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On Counting Rules in Distributed Detection

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On Counting Rules in Distributed Detection

R. VISWANATHAN AND V. AALO

Abstract—**A network of n sensors receiving independent and identical observations in R^N, regarding certain binary hypotheses, pass their decisions to a fusion center which then decides which one of the two hypotheses is true. We consider the situation where each sensor employs a likelihood ratio test with its own observation and a threshold, which is the same for all the sensors, and the fusion center decision based on k out of n decision rule. The asymptotic ($n \to \infty$) behavior of k out of n rules for finite k and finite $n - k$ are considered. For these rules, the error probability of making a wrong decision does not tend to zero as $n \to \infty$, unless the probability distributions under the hypotheses satisfy certain conditions. For a specific detection example, the asymptotic performances of the OR ($k = 1$) rule and the AND ($k = n$) rule are worse than that of a single sensor.**

I. Introduction

In decentralized processing involving a large number of sensors, each sensor processes its own observation and transmits condensed information to a fusion center. For the target detection problem, the fusion center decides the presence or the absence of the target based on the information received from different sensors [1]-[7]. If the problem is target tracking, the fusion center updates its estimates of the target position, velocity, etc., based on the received information. The decentralized detection problems are known to exhibit some surprise results such as the one that identical sensors receiving identical observations employ nonidentical tests in order that the performance at the fusion center is optimized [2], [11]. Also, certain rules, by any means not absurd, could have worse than anticipated performance (see [3] which shows AND is only as good as a single sensor; also see Section III).

In the distributed network shown in Fig. 1, each sensor receives observations which are identical and independently distributed given the hypothesis. We assume that each sensor employs the likelihood ratio test based on its own observation and a threshold, which is the same for all the sensors. The performance criterion is to minimize the probability of a miss for a given false alarm probability at the fusion center. Although the identical threshold may be suboptimal for finite n, Tsitsiklis has shown this to be optimal as $n \to \infty$ [2], [11]. With identical tests and identically distributed observations at the sensors, the optimal fusion rule is nothing but a counting rule. That is, if k or more of the sensors decide in favor of H_1, then the fusion center decides H_1 [7]. The specific value of k depends in general on the specified false alarm probability at the fusion center and the probability distribution under H_0 and H_1.

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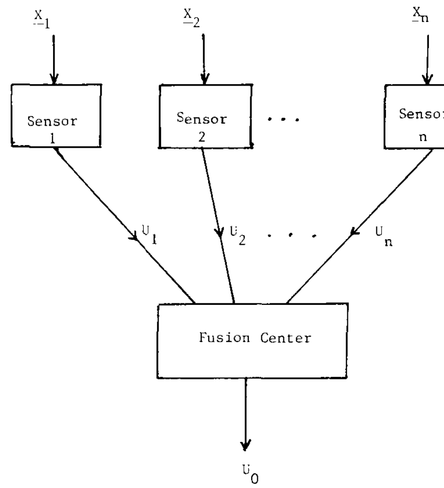

Fig. 1. Distributed decision fusion.

The purpose of this correspondence is to show that i) asymptotically the counting rules k out of n for finite k and finite $(n - k)$ are not optimal, ii) the probability of a miss for these rules need not approach 0 as $n \to \infty$ unless the distributions under H_0 and H_1 satisfy certain conditions, and iii) to provide an example where the OR and the AND rules perform worse than a single sensor. In Section II, the asymptotic error rates for the k out of n rules for finite k and finite $(n - k)$ are derived. A particular detection problem is considered in Section III.

II. Asymptotic Performance of k Out of n Rule

Let us denote the observation at the ith sensor as $X_i \in R^N$, $i = 1, \cdots, n$ and the probability density functions of X_i under H_1 and H_0 as $f_1(\ )$ and $f_0(\ )$, respectively. It is assumed that the support of the two density functions are identical. The likelihood ratio test at the ith sensor is given by

$$\Lambda(X_i) = \frac{f_1(X_i)}{f_0(X_i)} \underset{H_0}{\overset{H_1}{\gtrless}} t. \tag{1}$$

For the test (1), define the following probabilities:

$$\text{Probability of a detection,} \quad \theta = \int_t^\infty f_\Lambda(x|H_1)\,dx$$
$$\text{Probability of a false alarm,} \quad \alpha = \int_t^\infty f_\Lambda(x|H_0)\,dx$$
$$\text{Probability of a miss,} \quad \beta = 1 - \theta. \tag{2}$$

The performance of (1) is characterized by the plot of θ versus α, known as the receiver operating curve (ROC). Denote the ROC as $\theta = g(\alpha)$. The concavity and other properties of continuous ROC are well known [9].

For the k out of n rule at the fusion center, the false alarm and the detection probabilities at the fusion center are given by

$$\alpha_0 = \sum_{i=k}^{n} \binom{n}{i} \alpha^i (1 - \alpha)^{n-i}$$
$$\theta_0 = \sum_{i=k}^{n} \binom{n}{i} [g(\alpha)]^i [1 - g(\alpha)]^{n-i}$$
$$\beta_0 = 1 - \theta_0. \tag{3}$$

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As $n \to \infty$, it is optimal to have identical thresholds at the sensors [2]. The optimal threshold is arrived at by maximizing the Kullback–Liebler information number K_L, corresponding to the probability distributions of a decision U_i under the hypotheses H_0 and H_1. That is, each sensor operates with a false alarm of $\alpha \in (0, 1)$, as $n \to \infty$. From (3), by applying Demoivre Laplace Theorem [8], the optimal k for a fixed false alarm α_0 at the fusion center is given by

$$k = \sqrt{n}h + n\alpha \tag{4}$$

where h is a constant that depends on α and α_0. Therefore, the k out of n rules for finite k and finite $(n - k)$ are only suboptimal. In fact, the optimal fusion center rule (4) has $\beta_0 \to \exp(-K_L n)$, as $n \to \infty$. For finite k and finite $(n - k)$ rules, β_0 need not approach zero as the following results show.

i) Necessary and sufficient condition for k out of n rules for finite $(n - k)$ (finite k) to have vanishingly small β_0 as $n \to \infty$.

Consider the k out of n rule with $n - k = K < \infty$. Equation (3) can be rewritten as

$$\alpha_0 = \alpha^n + \sum_{i=1}^{K} \alpha^{n-i}(1 - \alpha)^i \binom{n}{i}. \tag{5}$$

K equals zero implies AND rule and for this rule α equals $\alpha_0^{1/n}$. For $K > 0$, guess a solution to (5) of the form

$$\alpha^n = C_0,\ 0 < C_0 < \alpha_0. \tag{6}$$

Upon denoting the ratio of the jth term to the $(j - 1)$th term inside the summation of (5) as R_j, and using (6),

$$R_j = \frac{1 - C_0^{1/n}}{C_0^{1/n}} \cdot \frac{n - j + 1}{j}.$$

As $n \to \infty$, R_j tends to the limit,

$$(-\ln C_0)/j, \qquad \text{for } j = 1, \cdots, K.$$

Hence, the right-hand side of (5) equals

$$C_0 + R_1 C_0 + R_2 R_1 C_0 + \cdots + R_K R_{K-1} \cdots R_1 C_0.$$

Therefore, a solution C_0 of the following equation is sought:

$$\alpha_0 = C_0 + C_0 \sum_{j=1}^{K} (-\ln C_0)^j / j!. \tag{7}$$

Rewriting (7) gives

$$\alpha_0 = e^{-a} + e^{-a} \sum_{j=1}^{K} a^j/j! \qquad \text{where } a = -\ln C_0. \tag{8}$$

The right-hand side of (8) is a monotone decreasing function of a and has a value greater than α_0 at $a = -\ln \alpha_0$. Hence, a unique solution of (8) exists. The probability of detection θ_0 is given by

$$\theta_0 = \sum_{i=0}^{K} [g(\alpha)]^{n-i} [1 - g(\alpha)]^i \binom{n}{i}. \tag{9}$$

As $n \to \infty$, since $g(\alpha) \geq \alpha$ and $\alpha^n \to C_0$, let $[g(\alpha)]^n$ approach a constant $d_0 \geq C_0$. Proceeding as before,

$$\underset{n \to \infty}{Lt}\ \theta_0 = e^{-b} + e^{-b} \sum_{j=1}^{K} b^j/j! \qquad \text{where } b = -\ln d_0. \tag{10}$$

The right-hand side of (10) is a monotone decreasing function of b and assumes the largest value of 1 when b equals 0. This requires that $-n \ln(g(\alpha))$ tends to zero or a necessary and sufficient condition for the k out of n rules with finite $(n - k)$ to have vanishingly small β_0 is that

$$\left.\frac{dg(\alpha)}{d\alpha}\right|_{\alpha=1} = 0. \tag{11}$$

For the finite k case, a similar derivation yields a necessary and sufficient condition as

$$\left.\frac{dg}{d\alpha}\right|_{\alpha=0} = \infty. \tag{12}$$

ii) An example of the signal detection problem follows.

Consider the detection of a constant signal in Generalized Gaussian noise [10]. This noise density function is given by

$$f(x) = \frac{\gamma^{1/c} c}{2\Gamma(1/c)} \exp(-\gamma |x|^c). \tag{13}$$

For $c = 2$, $f(x)$ becomes the standard Gaussian, $c = 1$, the density is Laplacian or double exponential and for $c < 1$, $f(x)$ represents a much heavier tail density. Without loss of generality, the variance of (13) is taken as unity. Assume a sample of size one at each sensor. Then $f_0(x)$ equals $f(x)$ and $f_1(x)$ equals $f(x - s)$, where s is the constant signal. For $c = 1$ and $c = 2$, the likelihood ratio is a monotonic function of x. Hence, for these cases, the conditions (11) and (12) can be checked easily. For other values of c, it is difficult to determine $dg/d\alpha$. Hence,

$$\left.\frac{dg}{d\alpha}\right|_{\alpha=0} = \begin{cases} \infty & c = 2 \\ \text{some finite, nonzero constant} & c = 1 \end{cases}$$
$$\left.\frac{dg}{d\alpha}\right|_{\alpha=1} = \begin{cases} 0 & c = 2 \\ \text{some finite, nonzero constant} & c = 1. \end{cases} \tag{14}$$

III. Constant Signal in Double Exponential Noise

With a sample of size one at each sensor, the ROC for the detection of a constant signal in double exponential noise is given by

$$g(\alpha) = \begin{cases} y\alpha & y \leq \dfrac{1}{2\alpha} \\ 1 - \dfrac{1}{4\alpha y} & y \geq \dfrac{1}{2\alpha} \end{cases} \quad \text{for } \alpha \leq 1/2$$
$$\left\{ 1 - \frac{1 - \alpha}{y} \right. \qquad \alpha > 1/2 \tag{15}$$

where $y = \exp(\sqrt{2}s)$. Therefore, the probabilities of detection for the AND and the OR rules are given by the following.

AND

$$\theta_0 = \begin{cases} (\alpha y)^n & y \leq \dfrac{1}{2\alpha} \\ \left(1 - \dfrac{1}{4\alpha y}\right)^n & y \geq \dfrac{1}{2\alpha} \end{cases} \quad \text{for } \alpha \leq 1/2$$
$$\left\{ \left(1 - \frac{1 - \alpha}{y}\right)^n \right. \qquad \text{for } \alpha > 1/2 \tag{16}$$

where $\alpha = \alpha_0^{1/n}$.

OR

$$\theta_0 = \begin{cases} 1 - (1 - \alpha y)^n & y \leq \dfrac{1}{2\alpha} \\ 1 - \left(\dfrac{1}{4\alpha y}\right)^n & y \geq \dfrac{1}{2\alpha} \end{cases} \quad \text{for } \alpha \leq 1/2$$
$$\left\{ 1 - \left(\frac{1 - \alpha}{y}\right)^n \right. \qquad \alpha > 1/2 \tag{17}$$

where $\alpha = 1 - (1 - \alpha_0)^{1/n}$.

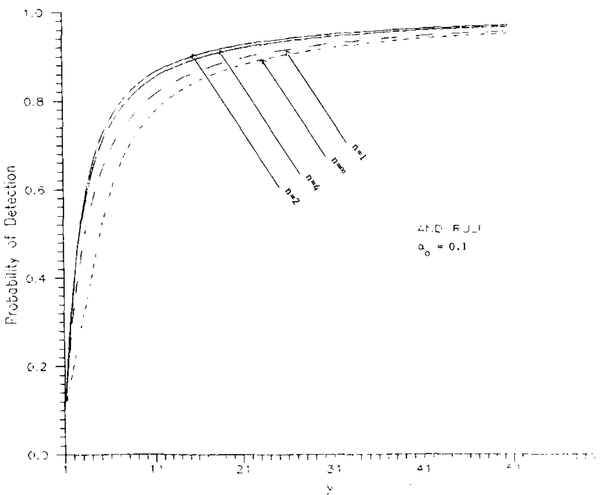

Fig. 2. Probability of detection versus signal level for the AND rule with $\alpha_0 = 0.1$.

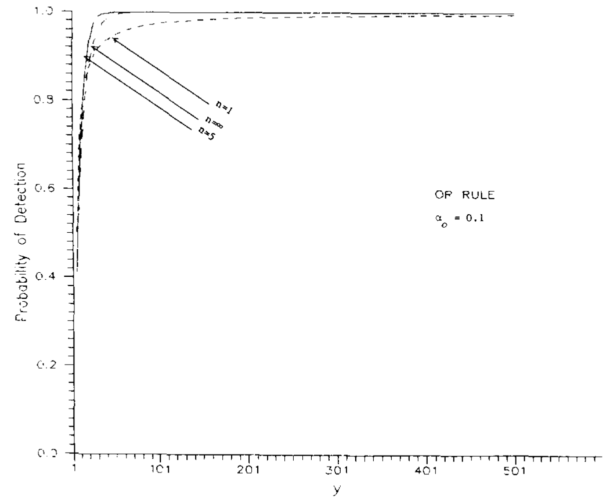

Fig. 4. Probability of detection versus signal level for the OR rule with $\alpha_0 = 0.1$.

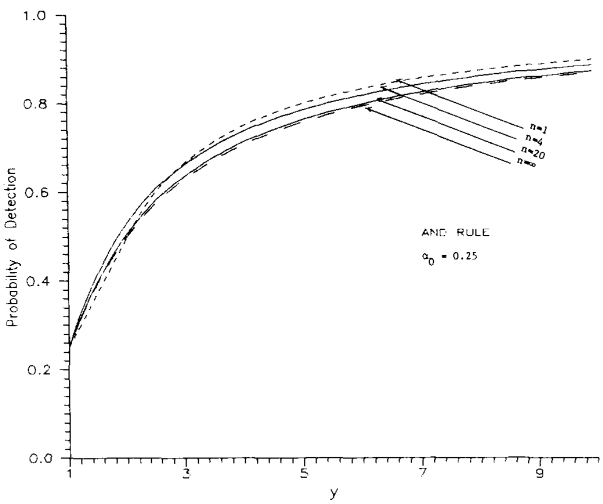

Fig. 3. Probability of detection versus signal level for the AND rule with $\alpha_0 = 0.25$.

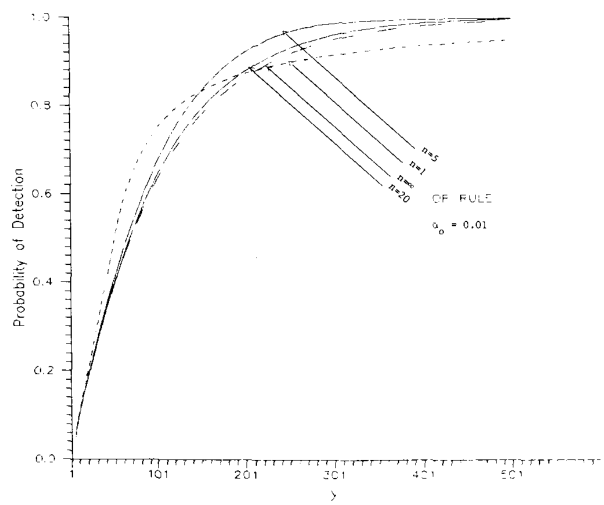

Fig. 5. Probability of detection versus signal level for the OR rule with $\alpha_0 = 0.01$.

As $n \rightarrow \infty$, (16) and (17) become

$$\text{AND} \qquad \theta_0 = \alpha_0^{1/y} \tag{18}$$

$$\text{OR} \qquad \theta_0 = 1 - (1 - \alpha_0)^y. \tag{19}$$

Plots of (15)–(19) are shown in Figs. 2–5, for some values of α_0 and n. There exist regions of y where the asymptotic performances of both OR and AND rules are worse than that of a single sensor. In the case of OR, this holds even for low values of α_0, which are usually of interest. Also, the asymptotic performance is reached even with a moderate number of sensors. Although OR is expected to be not optimal for large n, the poor performance of OR as compared to a single sensor is rather surprising. This shows that a suboptimal fusion center rule must be evaluated carefully for its performance. An example where a suboptimal test at the sensor leads to a poor performance at the fusion center is given in [2].

IV. Conclusions

A distributed network of n identical sensors sending their binary decisions to a fusion center is studied. The asymptotic performances of k out of n rules at the fusion center for finite k [finite $(n - k)$] are evaluated. When condition (12) [(11)] is not met, these rules do not give vanishingly small error probability even when n tends to infinity. In the example considered, the approach to asymptotic performance occurs even with moderate n values. Moreover, the performances of these rules could be below that of a single sensor. Hence, caution must be exercised in using k out of n rules with extreme k values near 1 (near n) when the condition (12) [(11)] is not satisfied.

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Distributed Detection of a Signal in Generalized Gaussian Noise

R. VISWANATHAN AND ARIF ANSARI

Abstract—**The problem of distributed detection of a signal in incompletely specified noise is considered. The noise assumed belongs to the generalized Gaussian family and the sensors in the distributed network employ the Wilcoxon test. The sensors pass the test statistics to a fusion center, where a hypothesis testing results in a decision regarding the presence or the absence of a signal. Three monotone and admissible fusion center tests are formulated. Restricted numerical evaluation over a certain parameter range of the noise distribution and the range of signal level indicates that these tests yield performances at comparable levels.**

I. Introduction

The problem of detection of a signal using a distributed network of sensors has been analyzed in the literature. In order to save transmission bandwidths, the sensors process the information they receive and pass condensed information, such as the test statistics or the decisions with regard to the presence or the absence of a signal, to the fusion center. For the best performance, it is essential that the processing at the sensors and at the fusion be optimized [1]-[9].

So far, the problem analyzed in the literature assumes a complete statistical knowledge of the received signal. However, in sonar and other underwater detection problems, the signal is embedded in a noise whose characteristics are not completely known and are changing with time. In such situations, the sensors' statistics must be based on some general characteristics of the noise density function rather than on some specific form of noise density function. In this correspondence, we consider the distributed detection of a constant signal in generalized Gaussian noise. Such a noise density function approximates physical noise encountered in different situations [10], [11].

In Section II we discuss test statistics at the sensors and at the fusion. In Section III we present the performance analysis of three different tests at the fusion center. Numerical results are shown for a three sensors network with three samples per sensor. We conclude our discussion in Section IV.

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II. The Generalized Gaussian Noise and Distributed Tests

The problem of detection of a constant signal in additive noise is described by the following hypotheses testing:

$$H_0:\quad X_j = n_j$$
$$H_1:\quad X_j = n_j + \theta, \quad j \text{ an integer.} \tag{1}$$

We assume that the noise n_j has a symmetric density function described by the following equation [11]:

$$f(n) = \frac{\alpha^{1/c} c}{2\Gamma(1/c)} \exp\left(-\alpha |n|^c\right). \tag{2}$$

The noise has unit variance and hence α satisfies the relation

$$\alpha^{2/c} = \Gamma(3/c)/\Gamma(1/c). \tag{3}$$

By varying the parameter c, we can control the tail of the noise density. When c equals 2 the noise reduces to the Gaussian, and for c equals 1 it becomes Laplace. In general, smaller values of c represent heavy tails. For detecting a signal in symmetric noise at a sensor, a variety of nonparametric tests such as the sign test and the Wilcoxon test exist [12]. Our choice of the Wilcoxon test is motivated by the fact that i) the Wilcoxon test is nonparametric, ii) its performance is comparable to other nonparametric tests, iii) it performs better than the sign test in most cases, and iv) the Wilcoxon statistic takes on a finite number of discrete values.

Fig. 1 shows the distributed network of sensors and the fusion center. The statistics $T_1, T_2, \cdots, T_N$ are the Wilcoxon statistics, and the test at the fusion is given as follows:

$$S(T_1, \cdots, T_N) \underset{H_0}{\overset{H_1}{\gtrless}} t. \tag{4}$$

Here S is a statistic based on $T_1, \cdots, T_N$. The observations $X_1 \cdots X_n$ at each sensor are assumed to be independent and identically distributed according to (1). Hence, the T_k's are i.i.d. A sensor performs the Wilcoxon test by ranking the absolute values of the X_i's and summing the ranks of the absolute values which are due to positive observations. The performance of the Wilcoxon test is well understood [12]. It is possible to obtain the distribution of T_k under H_0 and H_1 by enumeration. For large values of n, it is difficult to obtain the distribution. However, the mean and the variance can be found [12]:

$$E(T_k) = \sum_{i=1}^{n} i\lambda_i \tag{5}$$

$$\text{Var}(T_k) = \sum_{i=1}^{n} i^2\lambda_i(1 - \lambda_i) \tag{6}$$

$$\lambda_i = N\binom{N-1}{i-1} \int_0^\infty \left[F(u) - F(-u)\right]^{i-1} \cdot \left[1 - F(u) + F(-u)\right]^{n-i} f(u)\, du \tag{7}$$

where $f(\ )$ is the density of the observation X_i and $F(\ )$ is the corresponding CDF.

We consider three different statistics at the fusion. The minimum test is given by the rule

$$\text{Min}\,[T_1, \cdots, T_N] \underset{H_0}{\overset{H_1}{\gtrless}} t_m \tag{8}$$

where t_m is chosen to obtain a specific false alarm probability at the fusion center. However, when T_k's given the hypothesis are i.i.d., if any order statistic of $\{T_k\text{'s}\}$ is used as a test statistic at the fu-

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