2003

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Published in *Finite Fields and Their Applications*, 9, 286-299.

IRREDUCIBLE POLYNOMIALS OVER GF(2) WITH THREE PRESCRIBED COEFFICIENTS

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Abstract. For an odd positive integer n, we determine formulas for the number of irreducible polynomials of degree n over $GF(2)$ in which the coefficients of x^{n-1}, x^{n-2} and x^{n-3} are specified in advance. Formulas for the number of elements in $GF(2^n)$ with the first three traces specified are also given.

Let q be a prime power and let $GF(q)$ be a finite field with q elements. A classical result (see [6, 3.25]) gives the number, $P_q(n)$, of monic, irreducible polynomials of degree n over $GF(q)$:

$$P_q(n) = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d},$$

where μ is the Möbius function. This has been refined several times by counting the number $P_q(n, \epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ of monic irreducible polynomials over $GF(q)$ with the first k coefficients being the prescribed values $\epsilon_1, \ldots, \epsilon_k$. We are writing polynomials here as

$$p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n.$$

Carlitz [1] gave a formula for $P_q(n, \epsilon_1)$. Kuz'min [5] extended this to a formula for $P_q(n, \epsilon_1, \epsilon_2)$. This was re-discovered, for the case $q = 2$, in [2] which also introduced the connection with higher traces. The same connection was used in [8] to get a formula for $P_q(n, \epsilon_1, \epsilon_2, \epsilon_3)$ when $q = 2$ and n is even. We complete this case, getting a formula for $P_q(n, \epsilon_1, \epsilon_2, \epsilon_3)$ when $q = 2$ and n is odd. The proof is quite different and depends on computations with quadratic forms.

The higher traces are defined as follows. Let F be any field and let K/F be a separable extension of degree n. Let $\sigma_0, \ldots, \sigma_{n-1}$ be the monomorphisms from K into the algebraic closure of F. Then define for $\alpha \in K$:

$$\begin{aligned}
\mathrm{tr}_1(\alpha) &= \sum_{i=0}^{n-1} \sigma_i(\alpha) \\
\mathrm{tr}_2(\alpha) &= \sum_{0 \le i < j \le n-1} \sigma_i(\alpha)\sigma_j(\alpha) \\
\mathrm{tr}_3(\alpha) &= \sum_{0 \le i < j < k \le n-1} \sigma_i(\alpha)\sigma_j(\alpha)\sigma_k(\alpha)
\end{aligned}$$

In our case $(q = 2)$, $\sigma_i(x) = x^{2^i}$.

We fix odd $n = 2m+1$ and set $K = GF(2^n)$. We will only work over $GF(2)$ so we will drop the subscript on the P from $P_2(n, \epsilon_1, \epsilon_2, \epsilon_3)$. Let $F(n, \epsilon_1, \epsilon_2, \epsilon_3)$ denote the number of elements x in K with $\mathrm{tr}_i(x) = \epsilon_i$ for $1 \le i \le 3$ (note that each ϵ_i is 0 or 1). A Möbius inversion-type argument in [8] gives formulas for $P(n, \epsilon_1, \epsilon_2, \epsilon_3)$ in terms of $F(n, \epsilon_1, \epsilon_2, \epsilon_3)$ so we will concentrate on evaluating the F's.

1. Identities.

Set $Q = \mathrm{tr}_2 + \mathrm{tr}_3$. We also define maps $B_i : K \times K \to F$ as follows:

$$\begin{aligned}
B_2(\alpha, \beta) &= \mathrm{tr}_2(\alpha+\beta) + \mathrm{tr}_2(\alpha) + \mathrm{tr}_2(\beta)\\
B_3(\alpha, \beta) &= \mathrm{tr}_3(\alpha+\beta) + \mathrm{tr}_3(\alpha) + \mathrm{tr}_3(\beta)\\
B_Q(\alpha, \beta) &= Q(\alpha+\beta) + Q(\alpha) + Q(\beta) = B_2(\alpha, \beta) + B_3(\alpha, \beta).
\end{aligned}$$

Special cases of the following are known, see [4, 0.2] and [8, Proposition 10].

Lemma 1.1. *(1) $B_2(\alpha, \beta) = tr_1(\alpha)tr_1(\beta) + tr_1(\alpha\beta)$.*
(2) $B_3(\alpha, \beta) = tr_2(\alpha)tr_1(\beta) + tr_1(\alpha)tr_2(\beta) + tr_1(\alpha\beta^2 + \alpha^2\beta) + tr_1(\alpha\beta)tr_1(\alpha+\beta)$.

Proof. (1) To save on superscripts, we set $x_i = x^{2^i}$. Then

$$\begin{aligned}
B_2(\alpha, \beta) &= \sum_{0 \le i < j \le n-1} [(\alpha+\beta)_i(\alpha+\beta)_j + \alpha_i\alpha_j + \beta_i\beta_j]\\
&= \sum_{i \ne j} \alpha_i \beta_j\\
&= \sum_{i=0}^{n-1} \alpha_i \sum_{j \ne i} \beta_j\\
&= \sum_{i=0}^{n-1} \alpha_i(\mathrm{tr}_1(\beta) + \beta_i)\\
&= \mathrm{tr}_1(\alpha)\mathrm{tr}_1(\beta) + \mathrm{tr}_1(\alpha\beta).
\end{aligned}$$

(2)

$$\begin{aligned}
B_3(\alpha,\beta) &= \sum_{0\le i<j<k\le n-1} [\alpha_i\alpha_j\beta_k + \alpha_i\beta_j\alpha_k + \beta_i\alpha_j\alpha_k + \alpha_i\beta_j\beta_k + \beta_i\alpha_j\beta_k + \beta_i\beta_j\alpha_k] \\
&= \sum_{k=0}^{n-1}\Bigg(\sum_{\substack{i<j\\ i,j\neq k}} \alpha_i\alpha_j\Bigg)\beta_k + \sum_{i<j}\Bigg(\sum_{k\neq i,j}\alpha_k\Bigg)\beta_i\beta_j \\
&= \sum_{k=0}^{n-1}\Big[\mathrm{tr}_2(\alpha) + \alpha_k\sum_{i\neq k}\alpha_i\Big]\beta_k + \sum_{i<j}\Big[\mathrm{tr}_1(\alpha)+\alpha_i+\alpha_j\Big]\beta_i\beta_j \\
&= \mathrm{tr}_2(\alpha)\mathrm{tr}_1(\beta) + \mathrm{tr}_1(\alpha)\mathrm{tr}_1(\alpha\beta) + \mathrm{tr}_1(\alpha^2\beta) \\
&+ \mathrm{tr}_1(\alpha)\mathrm{tr}_2(\beta) + \mathrm{tr}_1(\alpha\beta^2) + \mathrm{tr}_1(\alpha\beta)\mathrm{tr}_1(\beta) \\
&= \mathrm{tr}_2(\alpha)\mathrm{tr}_1(\beta) + \mathrm{tr}_1(\alpha)\mathrm{tr}_2(\beta) + \mathrm{tr}_1(\alpha\beta^2+\alpha^2\beta) + \mathrm{tr}_1(\alpha\beta)\mathrm{tr}_1(\alpha+\beta).
\end{aligned}$$

□

Recall that K is a finite field of characteristic 2. In particular, $K = K^2$. Set $K_1 = \ker(\mathrm{tr}_1)$.

Definition. Let $\psi_2 : K_1 \to K$ be $\psi_2(\alpha) = \sqrt{\alpha} + \alpha^2$. Let $\psi_3 : K_1 \to K$ be $\psi_3(\alpha) = \sqrt{\alpha} + \alpha + \alpha^2$.

Lemma 1.2. *For $\alpha, \beta \in K_1$ we have:*

1. $B_2(\alpha,\beta) = tr_1(\alpha\beta)$.
2. $B_3(\alpha,\beta) = tr_1(\psi_2(\alpha)\beta)$
3. $B_Q(\alpha,\beta) = tr_1(\psi_3(\alpha)\beta)$.

Proof. (1) is clear form (1.1). For (2), (1.1) gives

$$\begin{aligned}
B_3(\alpha,\beta) &= \mathrm{tr}_1(\alpha^2\beta + \alpha\beta^2) \\
&= \mathrm{tr}_1(\alpha^2\beta + (\sqrt{\alpha}\beta)^2) \\
&= \mathrm{tr}_1(\alpha^2\beta + \sqrt{\alpha}\beta) \\
&= \mathrm{tr}_1(\psi_2(\alpha)\beta).
\end{aligned}$$

And lastly, $B_Q(\alpha,\beta) = \mathrm{tr}_1(\alpha\beta) + \mathrm{tr}_1(\psi_2(\alpha)\beta)$. □

We note that it is only for $GF(2)$ that ψ_2 and ψ_3 are linear.

Lemma 1.3.

1. *$\psi_2 : K_1 \to K_1$ is an isomorphism.*
2. *If 3 does not divide n then $\psi_3 : K_1 \to K_1$ is an isomorphism.*
3. *If 3 does divide n then $\ker(\psi_3)$ has order 4.*

Proof. (1) Since $\mathrm{tr}_1(\alpha) = \mathrm{tr}_1(\alpha^2)$ we have that ψ_2 maps into K_1. Say $\alpha \in \ker\psi_2$ and let $\beta^2 = \alpha$. Then $\beta + \beta^4 = 0$. But $x + x^4 = x(x+1)(x^2+x+1)$ and x^2+x+1 has no roots in K as $[K : F]$ is odd. Hence only 0 and 1 are sent to 0 by ψ_2 and $1 \notin K_1$. Thus ψ_2 is injective and so an isomorphism.

(2) First $\mathrm{tr}_1(\sqrt{\alpha} + \alpha + \alpha^2) = \mathrm{tr}_1(\alpha)$, so ψ_3 maps K_1 into K_1. Say $\alpha \in \ker\psi_3$ and let $\beta^2 = \alpha$. Then $\beta + \beta^2 + \beta^4 = 0$. But $x + x^2 + x^4 = x(1 + x + x^3)$ and the cubic has no roots in K if 3 does not divide n. So ψ_3 is an isomorphism.

(3) As above, $\ker(\psi_3)$ consists of the roots of $x + x^2 + x^4$ and so has order 4. □

Lemma 1.4. *For $\alpha \in K_1$, $tr_3(\alpha) = tr_1(\alpha^3)$.*

Proof. Again let α_i denote α^{2^i}. We first note that

$$\mathrm{tr}_3(\alpha) = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \mathrm{tr}_1(\alpha\alpha_i\alpha_j).$$

Namely, each term $\alpha_a\alpha_b\alpha_c$ occurs three times, once each in the sums for $\mathrm{tr}_1(\alpha\alpha_{b-a}\alpha_{c-a})$, $\mathrm{tr}_1(\alpha\alpha_{c-b}\alpha_{a+n-b})$ and $\mathrm{tr}_1(\alpha\alpha_{a+n-c}\alpha_{b+n-c})$. Thus

$$\begin{aligned}
\mathrm{tr}_3(\alpha) &= \mathrm{tr}_1(\alpha \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \alpha_i\alpha_j) \\
&= \mathrm{tr}_1(\alpha(\mathrm{tr}_2(\alpha) - \alpha \sum_{i=1}^{n-1} \alpha_i)) \\
&= \mathrm{tr}_1(\alpha(\mathrm{tr}_2(\alpha) - \alpha(\mathrm{tr}_1(\alpha) - \alpha))) \\
&= \mathrm{tr}_1(\alpha\mathrm{tr}_2(\alpha) + \alpha^3) \quad \text{since } \alpha \in K_1 \\
&= \mathrm{tr}_2(\alpha)\mathrm{tr}_1(\alpha) + \mathrm{tr}_1(\alpha^3) = \mathrm{tr}_1(\alpha^3).
\end{aligned}$$

□

2. Quadratic forms.

Over any field of characteristic 2 a *quadratic form* on an F-vector space V is a map $q : V \to F$ such that (1) $q(\lambda v) = \lambda^2 q(v)$ and (2) $b_q(v, w) \equiv q(v + w) - q(v) - q(w)$ is a symmetric bilinear form. We say q is *non-degenerate* if b_q is, namely, $b_q(v, w) = 0$ for all $w \in V$ implies $v = 0$. Note that b_q is *alternating*, namely that $b_q(v, v) = 0$ for all $v \in V$.

The non-degenerate , alternating, symmetric bilinear forms are necessarily even dimensional and have a symplectic basis $\{e_i, f_i\}$, $1 \le i \le m$, meaning

$$\begin{aligned}
b_q(e_i, e_j) &= 0 \\
b_q(e_i, f_j) &= \delta_{ij} \\
b_q(f_i, f_j) &= 0.
\end{aligned}$$

See [7, Chapter 9, Section 4] for further details.

We continue to assume $F = GF(2)$, since only in this case is condition (1) of a quadratic form satisfied by tr_3.

Lemma 2.1.

(1) *tr_2, tr_3 and Q are quadratic forms $K_1 \to GF(2)$.*
(2) *tr_2 and tr_3 are non-degenerate.*
(3) *Q is non-degenerate if 3 does not divide n. If 3 does divide n then the radical of Q is $C \equiv \ker \psi_3$ and Q is non-degenerate on K_1/C.*

Proof. (1) follows from (1.2). The trace form, $\alpha, \beta \to \mathrm{tr}_1(\alpha\beta)$ is non-degenerate by [6, 2.24]. Hence (2) and (3) follow from (1.3). $\square$

We use the notation $\mathrm{sp}(S)$ for the linear span of a set S.

Lemma 2.2. *Let q be a non-degenerate $2m$-dimensional quadratic form over $GF(2)$. Set $B = b_q$. Suppose U is an m-dimensional subspace with $B(u, u') = 0$ for all $u, u' \in U$. Then any basis of U can be extended to a symplectic basis $\{u_i, v_i\}$, $1 \leq i \leq m$. Moreover, v_1 can be taken to be any vector in $sp(u_2, \dots, u_m)^{\perp} \setminus U$.*

Proof. Let $u_1, \dots, u_m$ be a basis of U. Now $U \subset \mathrm{sp}(u_2, \dots, u_m)^{\perp}$ and $\dim \mathrm{sp}(u_2, \dots, u_m)^{\perp}$ is $m + 1$. So write

$$\mathrm{sp}(u_2, \dots, u_m)^{\perp} = U \oplus v,$$

for some v. Set $v_1 = v$. Then $B(u_i, v_1) = 0$ for all $i \geq 2$. Also $B(u_1, v_1) = 1$, else $v_1 \in U^{\perp} = U$, a contradiction.

Suppose we have constructed $v_1, \dots v_k$ with $B(v_i, v_j) = 0$ and $B(u_i, v_j) = \delta_{ij}$. As before,

$$\mathrm{sp}(u_1, \dots, u_k, u_{k+2}, \dots, u_m)^{\perp} = U \oplus r,$$

for some r. Set $S = \{i : 1 \leq i \leq k \quad B(v_i, r) = 1\}$ and let

$$v_{k+1} = r + \sum_{i \in S} u_i.$$

We check that this works. $B(u_i, v_{k+1}) = 0$ for all $i \neq k + 1$. Then $B(u_{k+1}, v_{k+1}) = 1$, else $v_{k+1} \in U^{\perp} = U$ while $r \notin U$. If $j \notin S$ then

$$B(v_j, v_{k+1}) = B(v_j, r) + \sum_{i \in S} B(v_i, u_j) = 0.$$

If $j \in S$ then

$$\begin{aligned} B(v_j, v_{k+1}) &= B(v_j, r) + \sum_{i \in S} B(v_i, u_j) \\ &= B(v_j, r) + B(v_j, u_j) = 1 + 1 = 0. \end{aligned}$$

$\square$

Let $N(f = a)$ denote the number of solutions to $f = a$. Let $mH = x_1 y_1 + \dots + x_m y_m$. We will use:

$$N(mH = \alpha) = \begin{cases} 2^{2m-1} + 2^{m-1}, & \text{if } \alpha = 0 \\ 2^{2m-1} - 2^{m-1}, & \text{if } \alpha = 1. \end{cases} \tag{2.3}$$

This is [6, 6.32]. It can be proven directly by a simple induction argument.

Lemma 2.4. *Let q be a $2m$-dimensional, non-degenerate quadratic form. Let U be an m-dimensional space with $b_q(u, u') = 0$ for all $u, u' \in U$. Suppose $\{u_1, \dots, u_m\}$ is a basis of U with $q(u_1) = 1$ and $q(u_i) = 0$ for $2 \le i \le m$. Let $v_1 \in sp(u_2, \dots, u_m)^{\perp} \setminus U$. Then:*

$$N(q=0) = \begin{cases} 2^{2m-1} + 2^{m-1}, & \text{if } q(v_1) = 0 \\ 2^{2m-1} - 2^{m-1}, & \text{if } q(v_1) = 1. \end{cases}$$

Proof. This can be deduced from [6, 6.32] but a direct proof is no more difficult. Extend $\{u_1, \dots, u_m, v_1\}$ to a symplectic basis $\{u_i, v_i\}$, which is possible by (2.2). For $z = \sum x_i u_i + \sum y_i v_i$ we have:

$$q(z) = x_1^2 + \sum_{i=1}^{m} x_i y_i + \sum_{i=1}^{m} q(v_i) y_i^2.$$

Note that x^2 and x are equal as functions over $GF(2)$ so that

$$q(z) = x_1 + x_1 y_1 + q(v_1) y_1 + \sum_{i=2}^{m} (x_i + q(v_i)) y_i.$$

If $q(v_1) = 0$ then $q(z) = x_1(1 + y_1) + \sum (x_i + q(v_i)) y_i$. Hence $N(q = 0) = N(mH = 0)$. Apply (2.3). If $q(v_1) = 1$ then

$$q(z) = 1 + (1 + x_1)(1 + y_1) + \sum_{i=2}^{m} (x_i + q(v_i)) y_i.$$

So $N(q = 0) = N(mH = 1)$. Apply (2.3). $\square$

We note that $q(v_1)$ is the Arf invariant of q, see [7, Chapter 9, section 4].

For $i = 2, 3, Q$ write $\mathrm{perp}_i(S)$ for $\{v \in K_1 : B_i(v, s) = 0 \quad \text{for all } s \in S\}$.

We will construct, in the next section, elements $u_1, \dots, u_m, x_1, y_2, z_1 \in K_1$ such that

(1) $B_2(u_i, u_j) = 0 = B_3(u_i, u_j)$ for all $i, j = 1, \dots, m$.
(2) $\mathrm{tr}_2(u_1) = \mathrm{tr}_3(u_2) = 1$.
(3) $\mathrm{tr}_3(u_1) = \mathrm{tr}_2(u_2) = 0$.
(4) $\mathrm{tr}_2(u_i) = 0 = \mathrm{tr}_3(u_i)$ for all $3 \le i \le m$.
(5) $x_1 \in \mathrm{perp}_2(u_2, \dots, u_m) \setminus U$, where U is the span of $u_1, \dots, u_m$.
(6) $y_2 \in \mathrm{perp}_3(u_1, u_3, \dots, u_m) \setminus U$.
(7) $z_1 \in \mathrm{perp}_Q(u_2, \dots, u_m) \setminus U$.

Now Q is degenerate if 3 divides n (2.1). Let $\bar{v}$ denote $v + C$ and let $\bar{Q}$ denote the map induced by Q on $\bar{K}_1 = K_1/C$. When 3 divides n we require two additional properties of our construction:

(8) $|C \cap U| = 2$ with the non-zero element γ of $C \cap U$ satisfying $\gamma + u_1 \in \mathrm{sp}(u_2, \dots u_m)$.
(9) $\bar{z}_2 \in \mathrm{perp}_{\bar{Q}}(\bar{u}_3, \dots, \bar{u}_m) \setminus \bar{U}$.

Proposition 2.5. *Let $n \geq 7$ and assume we have constructed elements in K_1 satisfying (1)-(9). If 3 does not divide n then:*

$$\begin{aligned}
F(n,0,0,0) &= 2^{2m-2} + 3 \cdot 2^{m-2} - (tr_2(x_1) + tr_3(y_2) + Q(z_1))2^{m-1} \\
F(n,0,0,1) &= 2^{2m-2} - 2^{m-2} + (-tr_2(x_1) + tr_3(y_2) + Q(z_1))2^{m-1} \\
F(n,0,1,0) &= 2^{2m-2} - 2^{m-2} + (tr_2(x_1) - tr_3(y_2) + Q(z_1))2^{m-1} \\
F(n,0,1,1) &= 2^{2m-2} - 2^{m-2} + (tr_2(x_1) + tr_3(y_2) - Q(z_1))2^{m-1}.
\end{aligned}$$

If 3 divides n then:

$$\begin{aligned}
F(n,0,0,0) &= 2^{2m-2} + 2^{m} - (tr_2(x_1) + tr_3(y_2) + 2\bar{Q}(\bar{z}_2))2^{m-1} \\
F(n,0,0,1) &= 2^{2m-2} - 2^{m-1} + (-tr_2(x_1) + tr_3(y_2) + 2\bar{Q}(\bar{z}_2))2^{m-1} \\
F(n,0,1,0) &= 2^{2m-2} - 2^{m-1} + (tr_2(x_1) - tr_3(y_2) + 2\bar{Q}(\bar{z}_2))2^{m-1} \\
F(n,0,1,1) &= 2^{2m-2} + (tr_2(x_1) + tr_3(y_2) - 2\bar{Q}(\bar{z}_2))2^{m-1}.
\end{aligned}$$

Proof. (1) We first note that

$$\begin{aligned}
\{u_1, \ldots, u_m, x_1\} &\quad \text{meets the hypotheses of (2.4) for } q = \mathrm{tr}_2 \\
\{u_2, u_1, u_3, \ldots, u_m, y_2\} &\quad \text{meets the hypotheses of (2.4) for } q = \mathrm{tr}_3 \\
\{u_1, u_1 + u_2, u_3 \ldots, u_m, z_1\} &\quad \text{meets the hypotheses of (2.4) for } q = Q.
\end{aligned}$$

Applying (2.4) yields

$$\begin{aligned}
F(n,0,0,0) + F(n,0,0,1) &= N(\mathrm{tr}_2 = 0) = 2^{2m-1} + 2^{m-1} - 2\mathrm{tr}_2(x_1)2^{m-1} \\
F(n,0,0,0) + F(n,0,1,0) &= N(\mathrm{tr}_3 = 0) = 2^{2m-1} + 2^{m-1} - 2\mathrm{tr}_3(y_2)2^{m-1} \\
F(n,0,0,0) + F(n,0,1,1) &= N(Q = 0) = 2^{2m-1} + 2^{m-1} - 2Q(z_1)2^{m-1} \\
F(n,0,0,0) + F(n,0,0,1) + F(n,0,1,0) + F(n,0,1,1) &= 2^{2m}.
\end{aligned}$$

The sum of the first three minus the fourth gives a formula for $2F(n,0,0,0)$. The others are easily found.

(2) Here Q is degenerate. Note that $\{\bar{u}_1, \bar{u}_3, \ldots, \bar{u}_m, \bar{z}_2\}$ meets the hypothesis of (2.4) for $q = \bar{Q}$. The two variables associated to C can take any value without affecting the value of Q. Hence

$$\begin{aligned}
N(Q = 0) &= 4N(\bar{Q} = 0) \\
&= 4(2^{2(m-1)-1} + 2^{(m-1)-1} - 2\bar{Q}(\bar{z}_2)2^{(m-1)-1}) \\
&= 2^{2m-1} + 2^{m} - 2\bar{Q}(\bar{z}_2)2^{m}.
\end{aligned}$$

Replace the right-hand side of the third equation above with this expression and solve. □

To complete the count we have:

Lemma 2.6.

$$F(n,0,\epsilon_2,\epsilon_3) = \begin{cases} F(n,1,\epsilon_2,\epsilon_2+\epsilon_3), & \text{if } m \text{ is even} \\ F(n,1,1+\epsilon_2,1+\epsilon_2+\epsilon_3), & \text{if } m \text{ is odd.} \end{cases}$$

Proof. From (1.1) we have for $\alpha \in K_1$

$$\begin{aligned} B_2(1,\alpha) &= \mathrm{tr}_1(1\cdot\alpha) + \mathrm{tr}_1(1)\mathrm{tr}_1(\alpha) = 0. \\ B_3(1,\alpha) &= \mathrm{tr}_2(1)\mathrm{tr}_1(\alpha) + \mathrm{tr}_2(\alpha)\mathrm{tr}_1(1) + \mathrm{tr}_1(\alpha^2+\alpha) \\ &= \mathrm{tr}_2(\alpha). \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{tr}_2(1+\alpha) &= \mathrm{tr}_2(1) + \mathrm{tr}_2(\alpha) \\ \mathrm{tr}_3(1+\alpha) &= \mathrm{tr}_3(1) + \mathrm{tr}_2(\alpha) + \mathrm{tr}_3(\alpha). \end{aligned}$$

Since

$$\mathrm{tr}_2(1) \equiv \binom{n}{2} \pmod 2 \quad \text{and} \quad \mathrm{tr}_3(1) \equiv \binom{n}{3} \pmod 2,$$

we have $\mathrm{tr}_2(1) = 1$ iff $\mathrm{tr}_3(1) = 1$ iff m is odd. The result follows. $\square$

3. The construction.

We will now give an explicit construction of $u_1, \dots, u_m, x_1, y_2, z_1$ and $\bar{z}_2$. Let $B = \{\alpha, \alpha^2, \dots, \alpha^{2^{n-1}}\}$ be a self-dual normal basis for K, see [3, 5.2.1] for the existence of such a basis. Here self-dual means that

$$\mathrm{tr}_1(\alpha^{2^i}\alpha^{2^j}) = \delta_{ij}.$$

We will use:

Proposition 3.1. *Let* $\gamma = c_0\alpha + c_1\alpha^2 + \cdots + c_{n-1}\alpha^{2^{n-1}} \in K_1$.

(1) $tr_1(\gamma) \equiv c_0 + c_1 + \cdots + c_{n-1} \pmod 2$ *is zero.*
(2) $tr_2(\gamma) \equiv \frac{1}{2}(c_0 + c_1 + \cdots + c_{n-1}) \pmod 2$.
(3) $tr_3(\gamma) \equiv c_{n-1}c_0 + c_0c_1 + c_1c_2 + \cdots + c_{n-2}c_{n-1} \pmod 2$.

Proof. (1) is [2, Lemma 9]. (2) is implicit in [2]. Namely, [2, Theorem 5] gives

$$\mathrm{tr}_2(\gamma) \equiv \sum_{0\le i<j<n} c_ic_j \pmod 2.$$

Now follow the proof of [2, Lemma 7]. Let k be the number of c_i equal to 1. The sum $\sum c_i c_j$ counts the number of pairs of 1's in the string $c_0 c_1 \dots c_{n-1}$. Thus

$$\sum_{0\le i<j<n} c_i c_j = \binom{k}{2}.$$

Since k is even by (1), we have $\mathrm{tr}_2(\gamma) = 0$ iff $k \equiv 0 \pmod 4$, which yields (2).

For (3) we have by (1.4)

$$\begin{aligned}\mathrm{tr}_3(\gamma) &= \mathrm{tr}_1(\gamma^3) = \mathrm{tr}_1(\gamma\gamma^2)\\ &= \mathrm{tr}_1((c_0\alpha + c_1\alpha^2 + \cdots + c_{n-1}\alpha^{2^{n-1}})(c_{n-1}\alpha + c_0\alpha^2 + \cdots + c_{n-2}\alpha^{2^{n-1}})).\end{aligned}$$

Since $\mathrm{tr}_1(\alpha^{2^i}\alpha^{2^j}) = \delta_{ij}$ we have the result. $\square$

Proposition 3.2. *Let* $\beta = b_0\alpha + b_1\alpha^2 + \cdots + b_{n-1}\alpha^{2^{n-1}}$ *and* $\gamma = c_0\alpha + c_1\alpha^2 + \cdots + c_{n-1}\alpha^{2^{n-1}}$ *be in* K_1.

(1) $B_2(\beta,\gamma) = b_0c_0 + b_1c_1 + \cdots + b_{n-1}c_{n-1} \pmod 2$.
(2) $B_3(\beta,\gamma) = b_0(c_{n-1}+c_1) + b_1(c_0+c_2) + \cdots + b_{n-1}(c_{n-2}+c_0) \pmod 2$.
(3) $B_Q(\beta,\gamma) = b_0(c_{n-1}+c_0+c_1) + b_1(c_0+c_1+c_2) + \cdots + b_{n-1}(c_{n-2}+c_{n-1}+c_0) \pmod 2$.

Proof. From (1.1), $B_2(\beta,\gamma) = \mathrm{tr}_1(\beta\gamma)$, $B_3(\beta,\gamma) = \mathrm{tr}_1(\beta\gamma^2+\beta^2\gamma)$ and $B_Q(\beta,\gamma) = \mathrm{tr}_1(\beta\gamma + \beta\gamma^2 + \beta^2\gamma)$. Now compute using the fact that $\mathrm{tr}_1(\alpha^{2^i}\alpha^{2^j}) = \delta_{ij}$. $\square$

For $\gamma = c_0\alpha + c_1\alpha^2 + \cdots + c_{n-1}\alpha^{2^{n-1}}$ we abuse notation and write $\gamma = (c_0c_1\dots c_{n-1})$. We use $*$ for concatenation and $n(s)$ for the concatenation of n copies of (s). We assume $n \ge 7$.

Let

$$\begin{aligned}
u_1 &= (00001) * (n-6)(0) * (1)\\
u_2 &= (1111) * (n-4)(0)\\
u_j &= (1001) * (j-3)(0) * (1) * (n-2j)(0) * (1) * (j-3)(0), \quad j = 3,\dots,m\\
x_1 &= (1100) * k(1) * (n-k-4)(0), \quad k = 2\left\lfloor \frac{n-3}{4} \right\rfloor\\
y_2 &= \begin{cases} (11101) * (2t-1)(1001), & \text{if } n = 8t+1\\ (110) * 2t(1100), & \text{if } n = 8t+3\\ (11101) * 2t(1001), & \text{if } n = 8t+5\\ (101) * (2t+1)(1100), & \text{if } n = 8t+7.\end{cases}
\end{aligned}$$

If 3 does not divide n then set

$$z_1 = \begin{cases} (1001) * (2t-1)(101) * 2t(100), & \text{if } n = 12t+1\\ (00) * (2t+1)(101) * 2t(001), & \text{if } n = 12t+5\\ (0000) * (2t+1)(110) * 2t(010), & \text{if } n = 12t+7\\ (11010) * (2t+1)(110) * (2t+1)(100), & \text{if } n = 12t+11.\end{cases}$$

If 3 does divide n then set

$$z_2 = \begin{cases} (000) * 2t(011) * 2t(010), & \text{if } n = 12t+3 \\ (000010) * 2t(110) * (2t+1)(100), & \text{if } n = 12t+9. \end{cases}$$

Proposition 3.3. *Let $n \geq 7$.*

(1) *$u_1, \ldots, u_m, x_1, y_2$ and z_1 satisfy conditions (1)-(7) of the last section.*

(2)
$$tr_2(x_1) = tr_3(y_2) = \begin{cases} 0, & \textit{if } m \equiv 0,3 \pmod 4 \\ 1, & \textit{if } m \equiv 1,2 \pmod 4. \end{cases}$$

(3) *If 3 does not divide n then $Q(z_1) = tr_2(x_1)$.*

(4) *If 3 does divide n then conditions (8) and (9) of the previous section hold. And $\bar{Q}(\bar{z}_2) = tr_2(x_1) + 1$.*

Proof. (1), (2) and (3) consist of several easy computations using (3.1) and (3.2). We do the computations involving x_1, namely condition (5) of the previous section and statement (2). Notice that $u_1 = \alpha^{16} + \alpha^{2^{n-1}}$, $u_2 = \alpha + \alpha^2 + \alpha^4 + \alpha^8$, $u_j = \alpha + \alpha^8 + \alpha^{2^{j+1}} + \alpha^{2^{n-j+2}}$, for $j = 3, \ldots, m$, and

$$x_1 = \alpha + \alpha^2 + \sum_{i=4}^{m+1} \alpha^{2^i} + \epsilon \alpha^{2^{m+2}},$$

where

$$\epsilon = \begin{cases} 0, & \text{if } m \text{ is even} \\ 1, & \text{if } m \text{ is odd.} \end{cases}$$

Now, x_1 and u_1 match only at α^{16} so by (3.2), $B_2(u_1, x_1) = 1$. In particular, $x_1 \notin U$. Next, x_1 and u_2 match only at α and α^2 so that $B_2(u_2, x_1) = 0$. Also, x_1 and u_j, $3 \leq j \leq m$, match only at α and $\alpha^{2^{j+1}}$ so that $B_2(u_j, x_1) = 0$. This proves condition (5). Finally, by (3.1),

$$\begin{aligned} \mathrm{tr}_2(x_1) &\equiv \tfrac{1}{2}(1 + 1 + (m-2) + \epsilon) \equiv \tfrac{1}{2}(m + \epsilon) \pmod 2 \\ &= \begin{cases} 0, & \text{if } m \equiv 0,3 \pmod 4 \\ 1, & \text{if } m \equiv 1,2 \pmod 4. \end{cases} \end{aligned}$$

Suppose 3 divides n. One checks that the non-zero elements of C are

$$\gamma_1 = \tfrac{n}{3}(011) \qquad \gamma_2 = \tfrac{n}{3}(101) \qquad \gamma_3 = \tfrac{n}{3}(110).$$

Now γ_2 and γ_3 are not in U since $B_2(\gamma_2, u_2) = B_2(\gamma_3, u_2) = 1$. But γ_1 is in U, in fact,

$$\gamma_1 = u_2 + \sum_{i \equiv 0,1 \pmod 3} u_i.$$

This also checks condition (8) of §2. For condition (9), take $\bar{z}_2 = z_2 + (C \cap U)$. $\square$

Now simply plug the values from (3.3)(2) and (3.3)(3) into the formulas of (2.5) and (2.6) to get:

Theorem 3.4. *(1) For $n = 2m+1$ odd, $n > 1$ and 3 not dividing n, we have*

$$F(n, \epsilon_1, \epsilon_2, \epsilon_3) = 2^{n-3} +$$

m	000	001	010	011	100	101	110	111
0	$3 \cdot 2^{m-2}$	-2^{m-2}	-2^{m-2}	-2^{m-2}	$3 \cdot 2^{m-2}$	-2^{m-2}	-2^{m-2}	-2^{m-2}
1	$-3 \cdot 2^{m-2}$	2^{m-2}	2^{m-2}	2^{m-2}	2^{m-2}	2^{m-2}	2^{m-2}	$-3 \cdot 2^{m-2}$
2	$-3 \cdot 2^{m-2}$	2^{m-2}	2^{m-2}	2^{m-2}	$-3 \cdot 2^{m-2}$	2^{m-2}	2^{m-2}	2^{m-2}
3	$3 \cdot 2^{m-2}$	-2^{m-2}	-2^{m-2}	-2^{m-2}	-2^{m-2}	-2^{m-2}	-2^{m-2}	$3 \cdot 2^{m-2}$

where the m is listed modulo 4.

(2) For $n = 2m+1$ odd, $n > 1$ and 3 dividing n, we have

$$F(n, \epsilon_1, \epsilon_2, \epsilon_3) = 2^{n-3} +$$

m	000	001	010	011	100	101	110	111
0	0	2^{m-1}	2^{m-1}	-2^m	0	2^{m-1}	-2^m	2^{m-1}
1	0	-2^{m-1}	-2^{m-1}	2^m	-2^{m-1}	2^m	-2^{m-1}	0
2	0	-2^{m-1}	-2^{m-1}	2^m	0	-2^{m-1}	2^m	-2^{m-1}
3	0	2^{m-1}	2^{m-1}	-2^m	2^{m-1}	-2^m	2^{m-1}	0

where again the m is listed modulo 4.

Note that our proof is only valid for $n \geq 7$. The above table however is also valid for $n = 3, 5$, which must be checked directly.

4. Irreducible polynomials.

We get formulas for the number of irreducible polynomials over $GF(2)$ with the first three coefficients prescribed, $P(n, \epsilon_1, \epsilon_2, \epsilon_3)$, from the inversion formulas of [8, Theorem 2]. For n odd these simplify slightly to:

$$P(n, 0, \epsilon_2, \epsilon_3) = \frac{1}{n} \sum_{d|n} \mu(d) F(n/d, 0, \epsilon_2, \epsilon_3)$$

$$P(n, 1, \epsilon_2, \epsilon_3) = \frac{1}{n} \sum_{\substack{d|n \\ d \equiv 1}} \mu(d) F(n/d, 1, \epsilon_2, \epsilon_3) + \frac{1}{n} \sum_{\substack{d|n \\ d \equiv 3}} \mu(d) F(n/d, 1, 1+\epsilon_2, 1+\epsilon_3).$$

The congruences here are modulo 4. The tables in (3.4) for F do not include the case $n = 1$ but these may arise in these inversion formulas. The values are $F(1, 0, 0, 0) = F(1, 1, 0, 0) = 1$ and the six others are 0.

As an example, suppose $n = 9$. The formulas become:

$$P(9,0,\epsilon_2,\epsilon_3) = \tfrac{1}{9}\left(F(9,0,\epsilon_2,\epsilon_3) - F(3,0,\epsilon_2,\epsilon_3)\right)$$
$$P(9,1,\epsilon_2,\epsilon_3) = \tfrac{1}{9}(F(9,1,\epsilon_2,\epsilon_3) - F(3,1,1+\epsilon_2,1+\epsilon_3)).$$

From the tables in (3.4) we get:

$$\begin{array}{ll} P(9,0,0,0) = 7 & P(9,1,0,0) = 7 \\ P(9,0,0,1) = 8 & P(9,1,0,1) = 8 \\ P(9,0,1,0) = 8 & P(9,1,1,0) = 5 \\ P(9,0,1,1) = 5 & P(9,1,1,1) = 8. \end{array}$$

These may be verified from Table C in [6, p. 553].

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