\infty$, we deduce that

$$\tilde{f}^{\langle m_0+l\rangle}(z) = \bar{f}^{\langle m_0+l\rangle}(z) + \left[\bar{f}^{\langle m_0\rangle}(z), \phi^{\langle l+1\rangle}(z)\right]. \tag{III.10}$$

Following arguments in the proof of Theorem II.1, we obtain

$$\frac{\partial^{\langle m_0+l-2\rangle}\tilde{a}^{\langle m_0+l\rangle j_*,s+2}}{\partial z_{r+1}^{i_{r+1}+l}\cdots\partial z_{n-s}^{i_{n-s}-2}} = \frac{\partial^{\langle m_0+l-2\rangle}\bar{a}^{\langle m_0+l\rangle j_*,s+2}}{\partial z_{r+1}^{i_{r+1}+l}\cdots\partial z_{n-s}^{i_{n-s}-2}} + K\mu_{l+1}(\bar{z}_r)\frac{\partial^{\langle m_0\rangle}\bar{f}_{j_*}^{\langle m_0\rangle}}{\partial z_{r+1}^{i_{r+1}}\cdots\partial z_{n-s}^{i_{n-s}}},$$

where $\bar{a}^{\langle m_0+l\rangle j_*,s+2}$ and $\tilde{a}^{\langle m_0+l\rangle j_*,s+2}$ are invariants associated, respectively, to the weighted homogeneous parts of degree $\langle m_0 + l\rangle$ of the systems $\Sigma_{1,CF}^\infty$ and $\Sigma_{2,CF}^\infty$.

Using Theorem 2 of [12], we can prove that the last identity implies $\mu_{l+1}(\bar{z}_r) = 0$, that is, $\phi^{\langle l+1\rangle} = 0$ and consequently we have $\alpha^{\langle l+1\rangle} = \beta^{\langle l\rangle} = 0$. Thus, the identity (III.10) reduces to

$$\tilde{f}^{\langle m_0+l\rangle} = \bar{f}^{\langle m_0+l\rangle},$$

which contradicts the definition of l. We conclude that the canonical forms $\Sigma_{1,CF}^\infty$ and $\Sigma_{2,CF}^\infty$ coincide on controllable parts. □

Example III.3 (**Kapitsa Pendulum**) We consider in this example the Kapitsa pendulum whose equations (see [2] and [3]) are given by

$$\begin{array}{rcl} \dot{\alpha} &=& p + \frac{w}{l}\sin\alpha \\ \dot{p} &=& (gl - \frac{w^2}{l^2}\cos\alpha)\sin\alpha - \frac{w}{l}p\cos\alpha \\ \dot{z} &=& w, \end{array}$$

where α denotes the angle of the pendulum with the vertical z-axis, w is the velocity of the suspension point z, p is proportional to the generalized impulsion, g is the gravity constant, l the length of the pendulum, and the control is the acceleration $\dot{w}$.

In [12] we showed that this system is feedback equivalent to the normal form

$$\begin{array}{rcl} \dot{x}_1 &=& \lambda x_1 + R_1(x_1,x_2) + x_3P_1(\bar{x}_3) + x_4^2Q_1(\bar{x}_3) \\ \dot{x}_2 &=& -\lambda x_2 + R_2(x_1,x_2) + x_3P_2(\bar{x}_3) + x_4^2Q_2(\bar{x}_3) \\ \dot{x}_3 &=& x_4 \\ \dot{x}_4 &=& u, \end{array}$$

where $\bar{x}_3 = (x_1, x_2, x_3)$ and

$$\begin{array}{rcl} R_1(x_1,x_2) &=& \sum\limits_{m=2}^{\infty} a_m x_1(x_1x_2)^{m-1} \\ R_2(x_1,x_2) &=& \sum\limits_{m=2}^{\infty} b_m x_2(x_1x_2)^{m-1} \end{array}$$

are resonant terms with $a_m, b_m \in \mathbb{R}$.

Let us assume that $Q_1(0) \neq 0$, that is

$$Q_1(x_1,x_2,x_3) = Q_{1,0}(x_1,x_2) + x_3Q_{1,1}(x_1,x_2,x_3),$$

with $Q_{1,0}(0) \neq 0$. Consider the weighted linear change of coordinates

$$\begin{array}{rcl} z_1 &=& x_1, \quad z_3 = \mu(x_1,x_2)x_3 \\ z_2 &=& x_2 \quad z_4 = \dot{z}_3 \end{array}$$

followed by the feedback $v = \dot{z}_4$. We have

$$z_4 = \mu(x_1, x_2)x_4 + x_3\frac{\partial \mu}{\partial x_1}\dot{x}_1 + x_3\frac{\partial \mu}{\partial x_2}\dot{x}_2.$$

Throughout the example, $O^k(z)$ will denote terms of degree k and higher in the variables z_3 and z_4 whose coefficients are functions of the variables z_1 and z_2.

This implies that

$$x_4^2 = \mu^{-2}(z_1, z_2)z_4^2 + z_3z_4\delta(\bar{z}_2) + z_3^2\theta^2(\bar{z}_2) + O^3(z).$$

Taking $\mu = \sqrt{Q_{1,0}}$, we transform the system into

$$\begin{array}{rcl}
\dot{z}_1 &=& \lambda z_1 + R_1(z_1, z_2) + z_3P_1(\bar{z}_3) + z_3z_4S_1(\bar{z}_3) \\
&+& z_4^2(1 + z_3Q_1(\bar{z}_3)) + z_4^3\tilde{Q}_1(\bar{z}_4) \\
\dot{z}_2 &=& -\lambda z_2 + R_2(z_1, z_2) + z_3P_2(\bar{z}_3) + z_3z_4S_2(\bar{z}_3) \\
&+& z_4^2Q_2(\bar{z}_4) \\
\dot{z}_3 &=& z_4 \\
\dot{z}_4 &=& v.
\end{array}$$

By a change of coordinates $\tilde{z}_1 = \phi_1(z_1, z_2, z_3)$ and $\tilde{z}_2 = \phi_2(z_1, z_2, z_3)$ we can always annihilate the terms $z_4S_1(\bar{z}_3)$ and $z_4S_2(\bar{z}_3)$ and thus without loss of generality we assume that the system is already in the form

$$\begin{array}{rcl}
\dot{x}_1 &=& \lambda x_1 + R_1(x_1, x_2) + x_3P_1(\bar{x}_3) \\
&+& x_4^2(1 + x_3^{l-1}Q_1(\bar{x}_3)) + x_4^3\tilde{Q}_1(\bar{x}_4) \\
\dot{x}_2 &=& -\lambda x_2 + R_2(x_1, x_2) + x_3P_2(\bar{x}_3) + x_4^2Q_2(\bar{x}_4) \\
\dot{x}_3 &=& x_4 \\
\dot{x}_4 &=& u,
\end{array}$$

for some $l \geq 2$. We decompose $1 + x_3^{l-1}Q_1(\bar{x}_3)$ as

$$1+x_3^{l-1}Q_1(\bar{x}_3) = 1+x_3^{l-1}Q_{1,0}(x_1, x_2)+x_3^lQ_{1,1}(x_1, x_2, x_3)$$

and we apply a transformation of the form

$$\begin{array}{rclcrcl}
z_1 &=& x_1, & & z_3 &=& x_3 + x_3^l\epsilon(x_1, x_2) \\
z_2 &=& x_2, & & z_4 &=& \dot{z}_3
\end{array}$$

followed by the feedback $v = \dot{z}_4$. We can check that

$$\begin{array}{rcl}
z_4 &=& (1 + lx_3^{l-1}\epsilon(x_1, x_2))x_4 + x_3^l\frac{\partial \epsilon}{\partial x_1}\dot{x}_1 + x_3^l\frac{\partial \epsilon}{\partial x_2}\dot{x}_2, \\
&=& x_4 + lx_3^{l-1}x_4\epsilon(x_1, x_2) + x_3^l\theta(x_1, x_2) + O^{l+1}(x)
\end{array}$$

whose inverse is of the form

$$\begin{array}{rclcrcl}
x_1 &=& z_1, & & x_3 &=& z_3 - z_3^l\epsilon(z_1, z_2) \\
x_2 &=& z_2, & & x_4 &=& z_4 - lz_3^{l-1}z_4\epsilon(\bar{z}_2) - z_3^l\theta(\bar{z}_2).
\end{array}$$

modulo $+O^{l+1}(z)$. This implies that

$$x_4^2 = z_4^2 - 2lz_3^{l-1}z_4^2\epsilon(z_1, z_2) - 2z_3^lz_4\theta(z_1, z_2) + O^{l+2}(z).$$

Taking $\epsilon = \frac{Q_{1,0}}{2l}$, we annihilate the terms $x_3^{l-1}x_4^2Q_{1,0}(\bar{x}_2)$ of the first component. Repeating the process we will arrive at the weighted canonical form

$$\begin{array}{rcl}
\dot{z}_1 &=& \lambda z_1 + R_1(z_1, z_2) + z_3P_1(\bar{z}_3) + z_4^2(1 + z_4Q_1(\bar{z}_4)) \\
\dot{z}_2 &=& -\lambda z_2 + R_2(z_1, z_2) + z_3P_2(\bar{z}_3) + z_4^2Q_2(\bar{z}_4) \\
\dot{z}_3 &=& z_4 \\
\dot{z}_4 &=& v.
\end{array}$$

We can remark that any diffeomorphism of the form

$$\begin{array}{rclcrcl}
\tilde{z}_1 &=& z_1, & & \tilde{z}_3 &=& z_3 \\
\tilde{z}_2 &=& \phi(z_1, z_2), & & \tilde{z}_4 &=& z_4
\end{array}$$

that preserves the form of the resonant terms $R_1(z_1, z_2), R_2(z_1, z_2)$ (but not necessarily the coefficients a_m and b_m; take, for example, $\tilde{z}_2 = kz_2$) transforms the above weighted canonical form into an analogous weighted canonical form with R_i, P_i, and Q_i being replaced by suitable $\tilde{R}_i$, $\tilde{P}_i$, and $\tilde{Q}_i$, for $i = 1, 2$. This illustrates Definition II.2 and justifies the name weighted canonical forms.

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