Explicit Symmetries of Strict Feedforward Control Systems

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Explicit Symmetries of Strict Feedforward Control Systems

Issa Amadou Tall and Witold Respondek

***Abstract*— We show that any symmetry of a smooth strict feedforward system is conjugated to a *scaling translation* and any 1-parameter family of symmetries to a family of scaling translations along the first variable. We compute explicitly those symmetries by finding the conjugating diffeomorphism. We deduce, in accordance with our previous work, that a smooth system is feedback equivalent to a strict feedforward form if and only if it gives rise to a sequence of systems, such that each element of the sequence, firstly, possesses an infinitesimal symmetry whose flow is conjugated to a 1-parameter families of scaling translations and, secondly, it is the factor system of the preceding one, that is, is reduced from the preceding one by its symmetry. We illustrate our results by computing the symmetries of the Cart-Pole system.**

I. INTRODUCTION

We consider smooth single-input nonlinear control-affine systems of the form

$$\Sigma : \ \dot{x} = f(x) + g(x)u,$$

where f and g are smooth vector fields on $\mathbb{R}^n$.

We will say that the system Σ is in *affine strict feedforward form*, (shortly Σ_{SFF}), if it is in the form

(SFF)

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_2,\ldots,x_n) + g_1(x_2,\ldots,x_n)u \\ & \cdots & \\ \dot{x}_{n-1} & = & f_{n-1}(x_n) + g_{n-1}(x_n)u \\ \dot{x}_n & = & f_n + g_n u, \end{array}$$

where f_n, $g_n \in \mathbb{R}$, $g_n \neq 0$.

A basic structural property of systems in strict feedforward form is that their solutions can be found by quadratures. Indeed, knowing $u(t)$ we integrate $f_n + g_n u(t)$ to get $x_n(t)$, then we integrate $f_{n-1}(x_n(t)) + g_{n-1}(x_n(t))u(t)$ to get $x_{n-1}(t)$, we keep doing that, and finally we integrate $f_1(x_2(t),\ldots,x_n(t)) + g_1(x_2(t),\ldots,x_n(t))u(t)$ to get $x_1(t)$.

Another property, crucial in applications, of systems in (strict) feedforward form is that we can construct for them a stabilizing feedback. This important result goes back to Teel [19] and has been followed by a growing literature on stabilization and tracking for systems in (strict) feedforward form (see e.g. [4], [6], [13], [20], [2], [7]).

Recently (see [11]), we have proved that feedback equivalence (resp. state-space equivalence) to the strict feedforward form can be characterized by the existence of a sequence of infinitesimal symmetries (resp. strong infinitesimal symmetries) of the system.

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In this paper we give a complete classification of symmetries of strict feedforward systems, and we restate the equivalence conditions obtained in [11] in terms of the symmetries of strict feedforward systems.

Notice that the problem of transforming a system, affine with respect to controls, into (strict) feedforward form via a nonlinear change of coordinates was studied in [5], and that a geometric description of systems in feedforward form has been given in [1]. We have also used another approach to propose a step-by-step constructive method to bring a system into a feedforward form in [15], [17] and strict feedforward form in [16].

The paper is organized as follows. Section II deals with notations and definitions. Section III contains the main results of the paper along with explicit examples. The proofs form the Section IV.

II. NOTATIONS AND DEFINITIONS

In this section we will give definitions concerning feedback equivalence of control systems and symmetries. The word smooth will mean throughout C^∞-smooth and all control systems are assumed to be smooth. For simplicity of notations we will consider here control-affine systems.

Two smooth control systems Σ and $\tilde{\Sigma}$ are called *feedback equivalent*, shortly *F-equivalent*, if there exist a smooth diffeomorphism $\phi : X \to \tilde{X}$ and smooth functions α, β, satisfying $\beta(\cdot) \neq 0$, such that

$$\phi_*(f + g\alpha) = \tilde{f} \quad \text{and} \quad \phi_*(g\beta) = \tilde{g}.$$

Recall that for any smooth vector field h on X and any smooth diffeomorphism $\tilde{x} = \phi(x)$ we denote

$$(\phi_* h)(\tilde{x}) = d\phi(\phi^{-1}(\tilde{x})) \cdot h(\phi^{-1}(\tilde{x})).$$

For the single-input control-affine system

$$\Sigma : \ \dot{x} = f(x) + g(x)u,$$

where $x \in X$, an open subset of $\mathbb{R}^n$, and $u \in U = \mathbb{R}$, and f and g are smooth vector fields on X, the *field of admissible velocities* is the following field of affine lines

$$\mathcal{A}(x) = \{f(x) + ug(x) \ : \ u \in \mathbb{R}\} \subset T_x X.$$

A diffeomorphism $\psi : X \longrightarrow X$ is a symmetry of Σ if it preserves the field of affine lines $\mathcal{A}$ (in other words, the affine distribution $\mathcal{A}$ of rank 1), that is, if $\psi_* \mathcal{A} = \mathcal{A}$.

A *local symmetry* at $p \in X$ is a local diffeomorphism ψ of X_0 onto X_1, where X_0 and X_1 are, respectively, neighborhoods of p and $\psi(p)$, such that

$$(\psi_* \mathcal{A})(q) = \mathcal{A}(q) \quad \text{for any } q \in X_1.$$

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A local symmetry ψ at p is called a *stationary symmetry* if $\psi(p) = p$ and a *nonstationary symmetry* if $\psi(p) \neq p$.

We say that a vector field v on an open subset $X \subset \mathbb{R}^n$ is an *infinitesimal symmetry* of the system Σ if the (local) flow γ_t^v of v is a local symmetry of Σ, for any t for which it exists.

An infinitesimal symmetry v is called stationary at $p \in X$ if $v(p) = 0$ and nonstationary if $v(p) \neq 0$.

III. Main Results

Consider the class of *smooth* single-input control systems in *strict feedforward form* (SFF)

$$\Sigma_{SFF} \;:\; \begin{cases} \dot{x} = f(x) + g(x)u, \\ f_j(x) = f_j(x_{j+1}, \dots, x_n), \quad 1 \le j \le n-1, \\ g_j(x) = g_j(x_{j+1}, \dots, x_n), \quad 1 \le j \le n-1 \\ f_n(x) = f_n \in \mathbb{R} \;\text{ and }\; g_n(x) = g_n \in \mathbb{R}^*. \end{cases}$$

Notice that for any $1 \le i \le n$, the subsystem Σ_{SFF}^i, defined as the projection of Σ_{SFF} onto $\mathbb{R}^{n-i+1}$ via $\pi_i(x_1, \dots, x_n) = (x_i, \dots, x_n)$, is a well defined system whose dynamics are given, for any $i \le j \le n$, by

$$\dot{x}_j = f_j(x_{j+1}, \dots, x_n) + g_j(x_{j+1}, \dots, x_n)u.$$

Define the linearizability index of Σ_{SFF} to be the largest integer p such that the subsystem Σ_{SFF}^{r+1}, where $p + r = n$, is feedback linearizable. Clearly, the linearizability index is feedback invariant and hence the linearizability indices of two feedback equivalent (SFF)-systems coincide. We will assume that the linear approximation around the origin is controllable which implies that $p \ge 2$.

For any nonzero real numbers $\lambda_1, \dots, \lambda_r, \lambda \in \mathbb{R}^*$ and any $c_1, \dots, c_{r+1} \in \mathbb{R}$, put $\Lambda = (\lambda_1, \dots, \lambda_r, \lambda, \dots, \lambda)$ and $C = (c_1, \dots, c_{r+1}, 0, \dots, 0)$ and define a *scaling translation* by

$$\mathbb{T}_{\Lambda,C}(x) = (\lambda_1 x_1 + c_1, \dots, \lambda_n x_n + c_n),$$

with $c_{r+2} = \dots = c_n = 0$ and $\lambda_{r+1} = \dots = \lambda_n = \lambda$.

Theorem III.1 *Consider a smooth system Σ_{SFF} in strict feedforward form with linearizability index $p = n - r$. Any symmetry ψ of Σ_{SFF} is of the form*

$$\psi = \sigma^{-1} \circ \mathbb{T}_{\Lambda,C} \circ \sigma,$$

for a fixed (Λ, C), where $z = \sigma(x)$ is the diffeomorphism of the transformation taking Σ_{SFF} into its strict feedforward normal form Σ_{SFNF} given by Definition III.2 below. Any local 1-parameter family of symmetries ψ_{c_1} of Σ_{SFF} is of the same form with $c_1 \in (-\epsilon_1, \epsilon_1)$.

Theorem III.1 says basically that strict feedforward systems have 1-parameter families of symmetries conjugated to *scaling translations*. Recall that in [9] we showed that any symmetry is conjugated to at most two 1-parameter families of translations along the first variable; those translations being the only symmetries of the canonical form.

The constant parameters $\lambda_1, \dots, \lambda_r, \lambda$ are likely to be either $+1$ or -1 and will be uniquely determined by $c_2, \dots, c_r$ (given by other equilibrium point) because, together, they should satisfy some strong conditions (SC), see below. The only free parameter is c_1. In Example III.8 we provide a case where some of the parameters $\lambda_1, \dots, \lambda_r, \lambda$ are not equal to $+1$ or -1 as well as some constants $c_2, \dots, c_{r+1}$ that are non zero. We then compare the results obtained here with those of [9], and show no ambiguity between them.

The importance of this result is that we can always put a (SFF)-system into a strict feedforward normal form (SFNF) via smooth feedback transformation while the canonical form is only guaranteed in the formal category. Moreover, the feedback transformation taking the system into its strict feedforward normal form (SFNF) can be constructed explicitly, for smooth systems, see Section IV.

The notion of strict feedforward normal form plays a crucial role in proving Theorem III.1 and is as follows.

Definition III.2 A smooth *strict feedforward normal form*, denoted Σ_{SFNF}, is a strict feedforward form

$$\begin{aligned} \dot{x}_1 &= \hat{F}_1(x_2, \dots, x_n) \\ &\cdots \\ \dot{x}_r &= \hat{F}_r(x_{r+1}, \dots, x_n) \\ \dot{x}_{r+1} &= x_{r+2} \\ &\cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= u \end{aligned}$$

for which $p = n - r$ is the linearizability index and

$$(SFNF)\;\; \hat{F}_j(x) = h_j(x_{j+1}) + \sum_{i=j+2}^{n} x_i^2 \hat{P}_{j,i}(x_{j+1}, \dots, x_i)$$

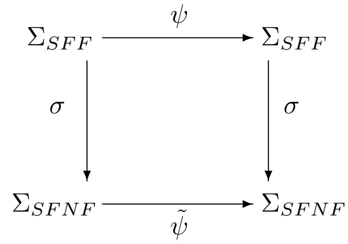
for any $1 \le j \le r$, where h_j and $\hat{P}_{j,i}$ are smooth functions of the indicated variables.

The above strict feedforward normal form Σ_{SFNF} was introduced in [12], where we proved the following:

Theorem III.3 *Any smooth strict feedforward form can be transformed into a strict feedforward normal form via smooth feedback transformation.*

Remark III.4 (i) In the proof of Theorem III.1, we will give an algorithm showing how to construct explicitly the feedback transformation (in particular, the diffeomorphism $z = \sigma(x)$) that takes a (SFF)-system into its (SFNF).

Then using the commutative diagram



where $\tilde{\psi}$ is a symmetry of the strict feedforward normal form Σ_{SFNF}, all we will have to prove is that all $\tilde{\psi}$'s are exhausted by *scaling translations* $\mathbb{T}_{\Lambda,C}$ defined above.

(ii) We will use this item to deduce, as a corollary, necessary and sufficient condition for a system to be brought to a strict feedforward form (see Theorem II.4 of [11]).

Corollary III.5 *Consider a smooth affine system Σ with linearizability index $p = n - r$. The following conditions are equivalent.*

(i) *Σ is, locally at $q \in X$, feedback equivalent to the affine strict feedforward form (SFF);*

(ii) *Each system Σ^1, Σ^2,…,Σ^r possesses an infinitesimal symmetry v_i, whose local flow $\gamma_{c_i}^{v_i}$ is conjugated to a* scaling translation

$$\gamma_{c_i}^{v_i} = \sigma_i^{-1} \circ \mathbb{T}_{\Lambda,C}^i \circ \sigma_i, \quad c_i \in (-\epsilon_i, \epsilon_i),$$

where Σ^1 is the restriction of Σ to a neighborhood X_q and

$$\Sigma^{i+1} = \Sigma^i/_{\sim_{v_i}}, \quad 1 \le i \le r-1.$$

Above, the equivalence relation $\sim_{v_i}$ is induced by the local action of the 1-parameter local group $\gamma_{c_i}^{v_i}$ defined by v_i, that is, such that $q_1 \sim_{v_i} q_2$ if and only if they belong to the same integral curve of v_i, and for any $1 \le i \le r-1$ the scaling translation $\mathbb{T}_{\Lambda,C}^i$ is the composition of $\mathbb{T}_{\Lambda,C}$ with the projection π_i:

$$\mathbb{T}_{\Lambda,C}^i(x) = (\lambda_i x_i + c_i, \ldots, \lambda_r x_r + c_r, \lambda x_{r+1}, \ldots, \lambda x_n).$$

EXAMPLES

Example III.6 Cart-Pole System. In this example we consider a cart-pole system that is represented by a cart with an inverted pendulum on it [8], [18]. The Lagrangian equations of motion for the cart-pole system are

$$\begin{aligned} (m_1+m_2)\ddot{q}_1 + m_2 l \cos(q_2)\ddot{q}_2 &= m_2 l \sin(q_2)\dot{q}_2^2 + F \\ \cos(q_2)\ddot{q}_1 + l\ddot{q}_2 &= g\sin(q_2), \end{aligned}$$

where m_1 and q_1 are the mass and position of the cart, m_2, l, $q_2 \in (-\pi/2, \pi/2)$ are the mass, length of the link, and angle of the pole, respectively.

Taking $\ddot{q}_2 = u$ and applying the feedback law (see [8])

$$\begin{aligned} F = &- ul(m_1 + m_2\sin^2(q_2))/\cos(q_2) \\ &+ (m_1+m_2)g\tan(q_2) - m_2 l \sin(q_2)\dot{q}_2^2 \end{aligned}$$

the dynamics of the cart-pole system are transformed into

$$\begin{array}{ll} \dot{x}_1 = x_2, & \dot{x}_2 = g\tan(x_3) - lu/cos(x_3) \\ \dot{x}_3 = x_4, & \dot{x}_4 = u, \end{array} \qquad \text{(III.1)}$$

where we take $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, and $x_4 = \dot{q}_2$.

This system is in strict feedforward form (SFF) with the linearizability index $p = 2$. We showed in [9] that the diffeomorphism

$$z = \sigma(x) = (\sigma_1(x), \sigma_2(x), \sigma_3(x), \sigma_4(x))$$

defined by

$$\begin{aligned} z_1 &= \sigma_1(x) = \mu x_1 + \mu l \int_0^{x_3} \frac{ds}{\cos s}, \\ z_2 &= \sigma_2(x) = \mu x_2 + \mu l \frac{x_4}{\cos x_3} \\ z_3 &= \sigma_3(x) = \mu g \tan x_3, \\ z_4 &= \sigma_4(x) = \mu g \frac{x_4}{\cos^2 x_3}. \end{aligned}$$

takes the system into its canonical form Σ_{SFCF}:

$$\begin{array}{ll} \dot{z}_1 = z_2, & \dot{z}_2 = z_3 + \dfrac{z_3}{(1+(g/l)z_3^2)^{3/2}} z_4^2 \\ \dot{z}_3 = z_4, & \dot{z}_4 = v. \end{array}$$

It is straightforward to verify that

$$\mathbb{T}_{c_1}^+(z) = (z_1 + c_1, z_2, z_3, z_4) \qquad \text{and}$$
$$\mathbb{T}_{c_1}^-(z) = (-z_1 + c_1, -z_2, -z_3, -z_4)$$

constitute two 1-parameter families of symmetries for the canonical form. By Theorem 4 (see [9]), they exhaust all possible symmetries of the canonical form.

The symmetries of (III.1) are obtained by computing

$$\psi(x) = \sigma^{-1} \circ \mathbb{T}_{c_1}^{\pm} \circ \sigma(x)$$

where the inverse $x = \eta(z) = \sigma^{-1}(z)$ is given by

$$\begin{aligned} x_1 &= \eta_1(z) = \tilde{\mu} g z_1 + \theta(z_3), \\ x_2 &= \eta_2(z) = \tilde{\mu} g z_2 - \tilde{\mu} l \frac{z_4}{\sqrt{1+(\tilde{\mu}z_3)^2}} \\ x_3 &= \eta_3(z) = \arctan(\tilde{\mu} z_3), \\ x_4 &= \eta_4(z) = \frac{\tilde{\mu} z_4}{1+(\tilde{\mu}z_3)^2} \end{aligned}$$

for a suitable function $\theta(z_3)$. It follows easily that

$$\sigma^{-1} \circ \mathbb{T}_{c_1}^+ \circ \sigma = \mathbb{T}_{b_1}^+ \;\text{ and }\; \sigma^{-1} \circ \mathbb{T}_{c_1}^- \circ \sigma(x) = \mathbb{T}_{d_1}^-$$

are both 1-parameter families of translations along the first component x_1 of (x_1, x_2, x_3, x_4).

Example III.7 Consider the system in $\mathbb{R}^4$ described by

$$\begin{array}{ll} \dot{x}_1 = \sin x_2 + x_4^2 \sin x_3, & \dot{x}_2 = \sin x_3 + x_4^3 \\ \dot{x}_3 = x_4, & \dot{x}_4 = u. \end{array}$$

This system is clearly in (SFNF) with linearizability index $p = 2$. It is easy to check that the forward and backward translations

$$\mathbb{T}_{c_1c_2c_3}^+(x) = (x_1 + c_1, x_2 + c_2, x_3 + c_3, x_4) \qquad \text{and}$$
$$\mathbb{T}_{c_1c_2c_3}^-(x) = (-x_1 + c_1, -x_2 + c_2, -x_3 + c_3, -x_4)$$

are symmetries, where c_2 and c_3 are any multiples of 2π.

Example III.8 Consider the system

$$\Sigma_{SFF}: \begin{array}{ll} \dot{x}_1 = x_2 + 2x_2 e^{x_3}\sin x_3 + 2x_2 e^{x_3} x_4^2, & \dot{x}_3 = x_4, \\ \dot{x}_2 = e^{x_3}\sin x_3 + e^{x_3} x_4^2, & \dot{x}_4 = u, \end{array}$$

in strict feedforward form with linearizability index $p = 2$. Due to the terms $2x_2 e^{x_3} \sin x_3$, this system is not in strict feedforward normal form. However, it is straightforward to check that the diffeomorphism $z = \sigma(x)$ defined by

$$z_1 = x_1 - x_2^2,\ z_2 = x_2,\ z_3 = x_3,\ z_4 = x_4$$

takes Σ_{SFF} into the strict feedforward normal form

$$\Sigma_{SFNF}: \begin{array}{ll} \dot{z}_1 = z_2, & \dot{z}_3 = z_4, \\ \dot{z}_2 = e^{z_3}\sin z_3 + e^{z_3} z_4^2, & \dot{z}_4 = u. \end{array}$$

We can notice that the scaling translations

$$\tilde{z} = \mathbb{T}_{\Lambda,C}(z) = (\lambda z_1 + c_1, \lambda z_2, z_3 + c_3, z_4)$$

with $c_3 = 2k\pi,\ k \in \mathbb{Z}$, and $\lambda = e^{c_3}$ form a family of symmetries of Σ_{SFNF} parameterized by c_1.

Indeed, it is easy to see that they map Σ_{SFNF} into Σ_{SFNF} given, around the equilibrium $q = (0, 0, c_3, 0)$, by

$$\Sigma^q_{SFNF}: \begin{array}{ll} \dot{\tilde{z}}_1 = \tilde{z}_2, & \dot{\tilde{z}}_3 = \tilde{z}_4, \\ \dot{\tilde{z}}_2 = e^{\tilde{z}_3}\sin \tilde{z}_3 + e^{\tilde{z}_3} \tilde{z}_4^2, & \dot{\tilde{z}}_4 = u. \end{array}$$

The composition $\tilde{x} = \sigma^{-1} \circ \mathbb{T}_{\Lambda,C} \circ \sigma(x)$ expresses the coordinates $\tilde{x}$ in terms of the coordinates x as follows

$$\begin{array}{ll} \tilde{x}_1 = \lambda x_1 + (\lambda^2 - \lambda)x_2^2 + c_1 & \tilde{x}_3 = x_3 + c_3 \\ \tilde{x}_2 = \lambda x_2 & \tilde{x}_4 = x_4, \end{array}$$

where $c_3 = 2\pi$ and $\lambda = e^{c_3}$.

A straightforward calculation shows that

$$\begin{array}{rcl} \dot{\tilde{x}}_1 & = & \lambda(x_2 + 2x_2 e^{x_3}\sin x_3 + 2x_2 e^{x_3} x_4^2) \\ & + & 2(\lambda^2 - \lambda)x_2(e^{x_3}\sin x_3 + e^{x_3}x_4^2) \\ & = & \lambda x_2 + 2\lambda^2 x_2 (e^{x_3}\sin x_3 + e^{x_3}x_4^2) \\ & = & \tilde{x} + 2\tilde{x}_2 e^{\tilde{x}_3}\sin\tilde{x}_3 + 2\tilde{x}_2 e^{\tilde{x}_3}\tilde{x}_4^2 \end{array}$$

because $\lambda x_2 = \tilde{x}_2$ and

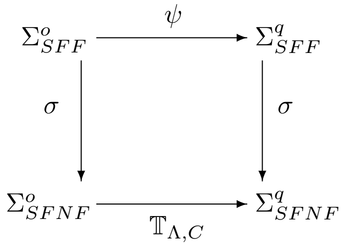
$$\lambda e^{x_3}\sin x_3 = e^{x_3 + c_3}\sin(x_3 + c_3) = e^{\tilde{x}_3}\sin\tilde{x}_3.$$

Similarly, we can show that

$$\dot{\tilde{x}}_2 = \lambda(e^{x_3}\sin x_3 + e^{x_3}x_4^2) = e^{\tilde{x}_3}\sin\tilde{x}_3 + e^{\tilde{x}_3}\tilde{x}_4^2.$$

Since $\dot{\tilde{x}}_3 = \tilde{x}_4$ and $\dot{\tilde{x}}_4 = u$, it follows that the composition $\tilde{x} = \sigma^{-1} \circ \mathbb{T}_{\Lambda,C} \circ \sigma(x)$ maps Σ_{SFF}, defined around the equilibrium $(0,0,0,0)$, into Σ_{SFF} described, around the equilibrium $q = (0, 0, 2\pi, 0)$, by the same dynamics

$$\Sigma_{SFF}: \begin{array}{ll} \dot{\tilde{x}}_1 = \tilde{x}_2 + 2\tilde{x}_2 e^{\tilde{x}_3}\sin\tilde{x}_3 + 2\tilde{x}_2 e^{\tilde{x}_3}\tilde{x}_4^2, & \dot{\tilde{x}}_3 = \tilde{x}_4, \\ \dot{\tilde{x}}_2 = e^{\tilde{x}_3}\sin\tilde{x}_3 + e^{\tilde{x}_3}\tilde{x}_4^2, & \dot{\tilde{x}}_4 = u. \end{array}$$



Hence $\tilde{x} = \psi(x) = \sigma^{-1} \circ \mathbb{T}_{\Lambda,C} \circ \sigma(x)$ is a 1-parameter family of symmetries of Σ_{SFF}.

For convenience of notation, we will denote Σ_{SFF}, defined around $(0,0,0,0)$, by Σ^o_{SFF} and the system Σ_{SFF}, defined around $q = (0, 0, 2\pi, 0)$, by Σ^q_{SFF}. The same notations apply to the systems Σ^o_{SFNF} and Σ^q_{SFNF}.

Now, in view of the results obtained in [12], we will compute the canonical form of Σ^o_{SFF} and the transformations taking Σ^o_{SFF} and Σ^q_{SFF} to this canonical form.

It is easy to verify that $y = \Phi(x)$, given by

$$\begin{array}{ll} y_1 = x_1 - x_2^2 & y_3 = e^{x_3}\sin x_3 \\ y_2 = x_2 & y_4 = e^{x_3}(\sin x_3 + \cos x_3)x_4, \end{array}$$

followed by an appropriate feedback, takes the system Σ^o_{SFF} into its canonical form

$$\Sigma_{SFCF}: \begin{array}{ll} \dot{y}_1 = y_2, & \dot{y}_3 = y_4, \\ \dot{y}_2 = y_3 + \Theta(y_3)y_4^2, & \dot{y}_4 = v, \end{array}$$

$$\text{where} \quad \Theta(y_3) = \left.\frac{1}{e^{x_3}(\sin x_3 + \cos x_3)^2}\right|_{x_3 = \theta^{-1}(y_3)},$$

with $\theta(x_3) = e^{x_3}\sin x_3$.

On the other hand, applying the translation

$$\hat{x} = T(\tilde{x}) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 - c_3, \tilde{x}_4)$$

to the system Σ^q_{SFF}, we can shift back the equilibrium point to $(0,0,0,0)$. In the new coordinates, Σ^q_{SFF} becomes

$$\tilde{\Sigma}^o_{SFF}: \begin{array}{ll} \dot{\hat{x}}_1 = \hat{x}_2 + 2\lambda\hat{x}_2\left(e^{\hat{x}_3}\sin\hat{x}_3 + e^{\hat{x}_3}\hat{x}_4^2\right), & \dot{\hat{x}}_3 = \hat{x}_4, \\ \dot{\hat{x}}_2 = \lambda\left(e^{\hat{x}_3}\sin\hat{x}_3 + e^{\hat{x}_3}\hat{x}_4^2\right), & \dot{\hat{x}}_4 = u, \end{array}$$

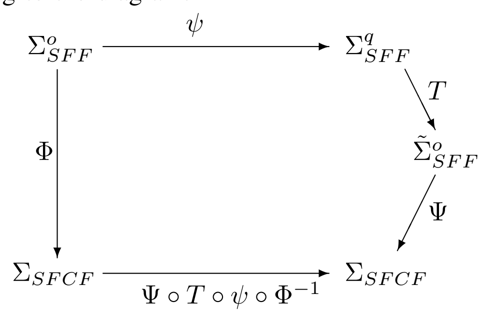
where $\lambda = e^{c_3}$. The diffeomorphism $\tilde{y} = \Psi(\hat{x})$ given by

$$\begin{array}{ll} \tilde{y}_1 = \lambda^{-1}(\hat{x}_1 - \hat{x}_2^2) & \tilde{y}_3 = e^{\hat{x}_3}\sin\hat{x}_3 \\ \tilde{y}_2 = \lambda^{-1}\hat{x}_2 & \tilde{y}_4 = e^{\hat{x}_3}(\sin\hat{x}_3 + \cos\hat{x}_3)\hat{x}_4, \end{array}$$

followed by an appropriate feedback, takes the system $\tilde{\Sigma}^o_{SFF}$ into its canonical form

$$\Sigma_{SFCF}: \begin{array}{ll} \dot{\tilde{y}}_1 = \tilde{y}_2, & \dot{\tilde{y}}_3 = \tilde{y}_4, \\ \dot{\tilde{y}}_2 = \tilde{y}_3 + \Theta(\tilde{y}_3)\tilde{y}_4^2, & \dot{\tilde{y}}_4 = v. \end{array}$$

It follows that the composition $\tilde{y} = \Psi \circ T \circ \psi \circ \Phi^{-1}(y)$ is a 1-parameter family of symmetries of the canonical form according to the diagram.



We explicitly find this family of symmetries by expressing the coordinates $\tilde{y}$ as functions of the coordinates y:

$$\begin{array}{rcl} \tilde{y}_1 & = & \lambda^{-1}(\hat{x}_1 - \hat{x}_2^2) = \lambda^{-1}(\tilde{x}_1 - \tilde{x}_2^2) \\ & = & \lambda^{-1}\left(\lambda x_1 + (\lambda^2 - \lambda)x_2^2 + c_1 - \lambda^2 x_2^2\right) \\ & = & x_1 - x_2^2 + \tilde{c}_1 = y_1 + \tilde{c}_1. \end{array}$$

Similarly, we get

$$\tilde{y}_2 = \lambda^{-1}\hat{x}_2 = \lambda^{-1}\tilde{x}_2 = \lambda^{-1}(\lambda x_2) = x_2 = y_2;$$

$$\tilde{y}_3 = e^{\hat{x}_3}\sin\hat{x}_3 = e^{\tilde{x}_3+2\pi}\sin(\tilde{x}_3+2\pi) = e^{x_3}\sin x_3 = y_3$$

and

$$\begin{aligned}\tilde{y}_4 &= e^{\hat{x}_3}(\sin\hat{x}_3+\cos\hat{x}_3)\hat{x}_4\\ &= e^{\tilde{x}_3+2\pi}(\sin(\tilde{x}_3+2\pi)+\cos(\tilde{x}_3+2\pi))\tilde{x}_4\\ &= e^{x_3}\sin x_3 + e^{x_3}x_4^2 = y_4.\end{aligned}$$

We conclude that the symmetries of the canonical form are exhausted here by a 1-parameter family of translations along the first variable. This is in concordance with the results in [9]. Notice that the composition $\Phi\circ\psi\circ\Phi^{-1}$ does not yield a symmetry for the canonical form. The reason is that, the system Σ^q_{SFF}, being defined around the equilibrium q, is not transformed into the canonical form Σ_{SFCF} by the same diffeomorphism Φ as Σ^o_{SFF} is.

IV. Proofs

In this section we will prove Theorem III.1. Let us consider a system Σ_{SFF} in strict feedforward form. Applying Theorem III.3, we can assume that the system Σ_{SFF} is in the strict feedforward normal form Σ_{SFNF}, given by definition III.2, (explicit transformations are given in the second part of this Section).

Notice that if $\tilde{x} = \psi(x)$ is a symmetry of Σ_{SFF} (in particular, of Σ_{SFNF}), then it preserves the structure of the strict feedforward form. Hence (see [17]), we have $\tilde{x}_j = \psi_j(x) = \psi_j(x_j,\ldots,x_{n-1})$ for $1\le j\le n-1$. This implies that $\pi_r(\psi) = (\psi_r(x),\ldots,\psi_n(x))$ is a symmetry of the projection Σ^r_{SFNF} of Σ_{SFF} whose dynamics are given by

$$\begin{aligned}\dot{x}_r &= h_r(x_{r+1}) + \sum_{i=r+2}^{n} x_i^2 P_{r,i}(x_{r+1},\ldots,x_i)\\ \dot{x}_{r+1} &= x_{r+2}\\ &\cdots\\ \dot{x}_{n-1} &= x_n\\ \dot{x}_n &= u.\end{aligned}$$

We claim that $\psi_j(x) = \psi_j(x_j)$ for any $r\le j\le n-1$. Indeed, we have $\psi_{n-1}(x) = \psi_{n-1}(x_{n-1})$. Let k be the largest integer, $r\le k\le n-2$, such that $\dfrac{\partial\psi_k}{\partial x_s}\neq 0$ for some $s\ge k+1$ (we can take s to be the largest integer that yields this property). Thus

$$\dot{\tilde{x}}_k = \frac{\partial\psi_k}{\partial x_k}\dot{x}_k + \cdots + \frac{\partial\psi_k}{\partial x_s}x_{s+1} = \tilde{x}_{k+1} = \psi_{k+1}(x)$$

gives a contradiction because $\psi_{k+1}(x) = \psi_{k+1}(x_{k+1})$. We conclude that $\psi_j(x) = \psi_j(x_j)$ for $r\le j\le n-1$. Since

$$\dot{\tilde{x}}_j = \psi'_j(x_j)x_{j+1} = \tilde{x}_{j+1} = \psi_{j+1}(x_{j+1}),$$

we deduce that $\psi_j(x_j) = \lambda_j x_j + c_j$ for all $r+1\le j\le n-1$. Similarly we get $\psi_r(x_r) = \lambda_r x_r + c_r$ and hence

$$\pi_r(\psi(x)) = (\lambda_r x_r + c_r, \lambda_{r+1}x_{r+1}+c_{r+1},\ldots,\lambda_n x_n + c_n).$$

In fact, it is easy to see that $\lambda_{r+1} = \cdots = \lambda_n = \lambda$ and $c_{r+2} = \cdots = c_n = 0$ but for homogeneity of notation, we will carry those constants as such.

Notice that λ_r, and the pairs (λ_k, c_k), $r+1\le k\le n$ should satisfy the strong condition:

$$(\mathrm{SC})_r \qquad \hat{F}_r(\cdot) = \lambda_r\hat{F}_r(x_{r+1},\ldots,x_n),$$

where $(\cdot) = (\lambda_{r+1}x_{r+1}+c_{r+1},\ldots,\lambda_n x_n + c_n)$ and

$$\hat{F}_r(x_{r+1},\ldots,x_n) = h_r(x_{r+1}) + \sum_{i=r+2}^{n} x_i^2\hat{P}_{r,i}(x_{r+1},\ldots,x_i).$$

We can remark that $(\mathrm{SC})_r$ is equivalent to the conditions

$$(\mathrm{SC})_a \qquad h_r(\lambda_{r+1}x_{r+1}+c_{r+1}) = \lambda_r h_r(x_{r+1})$$

$$(\mathrm{SC})_b \quad \hat{P}_{r,i}(\cdot) = \frac{\lambda_r}{\lambda_i^2}\hat{P}_{r,i}(x_{r+1},\ldots,x_i),\ \ r+2\le i\le n,$$

where $(\cdot) = (\lambda_{r+1}x_{r+1}+c_{r+1},\ldots,\lambda_i x_i + c_i)$.

A similar argument will imply that $\psi_j(x) = \psi(x_j)$ for all $1\le j\le r-1$. Taking $j = r-1$, we should have

$$\begin{aligned}\dot{\tilde{x}}_{r-1} &= \psi'_{r-1}(x_{r-1})\hat{F}_{r-1}(x_r,\ldots,x_n),\\ &= \hat{F}_{r-1}(\tilde{x}_r,\ldots,\tilde{x}_n)\end{aligned}$$

which implies that $\psi'_{r-1}(x_{r-1}) = \lambda_{r-1}$, and consequently, we have $\psi_{r-1}(x_{r-1}) = \lambda_{r-1}x_{r-1}+c_{r-1}$.

A straightforward recurrence shows that for any $1\le j\le r$, we have $\psi_j(x_j) = \lambda_j x_j + c_j$.

At each step, the constant λ_j is related to the pairs (λ_k, c_k), for $j+1\le k\le n$, by the strong conditions

$$(\mathrm{SC})_j \qquad \hat{F}_j(\cdot) = \lambda_j\hat{F}_j(x_{j+1},\ldots,x_n),$$

where $(\cdot) = (\lambda_{j+1}x_{j+1}+c_{j+1},\ldots,,\lambda_i x_n + c_n)$, and

$$\hat{F}_j(x_{j+1},\ldots,x_n) = h_j(x_{j+1}) + \sum_{i=j+2}^{n} x_i^2\hat{P}_{j,i}(x_{r+1},\ldots,x_i).$$

Notice that the constant c_1 can be chosen arbitrarily. To complete the proof of Theorem III.1, we will construct the diffeomorphism $z = \sigma(x)$ of the feedback transformation bringing Σ_{SFF} into its strict feedforward normal form.

Normalizing Coordinates

Consider a system Σ_{SFF} in strict feedforward form with linearizability index $p = n-r$. To simplify the proof, we will suppose here that $p = 2$, and without loss of generality we can assume the system in the form

$$\begin{aligned}\dot{x}_1 &= h_1(x_2) + F_1(x_2,\ldots,x_n)\\ \dot{x}_2 &= h_2(x_3) + F_2(x_3,\ldots,x_n)\\ &\cdots\\ \dot{x}_{n-2} &= h_{n-2}(x_{n-1}) + F_{n-2}(x_{n-1},x_n)\\ \dot{x}_{n-1} &= x_n\\ \dot{x}_n &= u,\end{aligned} \qquad \text{(IV.1)}$$

where h_j, and F_j are smooth functions such that

$$\begin{aligned}h_j(x_{j+1}) &= x_{j+1} + x_{j+1}^2 b_j(x_{j+1})\\ F_j(x_{j+1},0,\ldots,0) &= 0\end{aligned} \qquad \text{(IV.2)}$$

for any $1 \le j \le n-2$.

Denote the system (IV.1)-(IV.2) by Σ_n and let us suppose that for some $3 \le k \le n$, the system Σ_n has been transformed via a series of transformations into Σ_k, defined by (IV.1)-(IV.2), where, in addition, the components F_j are

$$F_j(x_{j+1},\dots,x_n) = \tilde{F}_j(x_{j+1},\dots,x_k) + \sum_{i=k+1}^{n} x_i^2 P_{j,i}(x_{j+1},\dots,x_i) \quad \text{(IV.3)}$$

for any $1 \le j \le n-2$ with $\tilde{F}_j(x_{j+1},0,\dots,0)=0$. (This is always true for $k=n$ with the identity transformation).

Notice that, when $k \le j$, the components $P_{j,i}$ are identically zero for all $k+1 \le i \le j+1$. Moreover, $\tilde{F}_j(x_{j+1},\dots,x_k)=0$ if $k \le j+1$.

Now, let us decompose $\tilde{F}_j(x_{j+1},\dots,x_k)$ uniquely as

$$\begin{aligned}\tilde{F}_j(x_{j+1},\dots,x_k) &= \bar{F}_j(x_{j+1},\dots,x_{k-1}) \\ &+ x_k\Theta_{j,k}(x_{j+1},\dots,x_{k-1}) \\ &+ x_k^2 P_{j,k}(x_{j+1},\dots,x_k)\end{aligned}$$

with $\bar{F}_j(x_{j+1},0,\dots,0)=0$.

The diffeomorphism $z=\sigma^k(x)$ whose components are

$$\begin{aligned} z_j &= \sigma_j^k(x) = x_j - \int_0^{x_{k-1}} \Theta_{j,k}(x_{j+1},\dots,x_{k-2},s)\,ds, \\ &\quad \text{if } 1 \le j \le k-1 \qquad \text{(IV.4)} \\ z_j &= \sigma_j^{k-1}(x) = x_j, \ \text{if } k \le j \le n \end{aligned}$$

takes the system Σ_k into a system Σ_{k-1} of the form

$$\begin{aligned} \dot{z}_1 &= h_1(z_2) + F_1(z_2,\dots,z_n) \\ \dot{z}_2 &= h_2(z_3) + F_2(z_3,\dots,z_n) \\ &\cdots \\ \dot{z}_{n-2} &= h_{n-2}(z_{n-1}) + F_{n-2}(z_{n-1},z_n) \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= u, \end{aligned}$$

where for any $1 \le j \le n-2$

$$F_j(x_{j+1},\dots,x_n) = \tilde{F}_j(x_{j+1},\dots,x_{k-1}) + \sum_{i=k}^{n} x_i^2 P_{j,i}(x_{j+1},\dots,x_i)$$

with $\tilde{F}_j(x_{j+1},0,\dots,0)=0$.

Starting from the original system Σ_n, we then define a successive sequence of diffeomorphisms σ^k given by (IV.4) for $k=n,n-1,\dots,3$ yielding a successive sequence of strict feedforward systems $\Sigma_n,\Sigma_{n-1},\dots,\Sigma_2$, where for any $3 \le k \le n$, the system Σ_{k-1} is the transform of Σ_k via σ^k. Moreover, each system Σ_k is in the form (IV.1)-(IV.3).

The composition $\sigma(x) = \sigma^3 \circ \cdots \circ \sigma^n(x)$ of these diffeomorphisms transforms (IV.1)-(IV.2) into its strict feedforward normal form, which indeed coincides with Σ_2.

Remark that there is a finite number of coordinates changes (actually $n-2$) and all changes are smooth.

If the diffeomorphism σ is not unique, say there is a diffeomorphism η that also takes Σ_{SFF} into Σ_{SFNF}, then $\eta \circ \sigma^{-1}$ would be a symmetry of Σ_{SFNF}. Hence

$$\eta \circ \sigma^{-1}(x) = \mathbb{T}_{\bar{\Lambda},\bar{C}}(x) = (\bar{\lambda}_1 x_1 + \bar{c}_1, \dots, \bar{\lambda}_n x_n + \bar{c}_n)$$

with $\bar{\lambda}_{r+1} = \cdots = \bar{\lambda}_n = \bar{\lambda}$ and $\bar{c}_{r+2} = \cdots = \bar{c}_n = 0$.

It follows that

$$\begin{aligned} \psi &= \sigma^{-1} \circ \mathbb{T}_{\Lambda,C} \circ \sigma \\ &= \eta^{-1} \circ \mathbb{T}_{\tilde{\Lambda},\tilde{C}} \circ \eta, \end{aligned}$$

where $\mathbb{T}_{\tilde{\Lambda},\tilde{C}} = \mathbb{T}^{-1}_{\bar{\Lambda},\bar{C}} \circ \mathbb{T}_{\Lambda,C} \circ \mathbb{T}_{\bar{\Lambda},\bar{C}}$. □

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