Normal Forms for Two-Inputs Nonlinear Control Systems

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Abstract

We study the feedback group action on two-inputs nonlinear control systems. We follow an approach proposed by Kang and Krener which consists of analyzing the system and the feedback group step by step. We construct a normal form which generalizes that obtained in the single-input case. We also give homogeneous m-invariants of the action of the group of homogeneous transformations on the homogeneous systems of the same degree. We illustrate our results by analyzing the normal form and invariants of homogeneous systems of degree two.

Keywords: normal forms, feedback transformation, homogeneous systems, invariants.

1 Introduction

During the last twenty years, the problem of transforming the nonlinear control system

$$\Sigma \ : \ \dot{\zeta} = f(\zeta) + g(\zeta)u, \quad \zeta \in \mathbb{R}^n, \quad u \in \mathbb{R}^m,$$

by a feedback transformation

$$\Gamma \ : \ \begin{array}{rcl} x & = & \phi(\zeta) \\ u & = & \alpha(\zeta) + \beta(\zeta)v \end{array}$$

to a simpler form has been extensively studied using various techniques. The transformation Γ brings Σ to the system

$$\tilde{\Sigma} \ : \ \dot{x} = \tilde{f}(x) + \tilde{g}(x)v,$$

whose dynamics are given by

$$\begin{array}{rcl} \tilde{f} & = & \phi_*(f + g\alpha) \\ \tilde{g} & = & \phi_*(g\beta), \end{array}$$

where for any vector field f and any diffeomorphism ϕ we denote

$$(\phi_* f)(x) = d\phi(\phi^{-1}(x)) \cdot f(\phi^{-1}(x)).$$

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A natural question to ask is whether we can find a transformation Γ bringing Σ into a linear system, that is, whether we can linearize the system Σ via feedback. Necessary and sufficient geometric conditions for feedback linearizability have been given in [6] and [10]. Those conditions are, except for the planar control affine case, restrictive and a natural problem that arises is to find normal forms for non linearizable systems. Although being natural, this problem is very involved (because it necessarily involves functional invariants) and has been extensively studied during the last twenty years. Four basic methods have been proposed to study feedback equivalence problems. The first method is based on the theory of singularities of vector fields and distributions, and their invariants, and using that method a large variety of feedback classification problems have been solved, see e.g. [7], [10], [11], [15], [18], [19], [24]. The second approach, proposed by Gardner [3], uses Cartan's method of equivalence and describes the geometry of feedback equivalence, see [4], [5], [17]. The third method, inspired by the hamiltonian formalism for optimal control problems, has been developed by Bonnard and Jakubczyk [2], [8], [9] and has led to a very nice description of feedback invariants in terms of singular extremals. Finally, a very fruitful approach was proposed by Kang and Krener [14] and then followed by Kang [12], [13]. Their idea, which is closely related with classical Poincaré's technique for linearization of dynamical systems (see e.g. [1]), is to analyze the system Σ and the feedback transformation Γ step by step and, as a consequence, to produce a simpler equivalent system $\tilde{\Sigma}$ also step by step.

Recently, the results of Kang and Krener [12], [13], [14] have been completed by the authors [20], [21] who obtained canonical forms and dual canonical forms for single-input nonlinear control systems with controllable linearization. The authors also obtained normal forms for single-input nonlinear control systems with uncontrollable linearization [22] (see also [16]), as well as the corresponding homogeneous invariants. Thus the feedback classification of single-input nonlinear control systems is almost complete and the aim of this paper is to deal with the multi-input nonlinear control systems. We will study two-inputs control systems and the general case will be given elsewhere.

In this paper we construct a normal form for two-inputs nonlinear control systems with controllable linearization. We will state our result for the case of equal controllability indices and involutive distribution $\mathcal{G}$ spanned by the control vector fields g_1 and g_2. The general case (of unequal controllability indices, of noninvolutive distribution $\mathcal{G}$, and of general systems $\dot{x} = f(x, u)$ that are not affine with respect to controls) reduces to the problem considered in this paper by a suitable prolongation of the system and is treated in [23]. For the class of systems considered in this paper, we rectify the involutive distribution $\mathcal{G}$ and we group all non removable nonlinearities in the drift. This procedure leads to a normal form that generalizes, in the case of two-inputs, those obtained in the single-input case [12], [14], [20], [21]. Indeed, the normal form obtained in this paper exhibits, similarly to the single-input case, a triangular structure.

The second contribution of the paper is to give m-invariants of the feedback equivalence of homogeneous systems under homogeneous feedback transformations of the same degree. For proofs and for a detailed analysis of feedback equivalence of two-inputs systems, the reader is sent to [23].

The paper is organized as follows. In Section 2, we give basic definitions and fix notations. In Section 3 we construct our normal form for two-inputs nonlinear control systems with controllable linearization. In Section 4, we discuss invariants of the action of homogeneous transformations on homogeneous systems of the same degree. Finally, in Section 5 we consider quadratic systems and give a geometric interpretation of their invariants.

2 Notations and definitions

All objects, that is, functions, maps, vector fields, control systems, etc., are considered in a neighborhood of the point $0 \in \mathbb{R}^n$ and assumed to be C^∞-smooth. Let h be a smooth $\mathbb{R}$-valued function. By

$$h(\zeta) = h^{[0]}(\zeta) + h^{[1]}(\zeta) + h^{[2]}(\zeta) + \cdots = \sum_{m=0}^{\infty} h^{[m]}(\zeta)$$

we denote its Taylor series expansion at $0 \in \mathbb{R}^n$, where $h^{[m]}(\zeta)$ stands for a homogeneous polynomial of degree m.

Similarly, for a map ϕ of an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ (resp. for a vector field f on an open subset of $\mathbb{R}^n$) we will denote by $\phi^{[m]}$ (resp. by $f^{[m]}$) the term of degree m of its Taylor series expansion at $0 \in \mathbb{R}^n$, that is, each component $\phi_j^{[m]}$ of $\phi^{[m]}$ (resp. $f_j^{[m]}$ of $f^{[m]}$) is a homogeneous polynomial of degree m in ζ.

We will denote by $H^{[m]}(\zeta)$ the space of homogeneous polynomials of degree m of the variables $\zeta_1, \ldots, \zeta_n$ and by $H^{\geq m}(\zeta)$ the space of formal power series of the variables $\zeta_1, \ldots, \zeta_n$ starting from terms of degree m. Analogously, we will denote by $R^{[m]}(\zeta)$ the space of homogeneous vector fields whose components are in $H^{[m]}(\zeta)$ and by $R^{\geq m}(\zeta)$ the space of vector fields formal power series whose components are in $H^{\geq m}(\zeta)$.

We consider nonlinear control systems, with two-inputs, defined by

$$\Sigma : \ \dot{\zeta} = f(\zeta) + g(\zeta)u = f(\zeta) + g_1(\zeta)u_1 + g_2(\zeta)u_2,$$

where $\zeta \in \mathbb{R}^n$ and $u = (u_1, u_2)^T \in \mathbb{R}^2$. Throughout the paper we assume that $0 \in \mathbb{R}^n$ is an equilibrium for the drift f, that is $f(0) = 0$, and that the linear approximation at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^2$, given by

$$\dot{\zeta} = F\zeta + Gu = F\zeta + G_1u_1 + G_2u_2, \qquad (1)$$

where

$$F = \frac{\partial f}{\partial \zeta}(0), \ G_1 = g_1(0), \ G_2 = g_2(0),$$

is controllable, and that the matrix G is of rank 2, which means that $G_1 \wedge G_2 \neq 0$. The integers $1 \leq r_1 \leq r_2$, with $r_1 + r_2 = n$, forming the largest pair (r_1, r_2), in the lexicographic ordering, such that

$$\operatorname{span}\left\{F^j G_i, \ \ 0 \leq j \leq r_i - 1, \ 1 \leq i \leq 2\right\} = \mathbb{R}^n, \quad (2)$$

are called the *controllability indices* of the linear approximation (1). Throughout the paper they are assumed to be equal, that is, $r_1 = r_2 = r$, where $2r = n$.

The linear controllability assumption (2) implies

$$\operatorname{span}\left\{ad_f^j g_i(0), \ \ 0 \leq j \leq r - 1, \ 1 \leq i \leq 2\right\} = \mathbb{R}^n.$$

In particular, the distribution $\mathcal{G} = \operatorname{span}\{g_1, g_2\}$, spanned by the vector fields g_1 and g_2, is of constant rank 2 in a neighborhood of the point $0 \in \mathbb{R}^n$.

Our aim is to give a normal form for Σ under invertible feedback transformations of the form

$$\Gamma : \begin{array}{rcl} x & = & \phi(\zeta) \\ u & = & \alpha(\zeta) + \beta(\zeta)v, \end{array}$$

where $v = (v_1, v_2)^T$, and α and β are, respectively, $\mathbb{R}^2$-valued and $Gl(2, \mathbb{R})$-valued smooth functions.

Consider the Taylor series expansion of the system Σ given by

$$\Sigma^\infty : \dot{\zeta} = F\zeta + Gu + \sum_{m=2}^{\infty} \left(f^{[m]}(\zeta) + g^{[m-1]}(\zeta)u\right). \ (3)$$

Consider also the Taylor series expansion Γ^∞ of the feedback transformation Γ given by

$$\Gamma^\infty : \begin{array}{rcl} x & = & T\zeta + \sum_{m=2}^{\infty} \phi^{[m]}(\zeta) \\ u & = & K\zeta + Lv + \sum_{m=2}^{\infty} \left(\alpha^{[m]}(\zeta) + \beta^{[m-1]}(\zeta)v\right), \end{array} \qquad (4)$$

where T and L are invertible matrices. Let us analyze the action of Γ^∞ on the system Σ^∞.

We will use an approach proposed by Kang and Krener [14], (see also [12], [13], [20], [21]), which consists of applying the feedback transformation Γ^∞ step by step.

We first notice that, because of the controllability assumption (2), with $r_1 = r_2 = r$, there always exists a linear feedback transformation

$$\Gamma^1 : \begin{array}{rcl} x & = & T\zeta \\ u & = & K\zeta + Lv \end{array}$$

bringing the linear part

$$\dot{\zeta} = F\zeta + Gu = F\zeta + G_1 u_1 + G_2 u_2$$

into the Brunovský canonical form

$$\dot{x} = Ax + Bv = Ax + B_1 v_1 + B_2 v_2,$$

where

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}_{n\times n}, B = (B_1, B_2) = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_{n\times 2}$$

with (A_1, b_1) and (A_2, b_2) being Brunovský canonical single-input forms, each of dimension r.

Then we study, successively for $m \geq 2$, the action of the *homogeneous feedback transformations*

$$\Gamma^m : \begin{array}{rcl} x & = & \zeta + \phi^{[m]}(\zeta) \\ u & = & v + \alpha^{[m]}(\zeta) + \beta^{[m-1]}(\zeta)v \end{array} \quad (5)$$

on the *homogeneous systems*

$$\Sigma^{[m]} : \dot{\zeta} = A\zeta + Bu + f^{[m]}(\zeta) + g^{[m-1]}(\zeta)u. \quad (6)$$

A feedback transformation defined as a series of successive compositions of Γ^m, $m = 1, 2, \ldots$ will also be denoted by Γ^∞ because, as a formal power series, it is of the form (4). We will not address the problem of convergence and we will call such a series of successive compositions a *formal feedback transformation.*

We say that the homogeneous system $\Sigma^{[m]}$, given by (6), is feedback equivalent to the homogeneous system

$$\tilde{\Sigma}^{[m]} : \dot{x} = Ax + Bv + \tilde{f}^{[m]}(x) + \tilde{g}^{[m-1]}(x)v \quad (7)$$

if there exists a homogeneous feedback transformation Γ^m, of the form (5), which brings $\Sigma^{[m]}$ into $\tilde{\Sigma}^{[m]}$ modulo terms in $R^{\geq m+1}(x, v)$.

We will say that the homogeneous system $\Sigma^{[m]}$ has an *involutive distribution* $\mathcal{G}^{[m]}$ if the distribution

$$\mathcal{G}^{[m]} = \operatorname{span}\left\{B_1 + g_1^{[m-1]}, B_2 + g_2^{[m-1]}\right\}$$

is involutive modulo terms in $R^{\geq m-1}(\zeta)$.

3 Main Results.

In this section we will establish our main results. For any $k = 1, 2$ and for any $1 \leq i \leq r$, we denote

$$\bar{x}_{k,i} = (x_{k,1}, \cdots, x_{k,i})^T.$$

Together with the distribution $\mathcal{G} = \operatorname{span}\{g_1, g_2\}$, we will consider

$$\mathcal{G}^\infty = \operatorname{span}\left\{B_1 + \sum_{m=2}^{\infty} g_1^{[m-1]}, B_2 + \sum_{m=2}^{\infty} g_2^{[m-1]}\right\}.$$

The following result gives normal forms for two-inputs control-affine systems with equal controllability indices and involutive control distribution.

Theorem 1 (i) *The two-inputs homogeneous system $\Sigma^{[m]}$, defined by* (6), *with* involutive distribution $\mathcal{G}^{[m]}$, *is equivalent, by a homogeneous feedback transformation Γ^m of the form* (5), *to the following normal form*

$$\Sigma_{NF}^{[m]} : \dot{x} = Ax + Bv + \bar{f}^{[m]}(x),$$

where, for any $m \geq 2$, we have

$$\bar{f}^{[m]}(x) = \sum_{j=1}^{r-2}\left(\bar{f}_j^{1[m]}(x)\frac{\partial}{\partial x_{1,j}} + \bar{f}_j^{2[m]}(x)\frac{\partial}{\partial x_{2,j}}\right), \quad (8)$$

with

$$\begin{array}{rcl} \bar{f}_j^{k[m]}(x) & = & \sum\limits_{i=j+2}^{r} x_{1,i}^2 P_{j,i}^{k[m-2]}(\bar{x}_{1,i}, \bar{x}_{2,i-1}) \\ & + & \sum\limits_{i=j+2}^{r} x_{2,i}^2 Q_{j,i}^{k[m-2]}(\bar{x}_{1,i-1}, \bar{x}_{2,i}) \\ & + & \sum\limits_{i=j+2}^{r} x_{1,i}x_{2,i} R_{j,i}^{k[m-2]}(\bar{x}_{1,i}, \bar{x}_{2,i}) \\ & + & \sum\limits_{i=j+2}^{r} x_{1,i}x_{2,i-1} S_{j,i}^{k[m-2]}(\bar{x}_{1,i-1}, \bar{x}_{2,i-1}) \end{array} \quad (9)$$

for any $1 \leq j \leq r-2$, and any $k = 1, 2$.

(ii) *The two-inputs nonlinear control affine system Σ^∞, defined by* (3), *with* involutive distribution $\mathcal{G}^\infty$ *is feedback equivalent, by a formal feedback transformation Γ^∞ of the form* (4), *to the following normal form*

$$\Sigma_{NF}^\infty : \dot{x} = Ax + Bv + \sum_{m=2}^{\infty} \bar{f}^{[m]}(x),$$

where, for any $m \geq 2$, the vector field $\bar{f}^{[m]}$ is given by (8)-(9).

Notice that, if we take $k = 1$, and hence $r = n$, then $(x_{2,1}, x_{2,2}, \ldots, x_{2,r})$ are not present and the polynomials $Q_{j,i}^{k[m-2]}$, $R_{j,i}^{k[m-2]}$, $S_{j,i}^{k[m-2]}$ for $k = 1, 2$, are not present either, which reduces the equation (9) to

$$\bar{f}_j^{1[m]}(x) = \sum_{i=j+2}^{n} x_{1,i}^2 P_{j,i}^{1[m-2]}(\bar{x}_{1,i}),$$

for $1 \leq j \leq n-2$. This is the normal form of single-input control systems with controllable linearization obtained by Kang [12].

A proof of Theorem 1 is given in [23], where we also show that the general case (of unequal controllability indices, of noninvolutive distribution $\mathcal{G}$, and of general systems $\dot{x} = f(x,u)$ that are not affine with respect to controls) reduces to the problem considered in this paper by a suitable prolongation of the system.

4 m-Invariants

The aim of this section is to introduce m-invariants of the action of homogeneous feedback transformations Γ^m on homogeneous systems

$$\Sigma^{[m]} : \begin{array}{rcl} \dot{\zeta} & = & A\zeta + Bu + f^{[m]}(\zeta) + g^{[m-1]}(\zeta)u \\ & = & A\zeta + B_1u_1 + B_2u_2 \\ & & +f^{[m]}(\zeta) + g_1^{[m-1]}(\zeta)u_1 + g_2^{[m-1]}(\zeta)u_2. \end{array}$$

Recall that we consider systems with equal controllability indices and with involutive control distribution $\mathcal{G}^{[m]}$. For any $0 \leq i \leq r-1$, we define the polynomial vector fields $X_{1,i}^{m-1}$, and $X_{2,i}^{m-1}$ by

$$\begin{array}{rcl} X_{1,i}^{m-1} & = & (-1)^i ad^i_{A\zeta+f^{[m]}}(B_1 + g_1^{[m-1]}), \quad \text{and} \\ X_{2,i}^{m-1} & = & (-1)^i ad^i_{A\zeta+f^{[m]}}(B_2 + g_2^{[m-1]}). \end{array}$$

We introduce the following homogeneous vector fields

$$\begin{array}{rcl} X_{1,i}^{[m-1]} & = & (-1)^i ad^i_{A\zeta} g_1^{[m-1]} \\ & + & \sum_{k=1}^{i} (-1)^{i-k} ad^{i-k}_{A\zeta} ad_{A^{k-1}B_1} f^{[m]}, \\ X_{2,i}^{[m-1]} & = & (-1)^i ad^i_{A\zeta} g_2^{[m-1]} \\ & + & \sum_{k=1}^{i} (-1)^{i-k} ad^{i-k}_{A\zeta} ad_{A^{k-1}B_2} f^{[m]} \end{array}$$

to be, respectively, the homogeneous parts of degree $m-1$ of the polynomial vector fields $X_{1,i}^{m-1}$ and $X_{2,i}^{m-1}$.

For any $0 \leq i, j \leq r$, we denote by $\pi^i_j(\zeta)$ the projection on the subspace

$$\begin{array}{rl} \mathcal{W}_{i,j} = \{ & \zeta = (\zeta_{1,1}, \cdots, \zeta_{1,r}, \zeta_{2,1}, \cdots, \zeta_{2,r})^T \in \mathbb{R}^{2r} : \\ & \zeta_{1,r-i+1} = \cdots = \zeta_{1,r} = 0, \quad \text{and} \\ & \zeta_{2,r-j+1} = \cdots = \zeta_{2,r} = 0 \quad \}, \end{array}$$

that is,

$$\pi^i_j(\zeta) = (\zeta_{1,1}, \cdots, \zeta_{1,r-i}, 0, \cdots, 0, \zeta_{2,1}, \cdots, \zeta_{2,r-j}, 0, \cdots, 0)^T.$$

Let $C = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$ be a $2 \times 2r$ matrix whose rows are given by $C_1 = (1, 0, \cdots, 0)$ and $C_2 = (\overbrace{0, \cdots, 0}^{r}, \overbrace{1, 0, \cdots, 0}^{r})$, and define the subset Δ of $\mathbb{N}^2$ by

$$\Delta = \{ \ (j,i) \in \mathbb{N}^2 : \ 1 \leq j \leq r-2, \ 0 \leq i \leq r-j-2 \ \}.$$

Now, let us define, for any $(j,i) \in \Delta$, the $\mathbb{R}^2$-valued homogeneous polynomials $a_{1,1}^{[m]j,i+2}$, $a_{2,2}^{[m]j,i+2}$, $a_{1,2}^{[m]j,i+2}$, and $a_{2,1}^{[m]j,i+2}$ to be, respectively, the homogeneous parts of degree $m-2$ of the $\mathbb{R}^2$-valued functions

$$\begin{array}{l} CA^{j-1}[X_{1,i}^{m-1}, X_{1,i+1}^{m-1}](\pi^i_{i+1}(\zeta)), \\ CA^{j-1}[X_{2,i}^{m-1}, X_{2,i+1}^{m-1}](\pi^{i+1}_i(\zeta)), \\ CA^{j-1}[X_{2,i+1}^{m-1}, X_{1,i+1}^{m-1}](\pi^{i+1}_{i+1}(\zeta)), \quad \text{and} \\ CA^{j-1}[X_{1,i}^{m-1}, X_{2,i+1}^{m-1}](\pi^i_i(\zeta)) + \\ +L_{A^{i+1}B_2} \int_0^{\zeta_{2,r-i}} CA^{j-1}[X_{2,i}^{m-1}, X_{1,i}^{m-1}](\pi^i_i(\zeta)) d\zeta_{2,r-i}. \end{array}$$

We will take $a_{k,l}^{[m]j,i+2} = 0$, for any $1 \leq k,l \leq 2$ and for any $(j,i) \notin \Delta$.

The $\mathbb{R}^2$-valued homogeneous polynomials $a_{k,l}^{[m]j,i+2}$, for $1 \leq k,l \leq 2$ and $(j,i) \in \Delta$, will be called the *m-invariants* of the homogeneous system $\Sigma^{[m]}$ under the action of the homogeneous feedback transformation Γ^m.

The following result asserts that m-invariants $a_{k,l}^{[m]j,i+2}$, for $1 \leq k,l \leq 2$ and $(j,i) \in \Delta$, are complete invariants of homogeneous feedback and, moreover, illustrates their meaning for the homogeneous normal form $\Sigma_{NF}^{[m]}$. Consider two homogeneous systems $\Sigma^{[m]}$ and $\tilde{\Sigma}^{[m]}$ and let

$$\{ \, a_{k,l}^{[m]j,i+2} \ : \ (j,i) \in \Delta, \ \ 1 \leq k,l \leq 2 \ \} \ \text{ and}$$

$$\{ \, \tilde{a}_{k,l}^{[m]j,i+2} \ : \ (j,i) \in \Delta, \ \ 1 \leq k,l \leq 2 \ \}$$

denote, respectively, their m-invariants. We get the following result generalizing that proved by Kang for single-input systems.

Theorem 2 *The* m-invariants *have the following properties:*
(i) *Two homogeneous systems* $\Sigma^{[m]}$ *and* $\tilde{\Sigma}^{[m]}$ *are equivalent via a homogeneous feedback transformation* Γ^m *if and only if*

$$a_{k,l}^{[m]j,i+2} = \tilde{a}_{k,l}^{[m]j,i+2},$$

for any $(j,i) \in \Delta$ *and any* $1 \leq k,l \leq 2$.

(ii) *The* m-invariants $\bar{a}_{k,l}^{[m]j,i+2}$, *for* $1 \leq k,l \leq 2$ *and for* $(j,i) \in \Delta$, *of the homogeneous normal form*

$$\Sigma_{NF}^{[m]} \ : \ \dot{x} = Ax + Bv + \bar{f}^{[m]}(x),$$

with the vector field $\bar{f}^{[m]}(x)$ *defined by* (8)-(9), *are given* by

$$\bar{a}_{1,1}^{[m]j,i+2}(x) = CA^{j-1}\frac{\partial^2 \bar{f}^{[m]}}{\partial x_{1,r-i}^2}(\pi_{i+1}^i(x)),$$

$$\bar{a}_{2,2}^{[m]j,i+2}(x) = CA^{j-1}\frac{\partial^2 \bar{f}^{[m]}}{\partial x_{2,r-i}^2}(\pi_i^{i+1}(x))$$

$$\bar{a}_{1,2}^{[m]j,i+2}(x) = CA^{j-1}\frac{\partial^2 \bar{f}^{[m]}}{\partial x_{1,r-i}\partial x_{2,r-i-1}}(\pi_{i+1}^{i+1}(x)),$$

$$\bar{a}_{2,1}^{[m]j,i+2}(x) = CA^{j-1}\frac{\partial^2 \bar{f}^{[m]}}{\partial x_{1,r-i}\partial x_{2,r-i}}(\pi_i^i(x)).$$

(iii) *Two homogeneous normal forms $\Sigma_{NF}^{[m]}$ and $\tilde{\Sigma}_{NF}^{[m]}$ are equivalent via a homogeneous feedback transformation Γ^m if and only if*

$$P_{j,i}^{k[m-2]} = \tilde{P}_{j,i}^{k[m-2]}, \quad Q_{j,i}^{k[m-2]} = \tilde{Q}_{j,i}^{k[m-2]},$$
$$R_{j,i}^{k[m-2]} = \tilde{R}_{j,i}^{k[m-2]}, \quad S_{j,i}^{k[m-2]} = \tilde{S}_{j,i}^{k[m-2]},$$

for any $(j,i) \in \Delta$, and any $1 \le k \le 2$.

(iv) *The homogeneous system $\Sigma^{[m]}$ is feedback linearizable, modulo terms in $R^{\ge m+1}(x,v)$, if and only if*

$$a_{k,l}^{[m]j,i+2} = 0,$$

for any $(j,i) \in \Delta$ and any $1 \le k,l \le 2$.

5 Quadratic homogeneous systems

In this section we will illustrate our results by discussing quadratic homogeneous systems. Theorem 1(i) implies that the quadratic system $\Sigma^{[2]}$ can be brought to the following quadratic normal form $\Sigma_{NF}^{[2]}$, consisting of two r-dimensional subsystems, each evolving on $(x_{k,1}, x_{k,2}, \ldots, x_{k,r})$, for $k = 1,2$,

$$\begin{aligned}
\dot{x}_{k,1} &= x_{k,2} + \sum_{i=3}^{r}\big(x_{1,i}^2 p_{1,i}^k + x_{2,i}^2 q_{1,i}^k \\
&\quad + x_{1,i}x_{2,i}r_{1,i}^k + x_{1,i}x_{2,i-1}s_{1,i}^k\big) \\
&\cdots \\
\dot{x}_{k,r-2} &= x_{k,r-1} + x_{1,r}^2 p_{r-2,r}^k + x_{2,r}^2 q_{r-2,r}^k \\
&\quad + x_{1,r}x_{2,r}r_{r-2,r}^k + x_{1,r}x_{2,r-1}s_{r-2,r}^k \\
\dot{x}_{k,r-1} &= x_{k,r} \\
\dot{x}_{k,r} &= v_k,
\end{aligned} \tag{10}$$

where $p_{j,i}^k, q_{j,i}^k, r_{j,i}^k, s_{j,i}^k \in \mathbb{R}$, for $(j,i) \in \Delta$ and $k = 1,2$, are invariants of $\Sigma^{[2]}$ under homogeneous feedback transformations Γ^2. Observe the triangular structure

$$\begin{matrix}
p_{1,3}^k & p_{1,4}^k & \cdots & p_{1,r}^k \\
 & p_{2,4}^k & \cdots & p_{2,r}^k \\
 & & & \cdots \\
 & & & p_{r-2,r}^k
\end{matrix} \tag{11}$$

of the invariants $p_{j,i}^k$ (of course, $q_{j,i}^k$, $r_{j,i}^k$, $s_{j,i}^k$ exhibit analogous triangular structure). Notice that if we take $k=1$, and hence $r=n$, then $(x_{2,1}, x_{2,2}, \ldots, x_{2,r})$ are not present and $q_{j,i}^k$, $r_{j,i}^k$, $s_{j,i}^k$ are not present either. In this case, we thus rediscover in (10), the normal form constructed by Kang and Krener [14] in the single-input case and in (11) we rediscover their invariants.

Now we will discuss a geometric meaning of the invariants $p_{j,i}^k$, $q_{j,i}^k$, $r_{j,i}^k$, $s_{j,i}^k$, in particular, why the number of invariants grows so rapidly when passing from one to two inputs. To this end, consider the two-inputs system $\Sigma^{[2]}$ in the simplest case of $n=6$ and $r=3$:

$$\begin{aligned}
\dot{x}_{k,1} &= x_{k,2} + x_{1,3}^2 p_{1,3}^k + x_{2,3}^2 q_{1,3}^k \\
&\quad + x_{1,3}x_{2,3}r_{1,3}^k + x_{1,3}x_{2,2}s_{1,3}^k \\
\dot{x}_{k,2} &= x_{k,3} \\
\dot{x}_{k,3} &= v_k,
\end{aligned} \tag{12}$$

for $k=1,2$. Theorem 2(iv) implies that the system (12) is feedback linearizable if and only if $p_{j,i}^k = q_{j,i}^k = r_{j,i}^k = s_{j,i}^k = 0$, for all $(j,i) \in \Delta$ and $k = 1,2$. Let us consider (12) as a perturbation of the system in Brunovský canonical form on $\mathbb{R}^6$, with the parameters $p_{j,i}^k$, $q_{j,i}^k$, $r_{j,i}^k$, $s_{j,i}^k$ measuring the distance from the Brunovský form.

It is well known (see e.g [6] or [10]) that (12) is feedback linearizable if and only if the distribution

$$\mathcal{G}_2 = \operatorname{span}\{g_1, g_2, ad_f g_1, ad_f g_2\}$$

is involutive (clearly, $\mathcal{G}_1 = \mathcal{G}$ is involutive). Now the crucial observation is that the involutivity of $\mathcal{G}_2$ involves actually four conditions: $[g_1, ad_f g_1] \in \mathcal{G}_2$, $[g_2, ad_f g_2] \in \mathcal{G}_2$, $[g_1, ad_f g_2] \in \mathcal{G}_2$, and $[ad_f g_1, ad_f g_2] \in \mathcal{G}_2$. We have

$$\begin{aligned}
[g_1, ad_f g_1] &= -2p_{1,3}^1\frac{\partial}{\partial x_{1,1}} - 2p_{1,3}^2\frac{\partial}{\partial x_{2,1}} \\
[g_2, ad_f g_2] &= -2q_{1,3}^1\frac{\partial}{\partial x_{1,1}} - 2q_{1,3}^2\frac{\partial}{\partial x_{2,1}} \\
[g_1, ad_f g_2] &= -r_{1,3}^1\frac{\partial}{\partial x_{1,1}} - r_{1,3}^2\frac{\partial}{\partial x_{2,1}} \\
[ad_f g_1, ad_f g_2] &= -s_{1,3}^1\frac{\partial}{\partial x_{1,1}} - s_{1,3}^2\frac{\partial}{\partial x_{2,1}}
\end{aligned}$$

(last equality holds mod $R^{\ge 1}(x)$). Therefore the invariants $p_{j,i}^k$, $q_{j,i}^k$, $r_{j,i}^k$, $s_{j,i}^k$ correspond to the four independent ways of possible violating the involutivity of $\mathcal{G}_2$.

Notice a slightly different role played by the invariants $s_{j,i}^k$ with respect to the remaining ones. Firstly, instead of $[g_1, ad_f g_2] \in \mathcal{G}_2$ we can ask for $[g_2, ad_f g_1] \in \mathcal{G}_2$ and the two conditions are equivalent by the Jacobi identity. This implies that in the normal form $\Sigma_{NF}^{[2]}$, instead of terms $x_{1,3}x_{2,2}s_{1,3}^k$, we could have terms of the form $x_{1,2}x_{2,3}s_{1,3}^k$ (the same remains true for any dimension and any degree of homogeneity).

Secondly, the condition $[g_1, ad_f g_1] \in \mathcal{G}_2$ implies that $\Sigma^{[2]}$ is linear with respect to $x_{1,3}$, the condition

$[g_2, ad_f g_2] \in \mathcal{G}_2$ implies that $\Sigma^{[2]}$ is linear with respect to $x_{2,3}$, and together with $[g_1, ad_f g_2] \in \mathcal{G}_2$, the three conditions imply that $\Sigma^{[2]}$ is linear with respect to the pair $(x_{1,3}, x_{2,3})$. Therefore under these three conditions, we can consider $\Sigma^{[2]}$ as the prolongation (via pre-integration of each input $x_{k,3}$, $k = 1, 2$) of the system

$$\begin{aligned} \dot{x}_{k,1} &= x_{k,2} + x_{1,3}x_{2,2} \\ \dot{x}_{k,2} &= x_{k,3} \end{aligned} \tag{13}$$

on $\mathbb{R}^4$ and controlled by $(x_{1,3}, x_{2,3})$. Now we can observe that the fourth involutivity condition $[ad_f g_1, ad_f g_2] \in \mathcal{G}_2$ for (12) is just the condition of the involutivity of the control distribution $\mathcal{G}$ of the reduced system (13), which gives its geometric interpretation. Analogous interpretation holds in any dimension and for any degree of homogeneity m.

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