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Strict feedforward form and symmetries of nonlinear control systems

Witold Respondek and Issa A. Tall

Abstract—We establish a relation between strict feedforward form and symmetries of nonlinear control systems. We prove that a system is feedback equivalent to the strict feedforward form if and only if it gives rise to a sequence of systems, such that each element of the sequence, firstly, possesses an infinitesimal symmetry and, secondly, it is the factor system of the preceding one, i.e., is reduced from the preceding one by its symmetry. We also propose a strict feedforward normal form and prove that a smooth strict feedforward system can be smoothly brought to that form.

I. INTRODUCTION

A smooth single-input nonlinear control system of the form

$$\dot{z} = F(z, u),$$

where $z \in \mathbb{R}^n$ and $u \in \mathbb{R}$ is in *strict feedforward form* if we have

$$(SFF) \quad \begin{aligned} \dot{z}_1 &= F_1(z_2, \dots, z_n, u) \\ &\vdots \\ \dot{z}_{n-1} &= F_{n-1}(z_n, u) \\ \dot{z}_n &= F_n(u). \end{aligned}$$

We will be also dealing with control-affine systems

$$\dot{z} = f(z) + g(z)u,$$

where f and g are smooth vector fields on $\mathbb{R}^n$ and we will say that the system is in *affine strict feedforward form* if we have

$$(ASFF) \quad \begin{aligned} \dot{z}_1 &= f_1(z_2, \dots, z_n) + g_1(z_2, \dots, z_n)u \\ &\vdots \\ \dot{z}_{n-1} &= f_{n-1}(z_n) + g_{n-1}(z_n)u \\ \dot{z}_n &= f_n + g_n u, \end{aligned}$$

where $f_n, g_n \in \mathbb{R}$.

A basic structural property of systems in strict feedforward form is that their solutions can be found by quadratures. Indeed, knowing $u(t)$ we integrate $F_n(u(t))$ to get $z_n(t)$, then we integrate $F_{n-1}(z_n(t), u(t))$ to get $z_{n-1}(t)$, we keep doing that, and finally we integrate $F_1(z_2(t), \dots, z_n(t), u(t))$ to get $z_1(t)$.

Notice that, in view of the above, systems in the strict feedforward form can be considered as duals of flat systems. In the single-input case, flat systems are feedback

linearizable and are defined as systems for which we can find a function of the state that, together with its *derivatives*, gives all the states and the control of the system [3]. In a dual way, for systems in the strict feedforward form, we can find all states via a successive *integration* starting from a function of the control.

Another property, crucial in applications, of systems in (strict) feedforward form is that we can construct for them a stabilizing feedback. This important result goes back to Teel [18] and has been followed by a growing literature on stabilization and tracking for systems in (strict) feedforward form (see e.g. [5], [9], [12], [19], [2], [10]).

It is therefore natural to ask which systems are equivalent to (strict) feedforward form. In [8], the problem of transforming a system, affine with respect to controls, into (strict) feedforward form via a diffeomorphism, i.e., via a nonlinear change of coordinates, was studied. A geometric description of systems in feedforward form has been given in [1]. The conditions of [1], although being intrinsic, are not checkable. Another approach has been used by the authors who have proposed a step-by-step constructive method to bring a system into a feedforward form in [15], [17] and strict feedforward form in [16].

In the present paper we look at the problem in the spirit of [1] but we focus our attention on vector fields rather than on invariant distributions. It turns out that feedback equivalence (resp. state-space equivalence) to the strict feedforward form can be characterized by the existence of a sequence of infinitesimal symmetries (resp. strong infinitesimal symmetries) of the system.

In [16] we proposed a formal normal form for strict feedforward systems. In the second part of this paper, we introduce a smooth counterpart of that normal form and show that a smooth strict feedforward system can always be transformed via feedback and coordinate transformation to our normal form.

The paper is organized as follows. Section II contains the first main result of the paper, namely a characterization of the strict feedforward form in terms of infinitesimal symmetries. For planar systems, the presented result leads to verifiable conditions, which we present in Section IV. We will show in Section III how the problem of transforming a general system to the strict feedforward form can be reduced by a preintegration to that for affine systems. The second main result of the paper, namely a smooth feedback transformation to the strict feedforward normal form is presented in Section V.

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II. SYMMETRIES AND STRICT FEEDFORWARD FORM

In this section we will establish results relating symmetries and strict feedforward forms. To start with, recall two basic notions of equivalence of control systems. The word smooth will mean throughout C^∞ -smooth and all control systems are assumed to be smooth (except for Section V, where we also consider analytic systems).

Two control systems

$$\Sigma : \dot{x} = f(x) + g(x)u,$$

where $x \in X$ and

$$\tilde{\Sigma} : \dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{u},$$

where $\tilde{x} \in \tilde{X}$, are called *state space equivalent*, shortly *S-equivalent*, if there exists a smooth diffeomorphism $\phi : X \rightarrow \tilde{X}$, such that

$$\phi_*f = \tilde{f} \quad \text{and} \quad \phi_*g = \tilde{g};$$

(we take $u = \tilde{u}$). Recall that for any smooth vector field f on X and any smooth diffeomorphism $\tilde{x} = \phi(x)$ we denote

$$(\phi_*f)(\tilde{x}) = d\phi(x) \cdot f(x),$$

with $x = \phi^{-1}(\tilde{x})$. Two control systems Σ and $\tilde{\Sigma}$ are called *feedback equivalent*, shortly *F-equivalent*, if there exists a smooth diffeomorphism $\phi : X \rightarrow \tilde{X}$ and smooth functions α, β , satisfying $\beta(\cdot) \neq 0$, such that

$$\phi_*(f + g\alpha) = \tilde{f} \quad \text{and} \quad \phi_*(g\beta) = \tilde{g}.$$

For the single-input control-affine system

$$\Sigma : \dot{x} = f(x) + g(x)u,$$

where $x \in X$, an open subset of $\mathbb{R}^n$, and $u \in U = \mathbb{R}$, and f and g are smooth vector fields on X , the *field of admissible velocities* is the following field of affine lines

$$\mathcal{A}(x) = \{f(x) + ug(x) : u \in \mathbb{R}\} \subset T_x X.$$

A diffeomorphism $\psi : X \rightarrow X$ is a symmetry of Σ if it preserves the field of affine lines $\mathcal{A}$ (in other words, the affine distribution $\mathcal{A}$ of rank 1), that is, if

$$\psi_*\mathcal{A} = \mathcal{A}.$$

A *local symmetry* at $p \in X$ is a local diffeomorphism ψ of X_0 onto X_1 , where X_0 and X_1 are, respectively, neighborhoods of p and $\psi(p)$, such that

$$(\psi_*\mathcal{A})(q) = \mathcal{A}(q)$$

for any $q \in X_1$.

We say that a vector field v on an open subset $X \subset \mathbb{R}^n$ is an *infinitesimal symmetry* of the system Σ if the (local) flow γ_t^v of v is a local symmetry of Σ , for any t for which it exists.

We will also be dealing with the following stronger notions. A diffeomorphism $\psi : X \rightarrow X$ is a *strong symmetry* of Σ if it preserves the vector fields f and g

(and not only the affine distribution $\mathcal{A}$ spanned by them), that is, if

$$\psi_*f = f \quad \text{and} \quad \psi_*g = g.$$

A local strong symmetry is a local diffeomorphism preserving f and g . We say that a vector field v on an open subset $X \subset \mathbb{R}^n$ is an *infinitesimal strong symmetry* of the system Σ if the (local) flow γ_t^v of v is a local strong symmetry of Σ , for any t for which it exists.

Consider the system Σ and denote by $\mathcal{G}$ the distribution spanned by the vector field g . We have the following characterization of infinitesimal symmetries and strong symmetries.

Proposition II.1 *A vector field v is an infinitesimal strong symmetry of Σ if and only if*

$$[v, g] = 0$$

$$[v, f] = 0.$$

A vector field v , such that $v(p) \neq 0$, is an infinitesimal symmetry of Σ , locally at p , if and only if

$$[v, g] = 0 \mod \mathcal{G}$$

$$[v, f] = 0 \mod \mathcal{G}.$$

in a neighborhood of p .

The second statement remains true even if $g(p) = 0$. In this case, we have to understand $\mathcal{G}$ as the module of vector fields generated by g over the ring of smooth functions.

A local symmetry ψ at p is called a *stationary symmetry* if $\psi(p) = p$ and a *nonstationary symmetry* if $\psi(p) \neq p$. An infinitesimal symmetry v is called stationary at $p \in X$ if $v(p) = 0$ and nonstationary if $v(p) \neq 0$.

Assume that v is a strong infinitesimal symmetry of Σ , nonstationary at $p \in X$. Then there exist a neighborhood X_p of p and the factor system $\Sigma/\sim_v$, where the equivalence relation $\sim_v$ is induced by the local action of the 1-parameter local group defined by v , that is, $q_1 \sim_v q_2$ if and only if they belong to the same integral curve of v (more precisely, to the same connected component of the intersection of an integral curve of v with X_p).

Theorem II.2 *The following condition are equivalent.*

- (i) Σ is, locally at $p \in X$, S-equivalent to the affine strict feedforward form (ASFF);
- (ii) Each system $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ possesses a strong infinitesimal nonstationary symmetry v_i , where Σ_1 is the restriction of Σ to a neighborhood X_p and

$$\Sigma_{i+1} = \Sigma_i / \sim_{v_i},$$

where $\sim_{v_i}$ is the equivalence relation defined by the local action of the 1-parameter group of v_i ;

- (iii) There exist smooth vector fields $w_1, \dots, w_n$, independent at $p \in X$, such that locally at p ,

$$[w_i, w_j] \in \mathcal{D}_{i-1}$$

$$[w_i, g] \in \mathcal{D}_{i-1}$$

$$[w_i, f] \in \mathcal{D}_{i-1},$$

- for any $1 \leq i \leq n$ and $j \leq i$, where $\mathcal{D}_0 = 0$ and $\mathcal{D}_i = \text{span}\{w_1, \dots, w_i\}$;
- (iv) There exist smooth vector fields $\tilde{w}_1, \dots, \tilde{w}_n$, independent at $p \in X$, such that locally at p ,

$$\begin{aligned} [\tilde{w}_i, \tilde{w}_j] &= 0 \\ [\tilde{w}_i, g] &\in \tilde{\mathcal{D}}_{i-1} \\ [\tilde{w}_i, f] &\in \tilde{\mathcal{D}}_{i-1}, \end{aligned}$$

for any $1 \leq i \leq n$ and $j \leq i$, where $\tilde{\mathcal{D}}_0 = 0$ and $\tilde{\mathcal{D}}_i = \text{span}\{\tilde{w}_1, \dots, \tilde{w}_i\}$.

In Section III we will show that the problem of transforming a general system to (SFF) can be reduced to the above theorem by a preintegration.

The above theorem implies that an invariant characterization of the affine strict feedforward form (ASSF) involves vector fields (forming a sequence of infinitesimal symmetries) rather than invariant distributions. To be more precise, a characterization of the *affine feedforward form (AFF)*

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, \dots, z_n) + g_1(z_1, \dots, z_n)u \\ \dot{z}_2 &= f_2(z_2, \dots, z_n) + g_2(z_2, \dots, z_n)u \\ &\vdots \\ \dot{z}_{n-1} &= f_{n-1}(z_{n-1}, z_n) + g_{n-1}(z_{n-1}, z_n)u \\ \dot{z}_n &= f_n(z_n) + g_n(z_n)u, \end{aligned}$$

was obtained by Astolfi and Mazenc [1] in terms of invariant distributions as follows:

Proposition II.3 *The system Σ is locally equivalent to the affine feedforward form (AFF) if and only if there exists a sequence of distributions*

$$\mathcal{D}_1 \subset \dots \subset \mathcal{D}_n,$$

where $\mathcal{D}_i$ is involutive and of rank i , such that

$$\begin{aligned} [\mathcal{D}_i, g] &\subset \mathcal{D}_i \\ [\mathcal{D}_i, f] &\subset \mathcal{D}_i. \end{aligned}$$

A first guess for a characterization of the affine strict feedforward form (ASFF) could be (compare [1]) the existence of a nested sequence of involutive distributions $\mathcal{D}_i$, of constant rank i , satisfying

$$\begin{aligned} [\mathcal{D}_i, g] &\subset \mathcal{D}_{i-1} \\ [\mathcal{D}_i, f] &\subset \mathcal{D}_{i-1}. \end{aligned}$$

This is not a correct answer for two reasons. Firstly, the latter conditions are not invariant, that is, even if they are satisfied for some vector fields $w_1, \dots, w_i$ spanning $\mathcal{D}_i$ then, in general, for other generators of the same distribution $\mathcal{D}_i$, we will have on the right the inclusion in $\mathcal{D}_i$ (and not in $\mathcal{D}_{i-1}$). Secondly, the above conditions, even reformulated in terms of vector fields, are not sufficient for equivalence

to (ASSF). Indeed, the condition that there exist linearly independent vector fields $w_1, \dots, w_n$ such that

$$\begin{aligned} [w_i, g] &\in \mathcal{D}_{i-1} \\ [w_i, f] &\in \mathcal{D}_{i-1}, \end{aligned}$$

for any $1 \leq i \leq n$, where $\mathcal{D}_0 = 0$ and $\mathcal{D}_i = \text{span}\{w_1, \dots, w_i\}$ are involutive, does not imply S -equivalence to (ASFF) unless we assume an additional property on the w'_i s: like the first condition of (iii) (which is the weakest possible) or the first condition of (iv), which is the strongest one.

Proof. We will prove that (i) $\Leftrightarrow$ (ii) and then that (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Assume that $\Sigma = \Sigma_1$ has the affine strict feedforward form (ASFF) in an open subset $X_1 = X_p \subset \mathbb{R}^n$. Then, clearly, $v_1 = \frac{\partial}{\partial x_1}$ is a strong infinitesimal symmetry of Σ_1 and the reduced system $\Sigma_2 = \Sigma_1 / \sim_{v_1}$ is defined on $X_2 = \pi_1(X_1)$, where $\pi_1(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n)$, by

$$\begin{aligned} \Sigma_2 : \quad \dot{x}_2 &= f_2(x_3, \dots, x_n) + g_2(x_3, \dots, x_n)u \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(x_n) + g_{n-1}(x_n)u \\ \dot{x}_n &= f_n + g_n u. \end{aligned}$$

Obviously, the vector field $v_2 = \frac{\partial}{\partial x_2}$ on $X_2 \subset \mathbb{R}^{n-1}$ is a strong infinitesimal nonstationary symmetry of Σ_2 . Repeating this, we easily conclude that each system

$$\Sigma_{i+1} = \Sigma_i / \sim_{v_i},$$

for $1 \leq i \leq n-1$, possesses a strong infinitesimal nonstationary symmetry $v_{i+1} = \frac{\partial}{\partial x_{i+1}}$.

(ii) $\Rightarrow$ (i). Assume that $\Sigma = \Sigma_1$ possesses a strong infinitesimal nonstationary symmetry v_1 . Take a neighborhood X_1 of $p \in \mathbb{R}^n$ and local coordinates $(x_1, \dots, x_n)$ such that $v_1 = \frac{\partial}{\partial x_1}$ in X_1 . It follows from the first part of Proposition II.1 that in X_1 , the system Σ_1 takes the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_2, \dots, x_n) + g_1(x_2, \dots, x_n)u \\ \dot{x}_2 &= f_2(x_2, \dots, x_n) + g_2(x_2, \dots, x_n)u \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(x_2, \dots, x_n) + g_{n-1}(x_2, \dots, x_n)u \\ \dot{x}_n &= f_n(x_2, \dots, x_n) + g_n(x_2, \dots, x_n)u. \end{aligned}$$

The system $\Sigma_2 = \Sigma_1 / \sim_{v_1}$ is thus well defined on $X_2 = \pi_1(X_1)$, where $\pi_1(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n)$ by

$$\begin{aligned} \Sigma_2 : \quad \dot{x}_2 &= f_2(x_2, \dots, x_n) + g_2(x_2, \dots, x_n)u \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(x_2, \dots, x_n) + g_{n-1}(x_2, \dots, x_n)u \\ \dot{x}_n &= f_n(x_2, \dots, x_n) + g_n(x_2, \dots, x_n)u. \end{aligned}$$

By assumption, Σ_2 possesses a strong infinitesimal nonstationary symmetry v_2 . There exists an open subset $\tilde{X}_2 \subset X_2$ and a coordinate system $(\tilde{x}_2, \dots, \tilde{x}_n) = \phi_2(x_2, \dots, x_n)$ on

$\tilde{X}_2$ such that $v_2 = \frac{\partial}{\partial \tilde{x}_2}$ in $\tilde{X}_2$. In $(\tilde{x}_2, \dots, \tilde{x}_n)$ coordinates the system Σ_2 takes the form

$$\begin{aligned} \dot{\tilde{x}}_2 &= \tilde{f}_2(\tilde{x}_3, \dots, \tilde{x}_n) + \tilde{g}_2(\tilde{x}_3, \dots, \tilde{x}_n)u \\ \vdots \\ \dot{\tilde{x}}_{n-1} &= \tilde{f}_{n-1}(\tilde{x}_3, \dots, \tilde{x}_n) + \tilde{g}_{n-1}(\tilde{x}_3, \dots, \tilde{x}_n)u \\ \dot{\tilde{x}}_n &= \tilde{f}_n(\tilde{x}_3, \dots, \tilde{x}_n) + \tilde{f}_n(\tilde{x}_3, \dots, \tilde{x}_n)u. \end{aligned}$$

Complete the coordinates $(\tilde{x}_2, \dots, \tilde{x}_n) = \phi_2(x_2, \dots, x_n)$ by $\tilde{x}_1 = x_1$. Then in $\tilde{x}$ -coordinates the original system Σ has the form $\tilde{\Sigma}_1$

$$\begin{aligned} \dot{\tilde{x}}_1 &= f_1(\phi_2^{-1}(\tilde{x}_2, \dots, \tilde{x}_n)) + g_1(\phi_2^{-1}(\tilde{x}_2, \dots, \tilde{x}_n))u \\ \dot{\tilde{x}}_2 &= \tilde{f}_2(\tilde{x}_3, \dots, \tilde{x}_n) + \tilde{g}_2(\tilde{x}_3, \dots, \tilde{x}_n)u \\ \vdots \\ \dot{\tilde{x}}_{n-1} &= \tilde{f}_{n-1}(\tilde{x}_3, \dots, \tilde{x}_n) + \tilde{g}_{n-1}(\tilde{x}_3, \dots, \tilde{x}_n)u \\ \dot{\tilde{x}}_n &= \tilde{f}_n(\tilde{x}_3, \dots, \tilde{x}_n) + \tilde{f}_n(\tilde{x}_3, \dots, \tilde{x}_n)u. \end{aligned}$$

Changing successively $(x_i, \dots, x_n)$ (and completing them each time by identity on $(x_1, \dots, x_{i-1})$), we construct the (ASSF) form for Σ .

(i) $\Rightarrow$ (iii) Consider the strict feedforward form (ASSF) on an open subset $X \subset \mathbb{R}^n$. Put

$$w_i = \frac{\partial}{\partial x_i},$$

for $1 \leq i \leq n$. Then, clearly $[w_i, w_j] \in \mathcal{D}_{i-1}$, for $j < i$, as well as $[w_i, f] \in \mathcal{D}_{i-1}$, and $[w_i, g] \in \mathcal{D}_{i-1}$.

(iii) $\Rightarrow$ (iv) Fix an n -tuple of smooth vector fields $w_1, \dots, w_n$ satisfying the three conditions of (iii). Put $\tilde{w}_1 = w_1$ and choose local coordinates $(x_1, \dots, x_n)$ around $p \in X$ such that $\tilde{w}_1 = w_1 = \frac{\partial}{\partial x_1}$. We have $w_2 = \sum_{j=1}^n w_2^j \frac{\partial}{\partial x_j}$, for some smooth functions w_2^j . Put $\tilde{w}_2 = w_2 - w_2^1 w_1$. We have $\mathcal{D}_2 = \text{span}\{w_1, w_2\} = \tilde{\mathcal{D}}_2 = \text{span}\{\tilde{w}_1, \tilde{w}_2\}$. This and the definition of $\tilde{w}$ imply that $[\tilde{w}_1, \tilde{w}_2] = 0$ and that $[\tilde{w}_2, f] \in \tilde{\mathcal{D}}_1$ and $[\tilde{w}_2, g] \in \tilde{\mathcal{D}}_1$.

Assume that for some k we have constructed vector fields $\tilde{w}_1, \dots, \tilde{w}_{k-1}$ such that, for $1 \leq i, j \leq k-1$, we have $[\tilde{w}_i, \tilde{w}_j] = 0$ and, moreover, $\mathcal{D}_i = \tilde{\mathcal{D}}_i$ as well as $[\tilde{w}_i, f] \in \tilde{\mathcal{D}}_{i-1}$ and $[\tilde{w}_i, g] \in \tilde{\mathcal{D}}_{i-1}$. Choose local coordinates $(x_1, \dots, x_n)$ around $p \in X$ such that $\tilde{w}_i = \frac{\partial}{\partial x_i}$, for $1 \leq i \leq k-1$. We have $w_k = \sum_{j=1}^n w_k^j \frac{\partial}{\partial x_j}$, for some smooth functions w_k^j . Put $\tilde{w}_k = w_k - \sum_{j=1}^{k-1} w_k^j w_j$. It follows that $\mathcal{D}_k = \text{span}\{w_1, \dots, w_k\} = \tilde{\mathcal{D}}_k = \text{span}\{\tilde{w}_1, \dots, \tilde{w}_k\}$. This and the definition of $\tilde{w}_k$ imply that for $1 \leq i \leq k$, we have $[\tilde{w}_i, \tilde{w}_j] = 0$ as well as $[\tilde{w}_k, f] \in \tilde{\mathcal{D}}_{k-1}$ and $[\tilde{w}_k, g] \in \tilde{\mathcal{D}}_{k-1}$. Now the implication (iii) $\Rightarrow$ (iv) follows by an induction argument.

(iv) $\Rightarrow$ (i) There exists a neighborhood X_p of $p \in X$ and local coordinates $(x_1, \dots, x_n)$ such that $\tilde{w}_i = \frac{\partial}{\partial x_i}$, for any $1 \leq i \leq n$. The conditions $[\tilde{w}_i, f] \in \mathcal{D}_{i-1}$ and $[\tilde{w}_i, g] \in \mathcal{D}_{i-1}$ imply that the system Σ takes, in the coordinates $(x_1, \dots, x_n)$, the affine strict feedforward form (ASFF). $\square$

We have an analogous result for feedback equivalence to strict feedforward form, where the role of strong infinitesimal symmetries is replaced by that of infinitesimal symmetries. To state it, we need the following considerations. We will write $\Sigma(f, g)$, to denote the system Σ defined by the pair of vector fields (f, g) . Assume that v is an infinitesimal symmetry of $\Sigma(f, g)$, nonstationary at $p \in X$, that is, such that $v(p) \neq 0$. Then the second part of Proposition II.1 implies that there exists a feedback pair (α, β) such that v is a strong infinitesimal symmetry of the system $\tilde{\Sigma}(\tilde{f}, \tilde{g})$, where $\tilde{f} = f + g\alpha$ and $\tilde{g} = g\beta$. Thus there exists a neighborhood X_p of p in which the factor system $\tilde{\Sigma}/\sim_v$ system is well defined, where the equivalence relation $\sim_v$ is induced by the local action of the 1-parameter local group defined by v . Notice that given a system Σ , there are many systems $\tilde{\Sigma}(\tilde{f}, \tilde{g})$, feedback equivalent to Σ , and such that v is a strong infinitesimal symmetry of $\tilde{\Sigma}$. We will denote by $\tilde{\Sigma}$ any of those systems. Actually, any two such systems are equivalent by a feedback pair $(\tilde{\alpha}, \tilde{\beta})$, where the functions $\tilde{\alpha}$ and $\tilde{\beta}$ are constant on the trajectories of v .

Theorem II.4 *The following condition are equivalent.*

- (i) Σ is, locally at $p \in X$, F -equivalent to the affine strict feedforward form (ASFF) satisfying $g_n \neq 0$;
- (ii) Each system $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ possesses an infinitesimal symmetry v_i , where Σ_1 is the restriction of Σ to a neighborhood X_p and

$$\Sigma_{i+1} = \tilde{\Sigma}_i / \sim_{v_i},$$

where $\sim_{v_i}$ is the equivalence relation induced by the local action of the 1-parameter group of v_i , and such that v_i and the control vector field g_i of Σ_i are independent, for $1 \leq i \leq n-1$;

- (iii) There exist smooth vector fields $w_1, \dots, w_n$, independent at $p \in X$, such that, locally at p ,

$$\begin{aligned} [w_i, w_j] &\in \mathcal{D}_{i-1} \\ [w_i, g] &\in \mathcal{D}_{i-1} + \mathcal{G} \\ [w_i, f] &\in \mathcal{D}_{i-1} + \mathcal{G}, \end{aligned}$$

for any $1 \leq i \leq n$ and $j \leq i$, where $\mathcal{D}_0 = 0$ and $\mathcal{D}_i = \text{span}\{w_1, \dots, w_i\}$ and, moreover, $g(p) \notin \mathcal{D}_{n-1}(p)$;

- (iv) There exist smooth vector fields $\tilde{w}_1, \dots, \tilde{w}_n$, independent at $p \in X$, such that, locally at p ,

$$\begin{aligned} [\tilde{w}_i, \tilde{w}_j] &= 0 \\ [\tilde{w}_i, g] &\in \tilde{\mathcal{D}}_{i-1} + \mathcal{G} \\ [\tilde{w}_i, f] &\in \tilde{\mathcal{D}}_{i-1} + \mathcal{G}, \end{aligned}$$

for any $1 \leq i \leq n$ and $j \leq i$, where $\tilde{\mathcal{D}}_0 = 0$ and $\tilde{\mathcal{D}}_i = \text{span}\{\tilde{w}_1, \dots, \tilde{w}_i\}$ and, moreover, $g(p) \notin \tilde{\mathcal{D}}_{n-1}(p)$.

The assumption $g(p) \notin \mathcal{D}_{n-1}(p)$ can be dropped (equivalently, we allow for $g_n = 0$) if we understand the conditions (iii) and (iv) as well as those of the second part

of Proposition II.1 in the sense of module of vector fields and not of distributions.

A proof of the above theorem follows the same line as that of Theorem II.2, the only difference is to show that in the successive steps, the existence of infinitesimal symmetries does not depend on the choice of $\tilde{\Sigma}_i$ in $\Sigma_{i+1} = \Sigma_i / \sim_{v_i}$.

III. STRICT FEEDFORWARD FORM: AFFINE VERSUS GENERAL

In this section we will show that the problem of transforming a general control system to the strict feedforward form can be reduced to that for affine systems by taking the preintegration. The same procedure of extension has been already used for the problems of linearization and decoupling [13] and equivalence to the p -normal form [11].

Consider a general nonlinear control system

$$\Pi : \dot{x} = f(x, u),$$

where $x \in X$, an open subset of $\mathbb{R}^n$, $u \in \mathbb{R}$. Together with Σ , we consider its extension (preintegration)

$$\Pi^e : \dot{x}^e = f^e(x^e) + g^e(x^e)w,$$

where $x^e = (x, u) \in X \times \mathbb{R}^1$, $w \in \mathbb{R}$, and the dynamics are given by $f^e(x^e) = f(x, u) + 0 \cdot \frac{\partial}{\partial u}$ and $g^e(x^e) = \frac{\partial}{\partial u}$. Notice that Σ^e is a control-affine system controlled by the derivative $\dot{u} = w$ of the original control u .

Recall that $\mathcal{L}_0$ denotes the Lie ideal generated by $\{f_u - f_{\bar{u}}\}$, $u, \bar{u} \in U$, in the Lie algebra $\mathcal{L}$ of the system Π . Assume that $\dim \mathcal{L}_0(p) = n$.

Proposition III.1 *The system Π is S -equivalent (resp. F -equivalent), locally at (x_0, u_0) , to the strict feedforward form (SFF) if and only if the extension Π^e is, locally at $x_0^e = (x_0, u_0)$, S -equivalent (resp. F -equivalent) to the affine strict feedforward form (ASFF).*

The proof is based on showing that a diffeomorphism bringing Π^e into the (ASFF) is of a special form: states depend on states only and the control is preserved. In particular, we show the following statement, which is of independent interest.

Corollary III.2 *If the system Σ is in an affine strict feedforward form (ASFF) satisfying $g_n \neq 0$, then it is S -equivalent to another (ASFF), for which $g_1 = \dots = g_{n-1} = 0$.*

IV. STRICT FEEDFORWARD SYSTEMS ON THE PLANE

In this section we will describe strict feedforward systems on the plane. Consider a system Σ on an open subset X of $\mathbb{R}^2$ and suppose that $g(p) \neq 0$. We define the *multiplicity* of Σ at p as the smallest positive integer μ , such that g and $ad_g^\mu f$ are linearly independent at p . Notice that the notion of multiplicity is feedback invariant (see, e.g., [4]). If the multiplicity is $\mu = 1$, then the system is feedback

linearizable and thus feedback equivalent to (ASFF). The case of multiplicity $\mu \geq 2$ is described by the following:

Proposition IV.1 *Consider a system Σ on open subset X of $\mathbb{R}^2$ and suppose that $g(p) \neq 0$ and that it has multiplicity $\mu \geq 2$ at p .*

- (i) *If f and g are linearly dependent at p , then Σ is locally F -equivalent to the strict feedforward form (ASFF) if and only*

$$f = \gamma ad_g f \mod \mathcal{G},$$

where γ is a smooth function such that the smooth function φ defined by

$$f = \varphi ad_g^\mu f \mod \mathcal{G}$$

is divisible by γ^μ . Moreover, in this case Σ is locally F -equivalent to

$$\begin{aligned} \dot{z}_1 &= z_2^\mu \\ \dot{z}_2 &= v. \end{aligned}$$

- (ii) *If f and g are linearly independent at p , then Σ is locally F -equivalent to the strict feedforward form if and only*

$$ad_g f = \gamma ad_g^2 f \mod \mathcal{G},$$

where γ is a smooth function such that the smooth function ψ defined by

$$ad_g f = \psi ad_g^\mu f \mod \mathcal{G}$$

is divisible by $\gamma^{\mu-1}$. Moreover in this case Σ is locally F -equivalent to

$$\begin{aligned} \dot{z}_1 &= 1 + \epsilon z_2^\mu \\ \dot{z}_2 &= v. \end{aligned}$$

In [4] it is proved that any planar system with a finite multiplicity μ at p is locally feedback equivalent to the following system around $0 \in \mathbb{R}^2$:

$$\begin{aligned} \dot{z}_1 &= z_2^\mu + a_{\mu-2} z_2^{\mu-2} + \dots + a_1 z_2 + a_0, \\ \dot{z}_2 &= v, \end{aligned}$$

where the smooth functions a_i , for $0 \leq i \leq \mu - 2$, depend on z_1 only and satisfy $a_i(0) = 0$ (except for a_0 in the case f and g independent at p). Moreover, we can always normalize one of the functions a_i (in particular, we can take $a_0 = 1$ if $a_0(0) \neq 0$) and then the infinite jets of all remaining functions are feedback invariant. Proposition IV.1 implies that among all planar system only those are F -equivalent to the affine strict feedforward form for which all the above invariants are identically zero.

Proof of (i) (Necessity) After applying a local diffeomorphism and feedback, the system Σ takes the following (ASSF)

$$\begin{aligned} \dot{z}_1 &= f_1(z_2) + g_1(z_2)u \\ \dot{z}_2 &= u. \end{aligned}$$

Without loss of generality, we can assume that $g_1 = 0$ (see Corollary III.2 following Proposition III.1). By the definition of multiplicity, $f_1 = z_2^\mu \tilde{f}_1$, where $\tilde{f}_1$ is a smooth function of z_2 such that $\tilde{f}_1(0) \neq 0$. A direct computation shows that the conditions of (i) are satisfied for the above system and, since they are feedback invariant, they are necessary for bringing the system to the (ASSF).

(Sufficiency) Rectifying the vector field g and applying a suitable feedback, we get

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= u.\end{aligned}$$

By the definition of multiplicity and the assumptions of (i), it follows that

$$f_1 = \gamma^\mu \tilde{\varphi},$$

where the smooth function $\tilde{\varphi}$ satisfies $\tilde{\varphi}(0) \neq 0$.

Differentiating the condition $f = \gamma \text{ad}_g f \bmod \mathcal{G}$ μ -times with respect to g we get

$$\text{ad}_g^\mu f = \gamma \text{ad}_g^{\mu+1} f + \mu(L_g \gamma) \text{ad}_g^\mu f + \sum_{i=1}^{\mu-1} h_i \text{ad}_g^i f + h_0 g,$$

where the smooth functions h_i satisfy $h_i(p) = 0$. Using the definition of multiplicity we can conclude that $L_g \gamma(p) \neq 0$. Put $\epsilon = \text{sign}(\tilde{\varphi}(0))$. Introducing coordinates

$$\begin{aligned}z_1 &= \epsilon x_1 \\ z_2 &= \gamma(\epsilon \tilde{\varphi})^{1/\mu},\end{aligned}$$

followed by a suitable feedback, we get

$$\begin{aligned}\dot{z}_1 &= z_2^\mu \\ \dot{z}_2 &= v.\end{aligned}$$

Necessity of (ii) is obvious while the proof of sufficiency follows the same line as for (i). $\square$

V. STRICT FEEDFORWARD NORMAL FORM

Definition V.1 A smooth (resp. analytic) *strict feedforward normal form* is a smooth (resp. analytic) strict feedforward form

$$\begin{aligned}\dot{z}_1 &= \hat{F}_1(z_2, \dots, z_n, u) \\ &\vdots \\ \dot{z}_{n-1} &= \hat{F}_{n-1}(z_n, u) \\ \dot{z}_n &= \hat{F}_n(u)\end{aligned}$$

for which
(SFNF)

$$\hat{F}_j(z, u) = \hat{a}_j(z_{j+1}) + \sum_{i=j+2}^{n+1} z_i^2 \hat{P}_{j,i}(z_{j+1}, \dots, z_i)$$

for any $1 \leq j \leq n-1$, where $\hat{a}_j$ and $\hat{P}_{j,i}$ are smooth (resp. analytic) functions of the indicated variables and $z_{n+1} = u$.

The above strict feedforward normal form (SFNF) was introduced by the authors in [16] (see also [15]), where we proved that any strict feedforward system can be brought

formally (see [6], [7], and [14] for a study of formal feedback transformations) to that form. In this section we will give a smooth version of this result. We will suppose that the linearization, around the equilibrium point, is controllable for the class of systems under consideration.

Theorem V.2 A smooth (resp. analytic) system Π is feedback equivalent to a smooth (resp. analytic) strict feedforward form (SFF) if and only if it is feedback equivalent to a smooth (resp. analytic) strict feedforward normal form (SFNF).

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