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Explicit Symmetries of Strict Feedforward Control Systems

Issa Amadou Tall and Witold Respondek

Abstract—We show that any symmetry of a smooth strict feedforward system is conjugated to a scaling translation and any 1-parameter family of symmetries to a family of scaling translations along the first variable. We compute explicitly those symmetries by finding the conjugating diffeomorphism. We deduce, in accordance with our previous work, that a smooth system is feedback equivalent to a strict feedforward form if and only if it gives rise to a sequence of systems, such that each element of the sequence, firstly, possesses an infinitesimal symmetry whose flow is conjugated to a 1-parameter families of scaling translations and, secondly, is the factor system of the preceding one, that is, is reduced from the preceding one by its symmetry. We illustrate our results by computing the symmetries of the Cart-Pole system.

I. INTRODUCTION

We consider smooth single-input nonlinear control-affine systems of the form

\[ \Sigma: \dot{x} = f(x) + g(x)u, \]

where \( f \) and \( g \) are smooth vector fields on \( \mathbb{R}^n \).

We will say that the system \( \Sigma \) is in affine strict feedforward form, (shortly \( \Sigma \text{SFF} \)), if it is in the form

\[ \dot{x}_1 = f_1(x_2, \ldots, x_n) + g_1(x_2, \ldots, x_n)u \]

\[ \quad \ldots \]

\[ \dot{x}_{n-1} = f_{n-1}(x_n) + g_{n-1}(x_n)u \]

\[ \dot{x}_n = f_n + g_nu, \]

where \( f_n, g_n \in \mathbb{R} \), \( g_n \neq 0 \).

A basic structural property of systems in strict feedforward form is that their solutions can be found by quadratures. Indeed, knowing \( u(t) \) we integrate \( f_n + g_nu(t) \) to get \( x_n(t) \), then we integrate \( g_{n-1}(x_n(t))u(t) \) to get \( x_{n-1}(t) \), we keep doing that, and finally we integrate \( f_1(x_2(t), \ldots, x_n(t)) + g_1(x_2(t), \ldots, x_n(t))u(t) \) to get \( x_1(t) \).

Another property, crucial in applications, of systems in (strict) feedforward form is that their solutions can be found by quadratures. This important result goes back to Teel [19] and has been followed by a growing literature on stabilization and tracking for systems in (strict) feedforward form (see e.g. [4], [6], [13], [20], [2], [7]).

Recently (see [11]), we have proved that feedback equivalence (resp. state-space equivalence) to the strict feedforward form can be characterized by the existence of a sequence of infinitesimal symmetries (resp. strong infinitesimal symmetries) of the system.

In this paper we give a complete classification of symmetries of strict feedforward systems, and we restate the equivalence conditions obtained in [11] in terms of the symmetries of strict feedforward systems.

Notice that the problem of transforming a system, affine with respect to controls, into (strict) feedforward form via a nonlinear change of coordinates was studied in [5], and that a geometric description of systems in feedforward form has been given in [1]. We have also used another approach to propose a step-by-step constructive method to bring a system into a feedforward form in [15], [17] and strict feedforward form in [16].

The paper is organized as follows. Section II deals with notations and definitions. Section III contains the main results of the paper along with explicit examples. The proofs form the Section IV.

II. NOTATIONS AND DEFINITIONS

In this section we will give definitions concerning feedforward equivalence of control systems and symmetries. The word smooth will mean throughout \( C^\infty \)-smooth and all control systems are assumed to be smooth. For simplicity of notations we will consider here control-affine systems.

Two smooth control systems \( \Sigma \) and \( \bar{\Sigma} \) are called feedback equivalent, shortly \( F \)-equivalent, if there exist a smooth diffeomorphism \( \phi: X \rightarrow \bar{X} \) and smooth functions \( \alpha, \beta \), satisfying \( \beta(\cdot) \neq 0 \), such that

\[ \phi_*(f + g\alpha) = \bar{f} \quad \text{and} \quad \phi_*(g\beta) = \bar{g}. \]

Recall that for any smooth vector field \( h \) on \( X \) and any smooth diffeomorphism \( \bar{x} = \phi(x) \) we denote

\[ (\phi_*(h))(\bar{x}) = d\phi(\phi^{-1}(\bar{x})) \cdot h(\phi^{-1}(\bar{x})). \]

For the single-input control-affine system

\[ \Sigma: \dot{x} = f(x) + g(x)u, \]

where \( x \in X \), an open subset of \( \mathbb{R}^n \), and \( u \in U = \mathbb{R} \), and \( f \) and \( g \) are smooth vector fields on \( X \), the field of admissible velocities is the following field of affine lines

\[ A(x) = \{ f(x) + ug(x) : u \in \mathbb{R} \} \subset T_xX. \]

A diffeomorphism \( \psi: X \rightarrow X \) is a symmetry of \( \Sigma \) if it preserves the field of affine lines \( A \) (in other words, the affine distribution \( A \) of rank 1), that is, if \( \psi_*A = A \).

A local symmetry at \( p \in X \) is a local diffeomorphism \( \psi \) of \( X_0 \) onto \( X_1 \), where \( X_0 \) and \( X_1 \) are, respectively, neighborhoods of \( p \) and \( \psi(p) \), such that

\[ (\psi_*A)(q) = A(q) \]

for any \( q \in X_1 \).
A local symmetry $\psi$ at $p$ is called a stationary symmetry if $\psi(p) = p$ and a nonstationary symmetry if $\psi(p) \neq p$.

We say that a vector field $v$ on an open subset $X \subset \mathbb{R}^n$ is an infinitesimal symmetry of the system $\Sigma$ if the (local) flow $\gamma^v_t$ of $v$ is a local symmetry of $\Sigma$, for any $t$ for which it exists.

An infinitesimal symmetry $v$ is called stationary at $p \in X$ if $v(p) = 0$ and nonstationary if $v(p) \neq 0$.

### III. Main Results

Consider the class of smooth single-input control systems in strict feedforward form

$$
\Sigma_{SFF} : \begin{cases} 
\dot{x} = f(x) + g(x)u, \\
\sigma_j(x) = f_j(x_{j+1}, \ldots, x_n), \quad 1 \leq j \leq n - 1, \\
\sigma_n(x) = g(x_{j+1}, \ldots, x_n). 
\end{cases}
$$

Notice that for any $1 \leq i \leq n$, the subsystem $\Sigma_i^{SFF}$, defined as the projection of $\Sigma_{SFF}$ onto $\mathbb{R}^{n-i+1}$ via $\pi_i(x_1, \ldots, x_n) = (x_i, \ldots, x_n)$, is a well defined system whose dynamics are given, for any $i \leq j \leq n$, by

$$
\dot{x}_j = f_j(x_{j+1}, \ldots, x_n) + g_j(x_{j+1}, \ldots, x_n)u.
$$

Define the linearizability index of $\Sigma_{SFF}$ to be the largest integer $p$ such that the subsystem $\Sigma_i^{SFF}$, where $p + r = n$, is feedback linearizable. Clearly, the linearizability index is feedback invariant and hence the linearizability indices of two feedback equivalent (SFF)-systems coincide. We will assume that the linear approximation around the origin is controllable which implies that $p \geq 2$.

For any nonzero real numbers $\lambda_1, \ldots, \lambda_r, \lambda \in \mathbb{R}$ and any $c_1, \ldots, c_{r+1} \in \mathbb{R}$, put $\Lambda = (\lambda_1, \ldots, \lambda_r, \lambda, \lambda)$ and $C = (c_1, \ldots, c_{r+1}, 0, \ldots, 0)$ and define a scaling translation by

$$
T_{\Lambda, C}(x) = (\lambda_1 x_1 + c_1, \ldots, \lambda_n x_n + c_n),
$$

with $c_{r+2} = \cdots = c_n = 0$ and $\lambda_r + 1 = \cdots = \lambda_n = \lambda$.

**Theorem III.1** Consider a smooth system $\Sigma_{SFF}$ in strict feedforward form with linearizability index $p = n - r$. Any symmetry $\psi$ of $\Sigma_{SFF}$ is of the form

$$
\psi = \sigma^{-1} \circ T_{\Lambda, C} \circ \sigma,
$$

for a fixed $(\Lambda, C)$, where $z = \sigma(x)$ is the diffeomorphism of the transformation taking $\Sigma_{SFF}$ into its strict feedforward normal form $\Sigma_{SFNF}$ given by Definition III.2 below. Any local 1-parameter family of symmetries $\psi_{c_1}$ of $\Sigma_{SFF}$ is of the same form with $c_1 \in (-\epsilon_1, \epsilon_1)$.

Theorem III.1 says basically that strict feedforward systems have 1-parameter families of symmetries conjugated to scaling translations. Recall that in [9] we showed that any symmetry is conjugated to at most two 1-parameter families of translations along the first variable; those translations being the only symmetries of the canonical form.

The constant parameters $\lambda_1, \ldots, \lambda_r, \lambda$ are likely to be either $+1$ or $-1$ and will be uniquely determined by $c_2, \ldots, c_r$ (given by other equilibrium point) because, together, they should satisfy some strong conditions (SC), see below. The only free parameter is $c_1$. In Example III.8 we provide a case where some of the parameters $\lambda_1, \ldots, \lambda_r, \lambda$ are not equal to $+1$ or $-1$ as well as some constants $c_2, \ldots, c_{r+1}$ that are non zero. We then compare the results obtained here with those of [9], and show no ambiguity between them.

The importance of this result is that we can always put a (SFF)-system into a strict feedforward normal form (SFNF) via smooth feedback transformation while the canonical form is only guaranteed in the formal category. Moreover, the feedback transformation taking the system into its strict feedforward normal form (SFNF) can be constructed explicitly, for smooth systems, see Section IV.

The notion of strict feedforward normal form plays a crucial role in proving Theorem III.1 and is as follows.

**Definition III.2** A smooth strict feedforward normal form, denoted $\Sigma_{SFNF}$, is a strict feedforward form

$$
\dot{x}_1 = \hat{F}_1(x_2, \ldots, x_n) \\
\ldots \\
\dot{x}_r = \hat{F}_r(x_{r+1}, \ldots, x_n) \\
\dot{x}_{r+1} = \hat{x}_{r+2} \\
\ldots \\
\dot{x}_{n-1} = x_n \\
\dot{x}_n = u
$$

for which $p = n - r$ is the linearizability index and

$$(SFNF) \quad \hat{F}_j(x) = h_j(x_{j+1}) + \sum_{i=j+2}^{n} x_i^2 \hat{P}_{j,i}(x_{j+1}, \ldots, x_i)
$$

for any $1 \leq j \leq r$, where $h_j$ and $\hat{P}_{j,i}$ are smooth functions of the indicated variables.

The above strict feedforward normal form $\Sigma_{SFNF}$ was introduced in [12], where we proved the following:

**Theorem III.3** Any smooth strict feedforward form can be transformed into a strict feedforward normal form via smooth feedback transformation.

**Remark III.4** (i) In the proof of Theorem III.1, we will give an algorithm showing how to construct explicitly the feedback transformation (in particular, the diffeomorphism $z = \sigma(x)$) that takes a (SFF)-system into its (SFNF).

Then using the commutative diagram

$$
\begin{array}{ccc}
\Sigma_{SFF} & \xrightarrow{\psi} & \Sigma_{SFF} \\
\sigma & \downarrow & \sigma \\
\Sigma_{SFNF} & \xrightarrow{\psi} & \Sigma_{SFNF}
\end{array}
$$
where $\psi$ is a symmetry of the strict feedforward normal form $\Sigma_{SFF}$, all we will have to prove is that all $\psi$'s are exhausted by scaling translations $T_{\Lambda,C}$ defined above.

(ii) We will use this item to deduce, as a corollary, necessary and sufficient condition for a system to be brought to a strict feedforward form (see Theorem II.4 of [11]).

**Corollary III.5** Consider a smooth affine system $\Sigma$ with linearizability index $p = n - r$. The following conditions are equivalent.

(i) $\Sigma$ is, locally at $q \in X$, feedback equivalent to the affine strict feedforward form (SFF):

(ii) Each system $\Sigma^1$, $\Sigma^2$, ..., $\Sigma^r$ possesses an infinitesimal symmetry $v_i$, whose local flow $\gamma_i$ is conjugated to a scaling translation

$$\gamma_i = \sigma_i^{-1} \circ T_{\Lambda,C} \circ \sigma_i, \quad c_i \in (-\epsilon_i, \epsilon_i),$$

where $\Sigma^1$ is the restriction of $\Sigma$ to a neighborhood $X_q$ and

$$\Sigma^{i+1} = \Sigma^{i} / \sim, \quad 1 \leq i \leq r - 1.$$

Above, the equivalence relation $\sim$ is induced by the local action of the 1-parameter local group $\gamma_i$ defined by $v_i$, that is, such that $q_i \sim v_i q_2$ if and only if they belong to the same integral curve of $v_i$, and for any $1 \leq i \leq r - 1$ the scaling translation $T_{\Lambda,C}$ is the composition of $T_{\Lambda,C}$ with the projection $\pi_1$:

$$T_{\Lambda,C}(x) = (\lambda x_1 + c_i, \ldots, \lambda x_r + c_r, \lambda x_{r+1}, \ldots, \lambda x_n).$$

**EXAMPLES**

**Example III.6 Cart-Pole System.** In this example we consider a cart-pole system that is represented by a cart with an inverted pendulum on it [8], [18]. The Lagrangian equations of motion for the cart-pole system are

$$(m_1 + m_2)\ddot{q}_1 + m_2l \cos(q_2) \ddot{q}_2 = m_2l \sin(q_2) \dot{q}_2^2 + F \cos(q_2) \ddot{q}_1 + \ddot{q}_2 = g \sin(q_2),$$

where $m_1$ and $m_2$ are the mass and position of the cart, $m_2$, $l$, $q_2 \in (-\pi/2, \pi/2)$ are the mass, length of the link, and angle of the pole, respectively.

Taking $\ddot{q}_2 = u$ and applying the feedback law (see [8])

$$F = -ul(m_1 + m_2 \sin^2(q_2))/\cos(q_2) + (m_1 + m_2)g \tan(q_2) - m_2l \sin(q_2) \dot{q}_2^2$$

the dynamics of the cart-pole system are transformed into

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = g \tan(x_3) - lu/\cos(x_3)$$
$$\dot{x}_3 = x_4, \quad \dot{x}_4 = u,$$

where we take $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, and $x_4 = \dot{q}_2$.

This system is in strict feedforward form (SFF) with the linearizability index $p = 2$. We showed in [9] that the diffeomorphism

$$z = \sigma(x) = (\sigma_1(x), \sigma_2(x), \sigma_3(x), \sigma_4(x))$$

defined by

$$z_1 = \sigma_1(x) = \mu x_1 + \mu l \int_0^{x_3} \frac{ds}{\cos s}$$
$$z_2 = \sigma_2(x) = \mu x_2 + \mu l \frac{x_4}{\cos x_3}$$
$$z_3 = \sigma_3(x) = \mu g \tan x_3,$$
$$z_4 = \sigma_4(x) = \mu \frac{x_4}{\cos^2 x_3},$$

takes the system into its canonical form $\Sigma_{SFF}$:

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3 + \frac{z_3}{(1 + (g/l)^2)^{3/2}} z_4^2$$
$$\dot{z}_3 = z_4, \quad \dot{z}_4 = v.$$

It is straightforward to verify that

$$T_{\Lambda,C}(z) = (z_1 + c_1, z_2, z_3, z_4)$$
$$T_{\Lambda,C}^{-1}(z) = (-z_1 + c_1, -z_2, -z_3, -z_4)$$

constitute two 1-parameter families of symmetries for the canonical form. By Theorem 4 (see [9]), they exhaust all possible symmetries of the canonical form.

The symmetries of (III.1) are obtained by computing

$$\psi(x) = \sigma^{-1} \circ T_{\Lambda,C} \circ \sigma(x)$$

where the inverse $x = \eta(z) = \sigma^{-1}(z)$ is given by

$$x_1 = \eta_1(z) = \mu g z_1 + \theta(z_3),$$
$$x_2 = \eta_2(z) = \mu g z_2 - \mu l \frac{z_4}{\sqrt{1 + (\mu z_3)^2}},$$
$$x_3 = \eta_3(z) = \arctan(\mu z_3),$$
$$x_4 = \eta_4(z) = \frac{\mu z_4}{1 + (\mu z_3)^2}$$

for a suitable function $\theta(z_3)$. It follows easily that

$$\sigma^{-1} \circ T_{\Lambda,C} \circ \sigma = T_{\Lambda,C}$$

and

$$T_{\Lambda,C} \circ \sigma(x) = T_{\Lambda,C}$$

are both 1-parameter families of translations along the first component $x_1$ of $(x_1, x_2, x_3, x_4)$.

**Example III.7** Consider the system in $\mathbb{R}^4$ described by

$$\dot{x}_1 = \sin x_2 + x_2^3 \sin x_3, \quad \dot{x}_2 = \sin x_3 + x_4^3$$
$$\dot{x}_3 = x_4, \quad \dot{x}_4 = u.$$

This system is clearly in (SFNF) with linearizability index $p = 2$. It is easy to check that the forward and backward translations

$$T_{\Lambda,C}^{x_1} = (x_1 + c_1, x_2 + c_2, x_3 + c_3, x_4)$$

and

$$T_{\Lambda,C}^{x_1} = (-x_1 + c_1, -x_2 + c_2, -x_3 + c_3, -x_4)$$

are symmetries, where $c_2$ and $c_3$ are any multiples of $2\pi$.

**Example III.8** Consider the system

$$\Sigma_{SFF} : \dot{x}_1 = x_2 + 2x_2 e^{x_3} \sin x_3 + 2x_2 e^{x_3} x_4^2, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = u,$$

$$\dot{x}_2 = e^{x_3} \sin x_3 + e^{x_3} x_4^2, \quad \dot{x}_4 = u.$$
in strict feedforward form with linearizability index $p = 2$. Due to the terms $2x_2 e^{x_3} \sin x_3$, this system is not in strict feedforward normal form. However, it is straightforward to check that the diffeomorphism $z = \sigma(x)$ defined by

$$z_1 = x_1 - x_2^2, \quad z_2 = x_2, \quad z_3 = x_3, \quad z_4 = x_4$$

takes $\Sigma_{SFF}$ into the strict feedforward normal form

$$\Sigma_{SFNF} : \quad \dot{z}_1 = z_2, \quad \dot{z}_2 = e^{x_1} \sin z_3 + e^{x_3} z_4^2, \quad \dot{z}_3 = z_4, \quad \dot{z}_4 = u.$$

We can notice that the scaling translations

$$\tilde{z} = T_{\Lambda,C}(z) = (\lambda z_1 + c_1, \lambda z_2, z_3 + c_3, z_4)$$

with $c_3 = 2k\pi$, $k \in \mathbb{Z}$, and $\lambda = e^{c_3}$ form a family of symmetries of $\Sigma_{SFNF}$ parameterized by $c_1$.

Indeed, it is easy to see that they map $\Sigma_{SFNF}$ into $\Sigma_{SFF}$, given around the equilibrium $q = (0, 0, c_3, 0)$, by

$$\Sigma_{SFF} : \quad \dot{\tilde{z}}_1 = \tilde{z}_2, \quad \dot{\tilde{z}}_2 = e^{\lambda \tilde{z}_3} \sin \tilde{z}_4 + e^{x_3} \tilde{z}_4^2, \quad \dot{\tilde{z}}_3 = \tilde{z}_4, \quad \dot{\tilde{z}}_4 = u.$$

The composition $\tilde{x} = \sigma^{-1} \circ T_{\Lambda,C} \circ \sigma(x)$ expresses the coordinates $\tilde{x}$ in terms of the coordinates $x$ as follows

$$\tilde{x}_1 = \lambda x_1 + (\lambda^2 - \lambda) x_2^2 + c_1, \quad \tilde{x}_2 = x_3 + c_3, \quad \tilde{x}_4 = x_4,$$

where $c_3 = 2\pi$ and $\lambda = e^{c_3}$.

A straightforward calculation shows that

$$\dot{\tilde{x}}_1 = \lambda (x_2 + 2x_2 e^{x_3} \sin x_3 + 2x_2 e^{x_3} x_4^2) + 2(\lambda^2 - \lambda) x_2 e^{x_3} \sin x_3 + e^{x_3} x_4^2) = \lambda x_2 + 2\lambda^2 x_2 e^{x_3} \sin x_3 + e^{x_3} x_4^2 = \tilde{x}_2 + 2\tilde{x}_2 e^{\tilde{x}_3} \sin \tilde{x}_4 + 2\tilde{x}_2 e^{\tilde{x}_3} \tilde{x}_4^2$$

because $\lambda x_2 = \tilde{x}_2$ and

$$\lambda e^{x_3} \sin x_3 = e^{x_3 + c_3} \sin(x_3 + c_3) = e^{\tilde{x}_3} \sin \tilde{x}_3.$$

Similarly, we can show that

$$\dot{\tilde{x}}_2 = \lambda (e^{x_3} \sin x_3 + e^{x_3} x_4^2) = e^{\tilde{x}_3} \sin \tilde{x}_4 + e^{\tilde{x}_3} \tilde{x}_4^2.$$

Since $\tilde{x}_3 = \tilde{x}_4$ and $\dot{\tilde{x}}_4 = u$, it follows that the composition $\tilde{x} = \sigma^{-1} \circ T_{\Lambda,C} \circ \sigma(x)$ maps $\Sigma_{SFF}$, defined around the equilibrium $(0, 0, 0, 0)$, into $\Sigma_{SFF}$ described, around the equilibrium $q = (0, 0, 2\pi, 0)$, by the same dynamics

$$\Sigma_{SFF} : \quad \dot{x}_1 = \dot{x}_2 + 2\dot{x}_2 e^{x_3} \sin x_3 + 2\dot{x}_2 e^{x_3} x_4^2, \quad \dot{x}_3 = \dot{x}_4, \quad \dot{x}_4 = \dot{u}.$$

Hence $\tilde{x} = \psi(x) = \sigma^{-1} \circ T_{\Lambda,C} \circ \sigma(x)$ is a 1-parameter family of symmetries of $\Sigma_{SFF}$.

For convenience of notation, we will denote $\Sigma_{SFF}$, defined around $(0, 0, 0, 0)$, by $\Sigma_{SFF}$ and the system $\Sigma_{SFF}$, defined around $q = (0, 0, 2\pi, 0)$, by $\Sigma_{SFF}$. The same notations apply to the systems $\Sigma_{SFNF}$ and $\Sigma_{SFF}$.

Now, in view of the results obtained in [12], we will compute the canonical form of $\Sigma_{SFF}$ and the transformations taking $\Sigma_{SFF}$ and $\Sigma_{SFF}$ to this canonical form.

It is easy to verify that $y = \Phi(x)$, given by

$$y_1 = x_1 - x_2^2, \quad y_3 = e^{x_3} \sin x_3, \quad y_2 = x_2, \quad y_4 = e^{x_3} (\sin x_3 + \cos x_3) x_4,$$

followed by an appropriate feedback, takes the system $\Sigma_{SFF}$ into its canonical form

$$\Sigma_{SFCF} : \quad \dot{y}_1 = y_2, \quad \dot{y}_2 = y_3 + \Theta(y_3) y_4^2, \quad \dot{y}_3 = y_4, \quad \dot{y}_4 = v,$$

where $\Theta(y_3) = \frac{1}{e^{x_3} (\sin x_3 + \cos x_3)^2} \bigg|_{x_3 = 0}$.

On the other hand, applying the translation

$$\tilde{x} = T(x) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 - c_3, \tilde{x}_4)$$

to the system $\Sigma_{SFF}$, we can shift back the equilibrium point to $(0, 0, 0, 0)$. In the new coordinates, $\Sigma_{SFF}$ becomes

$$\Sigma_{SFF} : \quad \dot{\tilde{x}}_1 = \tilde{x}_2 + 2\lambda^2 \tilde{x}_2 e^{\tilde{x}_3} (\sin \tilde{x}_3 + e^{x_3} \tilde{x}_4), \quad \dot{\tilde{x}}_3 = \dot{\tilde{x}}_4, \quad \dot{\tilde{x}}_4 = \lambda e^{\tilde{x}_3} (\sin \tilde{x}_3 + e^{x_3} \tilde{x}_4),$$

where $\lambda = e^{c_3}$. The diffeomorphism $\tilde{y} = \Psi(\tilde{x})$ given by

$$\tilde{y}_1 = \lambda^{-1}(\tilde{x}_1 - \tilde{x}_2^2), \quad \tilde{y}_3 = e^{x_3} \sin \tilde{x}_3, \quad \tilde{y}_2 = \lambda^{-1} \tilde{x}_2, \quad \tilde{y}_4 = e^{x_3} (\sin \tilde{x}_3 + \cos \tilde{x}_3) \tilde{x}_4,$$

followed by an appropriate feedback, takes the system $\Sigma_{SFF}$ into its canonical form

$$\Sigma_{SFCF} : \quad \dot{\tilde{y}}_1 = \tilde{y}_2, \quad \dot{\tilde{y}}_2 = \tilde{y}_3 + \Theta(\tilde{y}_3) \tilde{y}_4^2, \quad \dot{\tilde{y}}_3 = \tilde{y}_4, \quad \dot{\tilde{y}}_4 = v.$$

It follows that the composition $\tilde{y} = \Psi \circ T \circ \psi \circ \Phi^{-1}(y)$ is a 1-parameter family of symmetries of the canonical form according to the diagram.

$$\Sigma_{SFF} \xrightarrow{\psi} \Sigma_{SFF} \xrightarrow{\Theta} \Sigma_{SFF} \xrightarrow{T} \Sigma_{SFF} \xrightarrow{\Psi} \Sigma_{SFCF}$$

We explicitly find this family of symmetries by expressing the coordinates $\tilde{y}$ as functions of the coordinates $y$:

$$\tilde{y}_1 = \lambda^{-1}(\tilde{x}_1 - \tilde{x}_2^2), \quad \tilde{y}_3 = \lambda^{-1}(\lambda x_1 + (\lambda^2 - \lambda) x_2^2 + c_1 - \lambda^2 x_2^2), \quad \tilde{y}_2 = x_1 - x_2^2 + \tilde{c}_1 = y_1 + \tilde{c}_1.$$
Similarly, we get
\[ \dot{y}_2 = \lambda^{-1}\dot{x}_2 = \lambda^{-1}\dot{x}_2 = \lambda^{-1}(\lambda x_2) = x_2 = y_2; \]
\[ \dot{y}_3 = e^{\epsilon_3}\sin{\dot{x}_3} = e^{\epsilon_3 + 2\pi}\sin{\dot{x}_3} + 2\pi = e^{\epsilon_3}\sin{x_3} = y_3 \]
and
\[ \dot{y}_4 = e^{\epsilon_3}(\sin{\dot{x}_3} + \cos{\dot{x}_3})\dot{x}_4 \]
\[ = e^{\epsilon_3 + 2\pi}(\sin{\dot{x}_3} + \cos{\dot{x}_3})\dot{x}_4 \]
\[ = e^{\epsilon_3}\sin{x_3} + e^{\epsilon_3}x_4^2 = y_4. \]

We conclude that the symmetries of the canonical form are exhausted here by a 1-parameter family of translations along the first variable. This is in concordance with the results in [9]. Notice that the composition $\Phi \circ \psi \circ \Phi^{-1}$ does not yield a symmetry for the canonical form. The reason is that, the system $\Sigma_{SF}^q$, being defined around the equilibrium $q$, is not transformed into the canonical form $\Sigma_{SFCF}$ by the same diffeomorphism $\Phi$ as $\Sigma_{SF}^q$ is.

IV. PROOFS

In this section we will prove Theorem III.1. Let us consider a system $\Sigma_{SF}^q$ in strict feedforward form. Applying Theorem III.3, we can assume that the system $\Sigma_{SF}^q$ is in the strict feedforward normal form $\Sigma_{SFNF}$, given by definition III.2. (explicit transformations are given in the second part of this Section).

Notice that if $\hat{x} = \psi(x)$ is a symmetry of $\Sigma_{SF}$ (in particular, of $\Sigma_{SF}^q$), then it preserves the structure of the strict feedforward form. Hence (see [17]), we have $\hat{x}_j = \psi_j(x) = \psi_j(x_1, \ldots, x_{n-1})$ for $1 \leq j \leq n - 1$. This implies that $\pi_r(\psi) = (\psi(x), \ldots, \psi(x))$ is a symmetry of the projection $\Sigma_{SF}^q$ of $\Sigma_{SF}^q$ whose dynamics are given by
\[ \dot{x}_r = h_r(x_{r+1} + \sum_{i=r+2}^n x_i^2 \hat{P}_{r,i}(x_{r+1}, \ldots, x_i) \]
\[ \dot{x}_{r+1} = x_{r+2} \]
\[ \ldots \]
\[ \dot{x}_{n-1} = x_n \]
\[ \dot{x}_n = u. \]

We claim that $\psi_j(x) = \psi_j(x_j)$ for any $r \leq j \leq n - 1$. Indeed, we have $\psi_{n-1}(x) = \psi_{n-1}(x_{n-1})$. Let $k$ be the largest integer, $r \leq k \leq n - 2$, such that $\frac{\partial \psi_k}{\partial x_k} \neq 0$ for some $s \geq k+1$ (we can take $s$ to be the largest integer that yields this property). Thus
\[ \dot{x}_k = \frac{\partial \psi_k}{\partial x_k} \dot{x}_k + \ldots + \frac{\partial \psi_k}{\partial x_{s+1}} \dot{x}_{s+1} = \dot{x}_k + 1 = \psi_{k+1}(x) \]
gives a contradiction because $\psi_{k+1}(x) = \psi_{k+1}(x_{k+1})$. We conclude that $\psi_j(x) = \psi_j(x)$ for $r \leq j \leq n - 1$. Since
\[ \dot{x}_j = \psi_j'(x_j)x_{j+1} = \dot{x}_j + 1 = \psi_j(1)(x_{j+1}), \]
we deduce that $\psi_j(x_j) = \lambda_r x_j + c_j$ for all $r+1 \leq j \leq n - 1$. Similarly we get $\psi_r(x_r) = \lambda_r x_r + c_r$ and hence
\[ \pi_r(\psi(x)) = (\lambda_r x_r + c_r, \lambda_{r+1} x_{r+1} + c_{r+1}, \ldots, \lambda_n x_n + c_n). \]

In fact, it is easy to see that $\lambda_{r+1} = \cdots = \lambda_n = \lambda$ and $c_{r+2} = \cdots = c_n = 0$ but for homogeneity of notation, we will carry those constants as such.

Notice that $\lambda_r$, and the pairs $(\lambda_k, c_k)$, $r + 1 \leq k \leq n$ should satisfy the strong condition:
\[ (SC)_r \quad \dot{F}_r(\cdot) = \lambda_r F_r(x_{r+1}, \ldots, x_n), \]
where $(\cdot) = (\lambda_{r+1} x_{r+1} + c_{r+1}, \ldots, \lambda_n x_n + c_n)$ and
\[ \dot{F}_r(x_{r+1}, \ldots, x_n) = h_r(x_{r+1}) + \sum_{i=r+2}^n x_i^2 \hat{P}_{r,i}(x_{r+1}, \ldots, x_i). \]

We can remark that $(SC)_r$ is equivalent to the conditions
\[ (SC)_a \quad h_r(\lambda_{r+1} x_{r+1} + c_{r+1}) = \lambda_r h_r(x_{r+1}) \]
\[ (SC)_b \quad \dot{P}_{r,i}(\cdot) = \frac{\lambda_r}{\lambda_i} P_{r,i}(x_{r+1}, \ldots, x_i), \quad r + 2 \leq i \leq n, \]
where $(\cdot) = (\lambda_{r+1} x_{r+1} + c_{r+1}, \ldots, \lambda_n x_n + c_n)$.

A straightforward argument will imply that $\psi_j(x) = \psi_j(x_j)$ for all $1 \leq j \leq r - 1$. Taking $j = r - 1$, we should have
\[ \dot{x}_{r-1} = \psi_{r-1}(x_{r-1})F_{r-1}(x_r, \ldots, x_n) \]
\[ = F_{r-1}(x_{r-1}, \ldots, x_n) \]
which implies that $\psi_{r-1}(x_{r-1}) = \lambda_{r-1}$, and consequently, we have $\psi_{r-1}(x_{r-1}) = \lambda_{r-1} x_{r-1} + c_{r-1}$.

A straightforward recurrence shows that for any $1 \leq j \leq r$, we have $\psi_j(x_j) = \lambda_j x_j + c_j$.

At each step, the constant $\lambda_j$ is related to the pairs $(\lambda_k, c_k)$, for $j + 1 \leq k \leq n$, by the strong conditions
\[ (SC)_j \quad \dot{F}_j(\cdot) = \lambda_j \dot{F}_j(x_{j+1}, \ldots, x_n), \]
where $(\cdot) = (\lambda_{j+1} x_{j+1} + c_{j+1}, \ldots, \lambda_n x_n + c_n)$, and
\[ \dot{F}_j(x_{j+1}, \ldots, x_n) = h_j(x_{j+1}) + \sum_{i=j+2}^n x_i^2 \hat{P}_{j,i}(x_{j+1}, \ldots, x_i). \]

Notice that the constant $c_1$ can be chosen arbitrarily. To complete the proof of Theorem III.1, we will construct the diffeomorphism $z = (x)(z)$ of the feedback transformation bringing $\Sigma_{SF}^q$ into its strict feedforward normal form.

NORMALIZING COORDINATES

Consider a system $\Sigma_{SF}^q$ in strict feedforward form with linearizability index $p = n - r$. To simplify the proof, we will suppose here that $p = 2$, and without loss of generality we can assume the system in the form
\[ \dot{x}_1 = h_1(x_2) + F_1(x_2, \ldots, x_n) \]
\[ \dot{x}_2 = h_2(x_3) + F_2(x_3, \ldots, x_n) \]
\[ \ldots \]
\[ \dot{x}_{n-2} = h_{n-2}(x_{n-1} + F_{n-2}(x_{n-1}, x_n) \]
\[ \dot{x}_{n-1} = x_n \]
\[ \dot{x}_n = u, \]

where $h_j$, and $F_j$ are smooth functions such that
\[ h_j(x_{j+1}) = x_{j+1} + x_{j+1}^2 \hat{b}_j(x_{j+1}) \]
\[ F_j(x_{j+1}, 0, \ldots, 0) = 0 \]
for any $1 \leq j \leq n - 2$.

Denote the system (IV.1)-(IV.2) by $\Sigma_n$ and let us suppose that for some $3 \leq k \leq n$, the system $\Sigma_n$ has been transformed via a series of transformations into $\Sigma_k$, defined by (IV.1)-(IV.2), where, in addition, the components $F_j$ are

$$F_j(x_{j+1}, \ldots, x_n) = \tilde{F}_j(x_{j+1}, \ldots, x_k)$$

$$+ \sum_{i=k+1}^{n} x_i^2 P_{j,i}(x_{j+1}, \ldots, x_i)$$

(IV.3)

for any $1 \leq j \leq n - 2$ with $\tilde{F}_j(x_{j+1}, 0, \ldots, 0) = 0$. (This is always true for $k = n$ with the identity transformation).

Notice that, when $k \leq j$, the components $P_{j,i}$ are identically zero for all $k + 1 \leq i \leq j + 1$. Moreover, $\tilde{F}_j(x_{j+1}, \ldots, x_k) = 0$ if $k \leq j + 1$.

Now, let us decompose $\tilde{F}_j(x_{j+1}, \ldots, x_k)$ uniquely as

$$\tilde{F}_j(x_{j+1}, \ldots, x_k) = \tilde{F}_j(x_{j+1}, \ldots, x_{k-1})$$

$$+ x_j \Theta_{j,k}(x_{j+1}, \ldots, x_{k-1})$$

$$+ x_i^2 P_{j,k}(x_{j+1}, \ldots, x_k)$$

with $\tilde{F}_j(x_{j+1}, 0, \ldots, 0) = 0$.

The diffeomorphism $z = \sigma^k(x)$ whose components are

$$z_j = \sigma^k_j(x) = x_j - \int_0^{x_{k-1}} \Theta_{j,k}(x_{j+1}, \ldots, x_{k-2}, s) \, ds$$

if $1 \leq j \leq k - 1$

$$z_j = \sigma^{k-1}_j(x) = x_j, \text{ if } k \leq j \leq n$$

(IV.4)

takes the system $\Sigma_k$ into a system $\Sigma_{k-1}$ of the form

$$\dot{z}_1 = h_1(z_2) + F_1(z_2, \ldots, z_n)$$

$$\dot{z}_2 = h_2(z_3) + F_2(z_3, \ldots, z_n)$$

$$\ldots$$

$$\dot{z}_{n-2} = h_{n-2}(z_{n-1}) + F_{n-2}(z_{n-1}, z_n)$$

$$\dot{z}_{n-1} = z_n$$

$$\dot{z}_n = u,$$

where for any $1 \leq j \leq n - 2$

$$F_j(x_{j+1}, \ldots, x_n) = \tilde{F}_j(x_{j+1}, \ldots, x_{k-1})$$

$$+ \sum_{i=k}^{n} x_i^2 P_{j,i}(x_{j+1}, \ldots, x_i)$$

with $\tilde{F}_j(x_{j+1}, 0, \ldots, 0) = 0$.

Starting from the original system $\Sigma_n$, we then define a successive sequence of diffeomorphisms $\sigma^k$ given by (IV.4) for $k = n, n - 1, \ldots, 3$ yielding a successive sequence of strict feedforward systems $\Sigma_n, \Sigma_{n-1}, \ldots, \Sigma_2$, where for any $3 \leq k \leq n$, the system $\Sigma_{k-1}$ is the transform of $\Sigma_k$ via $\sigma^k$. Moreover, each system $\Sigma_k$ is in the form (IV.1)-(IV.3).

The composition $\sigma(x) = \sigma^3 \circ \cdots \circ \sigma^k(x)$ of these diffeomorphisms transforms (IV.1)-(IV.2) into its strict feedforward normal form, which indeed coincides with $\Sigma_2$.

Remark that there is a finite number of coordinates changes (actually $n - 2$) and all changes are smooth.

If the diffeomorphism $\sigma$ is not unique, say there is a diffeomorphism $\chi$ that also takes $\Sigma_{SF} \subset \Sigma_{SFNF}$, then $\eta \circ \sigma^{-1}$ would be a symmetry of $\Sigma_{SFNF}$. Hence $\eta \circ \sigma^{-1} = T_{\Lambda, C}(x) = (\Lambda_1 x_1 + c_1, \ldots, \Lambda_n x_n + c_n)$

with $\Lambda_{r+1} = \cdots = \Lambda_n = \lambda$ and $c_{r+2} = \cdots = c_n = 0$.

It follows that

$$\psi = \sigma^{-1} \circ T_{\Lambda, C} \circ \sigma = \eta^{-1} \circ T_{\Lambda, C} \circ \eta,$$

where $T_{\Lambda, C} = T_{\Lambda, C} \circ T_{\Lambda, C} \circ T_{\Lambda, C}$. □

**References**


