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K-Regular Witt Rings

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(R, G, q) will denote an abstract Witt ring in the sense of [4]. Nearly all examples of interest are Witt rings of non-singular quadratic forms over a field of characteristic not two,however using abstract Witt rings does simplify some proofs. The Witt ring is *k-regular* if there exists a 2-power k such that for all $1 \neq x \in G$ we have $|D\langle 1, -x\rangle| = k$. Such Witt rings were first studied in [1] primarily because the block design counting arguments there were perfectly suited to k-regular rings. However they remain unclassified.

We will always assume that G is finite; set $g = |G|$. If $k = g$ then R is totally degenerate and so classified by [4]. If $k = g/2$ then R is of local type [2] which are again classified in [4]. If $k = 2$ then R is a group ring extension of $\mathbb{Z}_2$ or $\mathbb{Z}_4$. If $2 < k < g/2$ then R is not of elementary type and no examples are known or even expected. We will always assume that $2 < k < g/2$ and call such k-regular Witt rings *exceptional*.

It was shown in [1] that exceptional k-regular Witt rings satisfy $8 \leq k$ and $2k^2 \leq g$. Kula [3] improved both bounds and added an upper bound, showing:

$$\begin{aligned} &16 \leq k \\ &8k^2 \leq g \leq k^4/4 \qquad \text{if } k \equiv 1 \pmod 3 \\ &8k^2 \leq g \leq k^4/8 \qquad \text{if } k \equiv 2 \pmod 3. \end{aligned}$$

Here we show that $k^3 \leq g$ and that if $k \equiv 1 \pmod 3$ then $g \equiv 1 \pmod 3$.

We fix some notation, which will agree with Kula's. G^* denotes $G \setminus \{1\}$. We set $e = \log_2 k$. For $a \in G^*$ and $i \geq 0$ set:

$$X_i(a) = \{x \in G : x \neq 1, a \text{ and } |Q(a) \cap Q(x)| = 2^i\},$$

where $Q(x) = \{q(x, y) : y \in G\}$. Now for $x \neq a, |Q(a) \cap Q(x)| = |D\langle 1, -ax\rangle|/|D\langle 1, -a\rangle \cap D\langle 1, -x\rangle| = k/|D\langle 1, -a\rangle \cap D\langle 1, -x\rangle|$. Thus we also have that:

$$X_i(a) = \{x \in G : x \neq 1, a \text{ and } |D\langle 1, -a\rangle \cap D\langle 1, -x\rangle| = 2^{e-i}\}.$$

In particular, we may assume $0 \leq i \leq e$. We further set $n_i(a) = |X_i(a)|$ and write $X(a)$ for $X_e(a)$. For a 2-fold Pfister form ρ we let $\rho\prime$ denote the pure part of ρ.

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We will use various equations derived by Kula:

$$\sum_{i=0}^{e-1}(2^{e-i}-1)n_i(a) = k^2-3k+2 \tag{1}$$

$$g + \sum_{1\neq\rho\in Q(a)} |D(\rho\prime)| = 1 + \frac{g}{k} + \sum_{i=0}^{e} 2^i n_i(a) \tag{2}$$

$$|X(a)\cap X(b)| \geq g-2k^2+6k-7 \geq g-2k^2, \tag{3}$$

where $a \neq b$ in G^* for (3). Equation (1) is [3,4.3b], (2) is equation (4.5.2) on [3,p.45] and the first inequality of (3) is equation (4.3.1) on [3,p.43]. The second inequality of (3) follows from our assumption that $k > 2$.

We will also use two simple equations:

$$\sum_{i=0}^{e} n_i(a) = g-2 \tag{4}$$

$$|D(\rho\prime)| < k^2 \qquad (\text{if } \rho \neq 1). \tag{5}$$

Both (4) and (5) appear in [3] but direct proofs are quick. (4) follows from $G \setminus \{1, a\}$ being the union of the $X_i(a)$. For (5), suppose $\rho\prime = \langle a, b, ab\rangle$. Then

$$D(\rho\prime) = a \cdot \cup_{x\in D\langle 1,a\rangle} D\langle 1, bx\rangle.$$

Since 1 occurs in each $D\langle 1, bx\rangle$ we have that $|D(\rho\prime)| < |D\langle 1, a\rangle| \cdot k = k^2$.

Using equation (4) to find $n_e(a)$ and equation (1) to find $n_{e-1}(a)$, equation (2) may be re-written (see [3, pp. 45-46]) as:

$$g + \sum_{1\neq\rho\in Q(a)} |D(\rho\prime)| = 1 + \frac{g}{k} + gk - \frac{k^3}{2} + \frac{3k^2}{2} - 3k + \sum_{i=0}^{e-2} 2^i(2^{e-i-1}-1)(2^{e-i}-1)n_i(a). \tag{6}$$

Proposition 1. *If* $k \equiv 1 \pmod 3$ *then* $g \equiv 1 \pmod 3$.

Proof. We may pick an $a \in G^*$ with $\langle\langle 1, 1\rangle\rangle \notin Q(a) \setminus \{1\}$ (otherwise $-G^* \subset D\langle 1, 1, 1\rangle$ while $|D\langle 1, 1, 1\rangle| < k^2$ by (5) and $|G^*| \geq 8k^2 - 1$ by [3,4.4]. Then for each anisotropic $\rho \in Q(a)$ we have that $|D(\rho\prime)| \equiv 0 \pmod 3$ by [3,2.9]. Also, since for each i, in equation (6) one of $e-i-1$ or $e-i$ is even, we have that one of $2^{e-i-1}-1$ or $2^{e-i}-1$ is divisible bt 3. Assuming $k \equiv 1 \pmod 3$, equation (6) gives:

$$g \equiv g + 1 + g - 2 \pmod 3,$$

and so $1 \equiv g \pmod 3$. $\square$

Theorem 1. $g \geq k^3$.

Proof. Suppose there exists an exceptional k-regular Witt ring (R,G) with $g < k^3$. Among all such Witt rings, choose one with minimal $h \equiv g/k^2$. Let a and b be distinct elements of G^*. Choose $x \in X(a) \cap X(b)$, which is possible by equation (3) and the fact that $g \geq 8k^2$ [3,4.4]. We use the equation (4.3.2) from [3,p.43]:

$$\begin{aligned} hk = g/k = |Q(x)| &\geq |(Q(x) \cap Q(a))(Q(x) \cap Q(b)| \\ &= \frac{k^2}{|Q(x) \cap Q(a) \cap Q(b)|} \\ &\geq \frac{k^2}{|Q(a) \cap Q(b)|} = k|D\langle 1, -a\rangle \cap D\langle 1, -b\rangle|. \end{aligned} \tag{7}$$

A simple consequence of (7) is that $|D\langle 1, -a\rangle \cap D\langle 1, -b\rangle| \leq h$. Pick minimal $s \geq 0$ so that there exists distinct a and b in G^* with $|D\langle 1, -a\rangle \cap D\langle 1, -b\rangle| = h/2^s$. Set $2^t = |Q(a) \cap Q(b)|$. then we have:

$$g \geq 2^{s+2}k^2 \qquad \text{and} \qquad t - s \geq 1 \tag{8}$$

Namely, if the first inequality failed then $h = g/k^2 \leq 2^{s+1}$. But then $|D\langle 1, -a\rangle \cap D\langle 1, -b\rangle| \leq 2$ for all distinct a and b in G^*. while as noted in the first sentence of [K,p.44] we can always find distinct a and b in G^* with $|D\langle 1, -a\rangle \cap D\langle 1, -b\rangle| \geq 4$. For the second inequality of (8) note that:

$$2^t = |Q(a) \cap Q(b)| = \frac{k}{|D\langle 1, -a\rangle \cap D\langle 1, -b\rangle|} = \frac{2^s k}{h}.$$

Thus $2^{t-s}h = k$. By the assumption that $g = hk^2 < k^3$ we have $2h \leq k$ and so $t - s \geq 1$.

For each $x \in X(a) \cap X(b)$ we can rewrite (7) as:

$$hk \geq \frac{k^2}{|Q(x) \cap Q(a) \cap Q(b)|} \geq \frac{k^2}{|Q(a) \cap Q(b)|} = \frac{hk}{2^s}. \tag{9}$$

Then

$$|Q(x) \cap Q(a) \cap Q(b)| \geq 2^{t-s}$$

since otherwise $|Q(x) \cap Q(a) \cap Q(b)| < 2^{t-s} = |Q(a) \cap Q(b)|/2^s$ and equation (9) becomes:

$$hk \geq \frac{k^2}{|Q(x) \cap Q(a) \cap Q(b)|} > \frac{2^s k^2}{|Q(a) \cap Q(b)|} = hk.$$

List the elements of $Q(a) \cap Q(b)$ as $1, \rho_2, \ldots, \rho_{2^t}$. We have that for each $x \in X(a) \cap X(b)$ that $2^{t-s} - 1$ of the ρ_i's lie in $Q(x)$, or equivalently, satisfy $-x \in D(\rho_i')$. Set T_x equal to the number of i's, $2 \leq i \leq 2^t$, such that $-x \in D(\rho_i')$. Then:

$$\sum_{x \in X(a) \cap X(b)} T_x \geq (2^{t-s} - 1)|X(a) \cap X(b)| \tag{10}$$

Now this sum counts the number of pairs (i,x) with $2 \leq i \leq 2^t, x \in X(a) \cap X(b)$ and $-x \in D(\rho_i')$. We can also count the number of such pairs by first fixing i. Namely:

$$\sum_{x \in X(a) \cap X(b)} T_x = \sum_{i=2}^{2^t} |D(\rho_i') \cap -(X(a) \cap X(b))|. \tag{11}$$

Now (11) implies that there exists an i, $\quad 2 \leq i \leq 2^t$, such that:

$$|D(\rho_i') \cap -(X(a) \cap X(b))| \geq \frac{1}{2^t - 1} \sum_{x \in X(a) \cap X(b)} T_x$$

and hence when combined with (9):

$$|D(\rho_i')| \geq \frac{2^{t-s} - 1}{2^t - 1} |X(a) \cap X(b)|.$$

Applying equations (5) and (3) yields:

$$k^2 > \frac{2^{t-s} - 1}{2^t - 1} (g - 2k^2) \tag{12}$$

If $s = 0$ then (12) becomes $k^2 > g - 2k^2$ which is impossible as $g \geq 8k^2$ [3,4.4]. Suppose then that $s \geq 1$. (11) is then:

$$(2^t + 2^{t-s+1} - 3)k^2 > (2^{t-s} - 1)g.$$

Use $g \geq 2^{s+2}k^2$ from the first part of (8) to get:

$$\begin{gathered} (2^t + 2^{t-s+1} - 3)k^2 > (2^{t+2} - 2^{s+2})k^2 \\ 2^{s+2} + 2^{t-s+1} - 3 > 3 \cdot 2^t. \end{gathered}$$

Lastly, using $t - 1 \geq s$ from the second part of (8) gives:

$$\begin{gathered} 2^{t+1} + 2^{t-s+1} - 3 > 3 \cdot 2^t, \\ 2^{t-s+1} - 3 > 2^t, \end{gathered}$$

which is impossible for $s \geq 1$. This contradiction shows $g \geq k^3$. $\square$

We combine these results with Kula's upper bound on g and bound on k.

Corollary 1. *For an exceptional k-regular Witt ring (R,G) with $g = |G|$: $\quad k \geq 16$ and*

(1) *if $k \equiv 1 \pmod 3$ then $g \equiv 1 \pmod 3$ and*

$$k^3 \leq g \leq \frac{1}{4} k^4,$$

(2) *if $k \equiv 2 \pmod 3$ then*

$$k^3 \leq g \leq \frac{1}{8} k^4.$$

$\square$

We note that the first open case is $k = 16$ and $g = 16^3 = 4096$.

Kula has shown that an exceptional k-regular Witt ring is non-formally real [3,Remark,p.41] so that $I^n R = 0$ for some n. We have:

Corollary 2. *If (R,G) is an exceptional k-regular Witt ring then* $I^3R \neq 0$. *In fact, for any anisotropic 2-fold Pfister form* $\rho, D(\rho) \neq G$.

Proof. $D(\rho) = \cup_{b \in D(\rho\prime)} D\langle 1, b\rangle$ so that $|D(\rho)| \leq k|D(\rho\prime)| < k^3$ by equation (5). Thus $D(\rho) \neq G$ by Theorem 1. $\square$

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