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HIGHLY DEGENERATE QUADRATIC FORMS OVER FINITE FIELDS OF CHARACTERISTIC 2

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ABSTRACT. Let K/F be an extension of finite fields of characteristic two. We consider quadratic forms written as the trace of xR(x), where R(x) is a linearized polynomial. We show all quadratic forms can be so written, in an essentially unique way. We classify those R, with coefficients 0 or 1, where the form has a codimension 2 radical. This is applied to maximal Artin-Schreier curves and factorizations of linearized polynomials.

Let q be a 2-power, $q = 2^t$. Set F = GF(q) and let $K = GF(q^k)$ be an extension. Let

$$R(x) = \sum_{j=0}^{h} \epsilon_j x^{q^j},$$

with each $\epsilon_j \in K$. We consider the quadratic forms $Q_R^K : K \to F$ given by $Q_R^K(x) = tr_{K/F}(xR(x))$.

These trace forms have appeared in a variety of contexts. They have been used to compute weight enumerators of certain binary codes [1,2], to construct curves with many rational points and the associated trace codes [8], as part of an authentication scheme [3], and to construct certain binary sequences in [5] and [4].

In each of these applications one wants the number of solutions (in K) to $Q_R^K(x) = 0$, denoted by $N(Q_R^K)$. This is easily worked out (see [7, 6.26,6.32]) in terms of the standard classification of quadratic forms:

$$N(Q_R^K) = \frac{1}{q}(q^k + \Lambda(Q_R^K)(q-1)\sqrt{q^{k+w}}).$$

where w is the dimension of the radical, v = (k - w)/2 and

$$\Lambda(Q_R^K) = \begin{cases}
0, & \text{if } Q_R^K \simeq z^2 + \sum_{i=1}^v x_i y_i \\
1, & \text{if } Q_R^K \simeq \sum_{i=1}^v x_i y_i \\
-1, & \text{if } Q_R^K \simeq x_1^2 + s y_1^2 + \sum_{i=1}^v x_i y_i.
\end{cases}$$

Here s is any element of F with $\operatorname{tr}_{F/GF(2)}(s) = 1$.

Key words and phrases. quadratic form, trace, linearized polynomial.

However, there is no simple way to determine the dimension of the radical or the invariant Λ . The one general result is due to Klapper [6] which only covers the case when R consists of a single term. In roughly half the applications ([1,2,8]) one wants highly degenerate forms, which give large $N(Q_R^K)$ when $\Lambda = 1$. We restrict to those R with all coefficients $\epsilon_i \in GF(2)$ as is the case in each of the cited papers except [8]. Our main result is to determine all such R, and all extensions K, such that the radical of Q_R^K has codimension (namely 2v) at most 2. We compute the invariant Λ in each case.

We first show that every quadratic form $Q: K \to F$ can be written as Q_R^K in an essentially unique way. Thus our result is more general than it appears. We apply our main result to obtain a classification of those R such that the number of points on the Artin-Schreier curve $y^q + y = xR(x)$ equals the Hasse-Weil bound. We also obtain results on the factors of self-reciprocal linearized polynomials.

1. Quadratic forms.

A quadratic form $Q: K \to F$ is a map such that

- (1) $Q(ax) = a^2 Q(x)$ for all $a \in F$ and $x \in K$, and
- (2) $B(x,y) :\equiv Q(x+y) + Q(x) + Q(y)$ is a bilinear map $K \times K \to F$.

The radical of Q is

$$rad Q = \{x \in K : B(x, y) = 0 \text{ for all } y \in K\}.$$

The codimension of the radical, $k - \dim \operatorname{rad} Q$ is always even.

To simplify notation, we write simply tr for $\operatorname{tr}_{K/F}$. We will write Tr_K for the absolute trace $\operatorname{tr}_{K/GF(2)}$.

Proposition 1.1. Let $Q: K \to F$ be a quadratic form. Let $m = \lfloor k/2 \rfloor$. Let $h = \frac{1}{2}codim\ radQ$.

(1) There exist $c, a_1, b_1, \ldots, a_h, b_h \in K$, independent over F, such that

$$Q(x) = \begin{cases} tr(cx)^2 + \sum_{i=1}^h tr(a_i x) tr(b_i x), & \text{if } \Lambda(Q) = 0\\ \sum_{i=1}^h tr(a_i x) tr(b_i x), & \text{if } \Lambda(Q) = 1\\ tr(a_1 x)^2 + tr(b_1 x)^2 + \sum_{i=1}^h tr(a_i x) tr(b_i x), & \text{if } \Lambda(Q) = -1. \end{cases}$$

(2) There exist $\epsilon_0, \epsilon_1, \ldots, \epsilon_m \in K$ such that

$$Q(x) = tr\left(x \cdot \sum_{i=0}^{m} \epsilon_i x^{q^i}\right).$$

Proof. (1) Suppose $\Lambda(Q) = 1$. Pick a basis of K over F and let M be the matrix of Q with respect to this basis. We apply the classification of quadratic forms. Let $H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and let N be the $k \times k$ matrix with k copies of K on the diagonal and the rest zero. Then

there exists an invertible $k \times k$ matrix P over F such that

$$M = P^{t}NP$$

$$Q(X) = X^{t}P^{t}NPX$$

$$Q(X) = \sum_{i=1}^{h} (r_{2i-1}X)(r_{2i}X),$$

where r_j is the jth row of P. As map from $K \to F$, rather than from $F^k \to F$, each $r_j X$ is linear and so equal to $\operatorname{tr}(d_j x)$ for some $d_j \in K$. The rows of P are independent over F so the d_j are also. This gives the desired representation of Q.

The cases when $\Lambda(Q) = 0$ or -1 are similar except the first copy of H in N is replaced by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

respectively.

(2) This proof is taken from [9,3.2, 5.1]. Our only change is to correct a slight error in the i = 1 term and to include the cases of $\Lambda(Q) = 0, -1$.

$$\operatorname{tr}(ax)\operatorname{tr}(bx) = \operatorname{tr}(\operatorname{tr}(ax)bx)$$
$$= \operatorname{tr}\left(\sum_{i=0}^{k-1} (ax)^{q^i}(bx)\right).$$

Now

$$\operatorname{tr}(a^{q^i}b) = \operatorname{tr}(ab^{q^{k-i}})$$

so that

$$\operatorname{tr}(ax)\operatorname{tr}(bx) = \begin{cases} \operatorname{tr}(abx^2 + \sum_{i=1}^{m} (a^{q^i}b + ab^{q^i})x^{q^i}), & \text{if } k = 2m+1 \text{ is odd} \\ \operatorname{tr}(abx^2 + \sum_{i=1}^{m} (a^{q^i}b + ab^{q^i})x^{q^i} + a^{q^m}bx^{q^m}), & \text{if } k = 2m \text{ is even.} \end{cases}$$

In either case,

$$\operatorname{tr}(ax)\operatorname{tr}(bx) = \operatorname{tr}\left(x \cdot \sum_{j=0}^{m} \epsilon'_{j} x^{q^{i}+1}\right),$$

for some $\epsilon'_j \in K$. Lastly,

$$\operatorname{tr}(cx)^2 = \operatorname{tr}((cx)^2) = \operatorname{tr}(x \cdot c^2 x).$$

Thus (1) implies (2). \square

The first representation of (1.1) is not unique. For instance,

$$tr(ax)^{2} + tr(ax)tr(bx) + tr(bx)^{2} = tr(ax)^{2} + tr(ax)tr((a+b)x) + tr((a+b)x)^{2}.$$

However, for the second representation we have:

Theorem 1.2. Let $Q: K \to F$ be a quadratic form and let $m = \lfloor k/2 \rfloor$. Then there exist unique $\epsilon_i \in K$, $0 \le i \le m$, such that

$$Q(x) = tr\left(x \cdot \sum_{i=0}^{m} \epsilon_i x^{q^i}\right),\,$$

except when k is even in which case ϵ_m is only unique modulo $GF(q^m)$.

Proof. We count. If we fix a basis of K over F then each quadratic form is represented uniquely by an upper triangular matrix. Hence there are $q^{k(k+1)/2}$ many quadratic forms.

Suppose k = 2m + 1. The number of $R(x) = \sum_{i=0}^{m} \epsilon_m x^{q^i}$ is $(q^k)^{m+1} = q^{k(k+1)/2}$. Since this is the number of quadratic forms, (1.1) implies the representation $Q(x) = \operatorname{tr}(xR(x))$ is unique.

Suppose k = 2m. Note that

$$(x^{q^m+1})^{q^m-1} = x^{q^k-1} \in GF(2^m)$$

for all $x \in K$. Thus if $\epsilon \in GF(q^m)$ then $\operatorname{tr}(x \cdot \epsilon x^{q^m}) = 0$ for all $x \in K$. The number of $R(x) = \sum_{i=0}^m \epsilon_i x^{q^i}$, with $\epsilon_0, \ldots, \epsilon_{m-1} \in K$ and $\epsilon_m \in K/GF(q^m)$ is

$$(q^k)^m \cdot (q^m) = q^{2m^2 + m} = q^{k(k+1)/2}.$$

As before, this shows the representation of Q(x) as $\operatorname{tr}(xR(x))$ is unique (taking ϵ_m modulo $GF(q^m)$). \square

Throughout the remainder of the paper we assume

$$R(x) = \sum_{j=0}^{h} \epsilon_j x^{q^j}$$
 with $\epsilon_j \in GF(2)$,

where $h = \lfloor (k-1)/2 \rfloor$. Here we have dropped the ϵ_m term when k = 2m as $\epsilon_m = 0$ or 1, both of which are in $GF(2^m)$.

Corollary 1.3. Let $R = \sum_{i=0}^{h} \epsilon_i x^{q^i}$, where each $\epsilon_i \in GF(2)$ and $h = \lfloor (k-1)/2 \rfloor$. Then Q_R^K has radical of codimension 2 iff there exist independent $a, b, c \in K$ such that

(Ei)
$$a^{q^i}b + ab^{q^i} = \epsilon_i \quad \text{for } 1 \le i \le h$$

and

(E0)
$$\epsilon_0 = \begin{cases} c^2 + ab, & \text{if } \Lambda(Q_R^K) = 0\\ ab, & \text{if } \Lambda(Q_R^K) = 1\\ a^2 + ab + sb^2, & \text{if } \Lambda(Q_R^K) = -1, \end{cases}$$

plus, if k=2m,

(Em)
$$a^{q^m}b \in GF(q^m).$$

Again here $s \in F$ is an element with $Tr_F(s) = 1$.

Proof. We have by (1.1)(1) that the quadratic forms with radical of codimension 2 are

$$\operatorname{tr}(cx)^2 + \operatorname{tr}(ax)\operatorname{tr}(bx) \quad \operatorname{tr}(ax)\operatorname{tr}(bx) \quad \operatorname{tr}(ax)^2 + \operatorname{tr}(ax)\operatorname{tr}(bx) + s\operatorname{tr}(bx)^2,$$

where the invariants are 0, 1, -1 respectively (see [9, 3.1]). The computation of tr(ax)tr(bx) in (1.1)(2) gives the equations (Ei) for $1 \le i \le h$. (Em) follows as

$$\operatorname{tr}(a^{q^m}bx^{q^m+1}) = 0 \quad \text{iff} \quad a^{q^m}b \in GF(q^m),$$

by (1.2). And $\operatorname{tr}(cx)^2 = \operatorname{tr}((cx)^2) = \operatorname{tr}(c^2x \cdot x)$ yields the three forms of (E0). \square

2. The Main Theorem.

We begin with three lemmas needed to solve the equations (Ei).

Lemma 2.1. Suppose $y^2 = y + z$. Then

$$y^{2^{i}} = y + z + z^{2} + z^{4} + \dots + z^{2^{i-1}}.$$

Proof. Induction. \square

The following identity is well-known and may be derived in many ways. For instance, one may take Waring's identity, expressing the sum of two *n*th powers in terms of a Dickson polynomial, modulo 2. We use instead a simple induction argument.

Lemma 2.2. Let u = x + y and v = xy. Then

$$x^{2^{n}+1} + y^{2^{n}+1} = u^{2^{n}+1} + \sum_{i=0}^{n-1} u^{2^{n}+1-2^{i-1}} v^{2^{i}}.$$

Proof. By induction,

$$\begin{split} x^{2^{n+1}+1} + y^{2^{n+1}+1} &= (x^{2^n} + y^{2^n})(x^{2^n+1} + y^{2^n+1}) + x^{2^n}y^{2^n+1} + x^{2^n+1}y^{2^n} \\ &= u^{2^n}(u^{2^n+1} + \sum_{i=0}^{n-1} u^{2^n+1-2^{i-1}}v^{2^i}) + uv^{2^n} \\ &= u^{2^{n+1}+1} + \sum_{i=0}^n u^{2^{n+1}+1-2^{i-1}}v^{2^i}, \end{split}$$

as desired. \square

The following highly technical lemma is need to compute the invariant Λ in one case.

Lemma 2.3. Let $v = 2^{3^r}$ and let

$$g_v(x) = x^{v+1}(1 + x^{-2} + x^{-4} + \dots + x^{-v}) + 1.$$

Let δ be a root of $g_v(x)$ in some extension of F. Then

- (1) $\delta \in GF(v^3) \setminus GF(v)$
- (2) $\delta^{2v} + \delta^{v+1} + \delta^2 = 1$
- (3) $\delta^{v^2+1} + \delta^{2v} = 1$
- (4) $\delta^2 + \delta^{v^2 + v} = 1$.

Proof. We have

$$1 + \delta^{-2} + \delta^{-4} + \dots + \delta^{-v} = \delta^{-(v+1)}$$
.

Add this to its square to get

$$\delta^{-2} + \delta^{-2v} = \delta^{-(v+1)} + \delta^{-2(v+1)}.$$

Multiply by $\delta^{2(v+1)}$ to get (2).

Re-write (2) by dividing by δ^2

(5)
$$\delta^{2(v-1)} + \delta^{v-1} + (1+\delta^{-2}) = 0.$$

This has the form $y^2 + y + z = 0$ with $y = \delta^{v-1}$ and $z = 1 + \delta^{-2}$. By (1.4)

$$\delta^{(v-1)v} = \delta^{v-1} + z + z^2 + \dots + z^{v/2}.$$

As v is an odd power of 2 there are an odd number of z^i terms. So

$$\delta^{(v-1)v} = \delta^{v-1} + 1 + \delta^{-2} + \delta^{-4} + \dots + \delta^{-v}$$
$$= \delta^{v-1} + \delta^{-(v+1)},$$

by the original equation. Then

$$\delta^{v^2+1} + \delta^{2v} = \delta^{v+1}(\delta^{v^2-v} + \delta^{v-1}) = \delta^{2v}\delta^{-2v} = 1,$$

giving (3).

Now multiply (5) by δ^{v-1} to get

$$\delta^{3(v-1)} = \delta^{2(v-1)} + \delta^{v-1} + \delta^{v-3}$$
$$= 1 + \delta^{-2} + \delta^{v-3},$$

using (5). Multiply by δ^{v+3} to get $\delta^{4v} = \delta^{v+3} + \delta^{v+1} + \delta^{2v}$. Apply (2), divide by δ^2 and apply (2) again:

$$\delta^{4v} + \delta^{v+3} = \delta^2 + 1$$

$$\delta^{4v-2} + \delta^{v+1} = 1 + \delta^{-2}$$

$$\delta^{4v-2} + \delta^{-2} = \delta^{2v} + \delta^2.$$
(6)

Next divide (3) by δ

(7)
$$\delta^{v^2} + \delta^{2v-1} = \delta^{-1}.$$

Square (7) and apply (6)

(8)
$$\delta^{2v^2} + \delta^{4v-2} = \delta^{-2}$$
$$\delta^{2v^2} = \delta^{2v} + \delta^2.$$

Now raise (7) to the vth power

$$\delta^{v^3} = \delta^{2v^2 - v} + \delta^{-v}$$

$$= \delta^{-v} (\delta^{2v^2} + 1)$$

$$= \delta^{-v} (\delta^{2v} + \delta^2 + 1) \quad \text{by (8)}$$

$$= \delta^{-v} \delta^{v+1} = \delta.$$

using (2). Hence $\delta \in GF(v^3)$, giving (1). If $\delta \in GF(v)$ then $\delta^{v^2+1} = \delta^2$ and $\delta^{2v} = \delta^2$ also which contradicts (3). Thus $\delta \notin GF(v)$.

Lastly, re-write (3)

$$1 = \delta^{v^2+1} + \delta^{2v} = \delta^{v^3+v^2} + \delta^{2v} = (\delta^{v^2+v} + \delta^2)^v.$$

Hence $\delta^{v^2+v} + \delta^2 = 1$, giving (4). \square

Set

$$Ad(x) = \sum_{\substack{j=1\\d\nmid j}}^{h} x^{q^j}$$

Theorem 2.4. Let $R = \sum_{i=0}^{h} \epsilon_i x^{q^i}$, where each $\epsilon_i \in GF(2)$ and $h = \lfloor (k-1)/2 \rfloor$. Then Q_R^K has radical of codimension 2 iff

- (1) 3|k and R = A3 or x + A3, or
- (2) 4|k and R = A2 or x + A2.

The classification in these cases (assuming the restriction on k) is

$$\begin{split} &\Lambda(Q_{A2}^K) = -1 \\ &\Lambda(Q_{x+A2}^K) = \left\{ \begin{array}{ll} 1, & \text{if t is odd} \\ -1, & \text{if t is even} \end{array} \right. \\ &\Lambda(Q_{A3}^K) = 0 \\ &\lambda(Q_{x+A3}^K) = \left\{ \begin{array}{ll} 1, & \text{if t is even} \\ -1, & \text{if t is odd.} \end{array} \right. \end{split}$$

Recall that $q = 2^t$.

Proof. In the first half of the proof we find all extensions K, all independent $a, b, c \in K$, and all ϵ_i , $i \ge 1$, that satisfy the equations (Ei), for $i \ge 1$, and (Em). We will see that R must be A2, x + A2, A3 or x + A3 with the desired restrictions on k.

Set $u = a^{q-1} + b^{q-1}$ and v = ab. Then (E1) is $uv = \epsilon_1$. If $\epsilon_1 = 0$ then either a = 0, b = 0 or $a^{q-1} = b^{q-1}$ (and so $a = \lambda b$ for some $\lambda \in F$), contradicting the independence of a, b over F. Thus $\epsilon_1 = 1$ and u = 1/v.

Now (E2) is

$$ab((a^{q-1})^{q+1} + (b^{q-1})^{q+1}) = \epsilon_2$$

$$v \left[u^{q+1} + \sum_{i=0}^{t-1} u^{q+1-2^{i+1}} (v^{q-1})^{2^i} \right] = \epsilon_2,$$

using (2.2). Replacing u by 1/v and multiplying by v^q yields

(2.5)
$$\sum_{i=0}^{t-1} v^{2^{i}(q+1)} = \epsilon_2 v^q.$$

We first treat the case of $\epsilon_2 = 0$. Set $w = v^{q+1}$. Then, by (2.5), $w^{q/2} + \cdots + w + 1 = 0$. Hence $w^q = w, w \in F$ and $Tr_F(w) = 1$.

Now the (q+1)st roots of $w \in F$ lie in $L = GF(q^2)$ since if z generates $GF(q^2)^*$ then z^{q+1} generates $GF(q)^*$. Thus $v = ab \in L$. As $\epsilon_2 = 0$ we have $(a/b)^{q^2-1} = 1$ and so $a/b \in L$ also. Thus $a, b \in L$. Now if $a, b \in F$ then they are dependent over F. Hence at least one of a, b is in $L \setminus F$. Say $a \in L \setminus F$. So if $a \in K$ then 2|k.

By construction, $\epsilon_1 = 1$ and $\epsilon_2 = 0$. As $a \in L$ we have

$$a^{q^i} = \begin{cases} a, & \text{if } i \text{ is even} \\ a^q, & \text{if } i \text{ is odd,} \end{cases}$$

and similarly for b. Hence for $i \geq 3$

$$\epsilon_i = a^{q^i}b + ab^{q^i} = \begin{cases} \epsilon_1, & \text{if } i \text{ is odd} \\ \epsilon_2, & \text{if } i \text{ is even.} \end{cases}$$

Thus R = A2 or x + A2.

Lastly, we know k=2m is even so we check (Em). If m is odd then

$$a^{q^m}b = a^qb = a^{q-1}v = a^{q-1}/u \in L \setminus F,$$

so that $a^{q^m}b \notin GF(q^m)$. And if m is even then $a^{q^m}b = ab \in GF(q^2) \subset GF(q^m)$. Thus to have a solution in K we require that m be even, that is, that 4|k.

We now treat the case of $\epsilon_2 = 1$. From (2.5) we have:

$$v^{(q+1)q/2} + v^{(q+1)q/4} + \dots + v^{q+1} + 1 = v^q$$
.

Squaring this gives

(2.6)
$$v^{(q+1)q} = v^{(q+1)q/2} + \dots + v^{(q+1)2} + 1 + v^{2q}$$
$$= v^{q+1} + v^q + v^{2q},$$

by (2.5). Divide this by v^q and then raise to the qth power:

(2.7)
$$v^{q^2} = 1 + v + v^q$$
$$v^{q^3} = 1 + v^q + v^{q^2} = v.$$

Thus $v \in E \equiv GF(q^3)$ and $\operatorname{tr}_{E/F}(v) = 1$.

Now $va^{q-1}=a^qb$ and $vb^{q-1}=ab^q$ sum to 1 and their product is $a^{q+1}b^{q+1}=v^{q+1}$. Thus va^{q-1} and vb^{q-1} are roots of $y^2+y+v^{q+1}\in E[y]$. Now

$$\operatorname{Tr}_{E}(v^{q+1}) = \sum_{i=0}^{t-1} v^{2^{i}(q+1)} + \sum_{i=0}^{t-1} v^{2^{i}(q+1)q} + \sum_{i=0}^{t-1} v^{2^{i}(q+1)q^{2}}$$
$$= (1+v^{q}) + (1+v^{q})^{q} + (1+v^{q})^{q^{2}} \quad \text{by (2.5)}$$
$$= 1 + v^{q} + v^{q^{2}} + v^{q^{3}} = 0,$$

by (2.7). Thus $y^2 + y + v^{q+1}$ has its roots in E, by [7, 3.79]. So a^{q-1} and b^{q-1} are in E. Next, by (2.1)

$$y^{q} = y + v^{q+1} + v^{2(q+1)} + \dots + v^{(q+1)q/2}$$

$$y^{q} = y + 1 + v^{q} \quad \text{by (2.5)}$$

$$y^{q^{2}} = y + v^{q} + v^{q^{2}}$$

$$y^{q^{2}+q} = y^{2} + y(1 + v^{q^{2}}) + v^{q} + v^{q^{2}} + v^{2q} + v^{q^{2}+q}$$

$$= yv^{q^{2}} + v^{q^{2}} \quad \text{by (2.7)}$$

$$y^{q^{2}+q+1} = yv^{q^{2}} + v^{q^{2}+q+1} + yv^{q^{2}} = v^{q^{2}+q+1}.$$

Hence, dividing by v^{q^2+q+1} yields $a^{q^3-1}=1=b^{q^3-1}$. Thus $a,b\in E$. In particular,

$$\epsilon_3 = a^{q^3}b + ab^{q^3} = ab + ab = 0.$$

By construction $\epsilon_1 = 1 = \epsilon_2$. And

$$a^{q^i} = \begin{cases} a, & \text{if } i \equiv 0 \pmod{3} \\ a^q, & \text{if } i \equiv 1 \pmod{3} \\ a^{q^2}, & \text{if } i \equiv 2 \pmod{3}. \end{cases}$$

Thus for $i \geq 3$, $\epsilon_i = \epsilon_j$ where $j \in \{1, 2, 3\}$ and $i \equiv j \pmod{3}$. Hence R = A3 or x + A3.

Again, if $a, b \in F$ then they are dependent over F. Hence at least one of a, b is in $E \setminus F$. Say $a \in E \setminus F$. So if $a \in K$ then 3|k. Finally, if k = 2m is even we must check (Em). But $a^{q^m}b \in E = GF(q^3) \subset GF(q^m)$, as 3|k, so (Em) is satisfied. This completes the first half

In the second half of the proof we show that each of R = A2, x + A2, A3 and x + A3does give a quadratic form with radical of codimension 2 (assuming the restrictions on k) and compute their invariants. We do this by finding explicit solutions to the equations (Ei). There are six cases.

First consider Q_{A2}^K when 4|k. Fix an $s \in F$ with $\mathrm{Tr}_F(s) = 1$. Then $y^2 + y + s \in F[y]$ is irreducible. Let $\alpha \in GF(q^2) \subset K$ be a root. Let β be a primitive element of $GF(q^2)$. Set $b = \beta^{q-1}$ and $a = \alpha b$. These are independent over F as $\alpha \notin F$. We compute

(E0)
$$a^{2} + ab + sb^{2} = b^{2}((a/b)^{2} + (a/b) + s) = 0$$

(E1)
$$a^q b + a b^q = (\alpha^q + \alpha) b^{q+1} = \text{Tr}_F(s) \delta^{q^2 - 1} = 1$$

(E2)
$$a^{q^2}b + ab^{q^2} = ab + ab = 0.$$

Also $ab \in GF(q^2)$ implies $\epsilon_{i+2} = \epsilon_i$ for $i \geq 1$. If k = 2m then $a^{q^m}b \in GF(q^2) \subset GF(q^m)$ as m is even, so that (Em) is satisfied. Hence

$$\operatorname{tr}(ax)^{2} + \operatorname{tr}(ax)\operatorname{tr}(bx) + \operatorname{str}(bx)^{2} = Q_{A2}^{K}(x).$$

By (1.3) Q_{A2}^K has radical with codimension 2 and invariant -1. Next consider Q_{x+A2}^K when 4|k and $q=2^t$ with t odd. Let $\beta\in GF(q^2)\subset K$ be primitive. As t is odd, 3|(q+1). Set

$$a = \beta^{(q-2)(q+1)} \beta^{(q+1)/3}$$
 $b = \beta^{2(q+1)/3}$.

Note that a and b are independent over F as $(b/a)^{q-1} = \beta^{(q^2-1)/3} \neq 1$ so that $b/a \notin F$. We compute

(E0)
$$ab = \beta^{(q-2)(q+1)}\beta^{q+1} = \beta^{q^2-1} = 1$$

(E1)
$$a^{q}b + ab^{q} = a^{q-1} + b^{q-1} = \beta^{(q^{2}-1)/3} + \beta^{2(q^{2}-1)/3} = 1$$

(E2)
$$a^{q^2}b + ab^{q^2} = ab + ab = 0.$$

As in the previous case $\epsilon_{i+2} = \epsilon_i$ for $i \geq 1$ and (Em) is satisfied. Hence $\operatorname{tr}(ax)\operatorname{tr}(bx) =$ $Q_{x+A2}^K(x)$ is, by (1.3), a form of codimension 2 radical and invariant 1.

Next consider Q_{x+A2}^K when 4|k and t even. Fix $s \in F$ with $\mathrm{Tr}_F(s) = 1$. Then $\mathrm{Tr}_F(s+1) = 1$ 1) = Tr_F(s) = 1 as t is even. Thus $x^2 + x + s + 1$ is irreducible over F. Let $\alpha \in GF(q^2) \subset K$ be a root. Set $a = \alpha$ and b = 1; they are independent over F as $\alpha \notin F$. Then

(E0)
$$a^{2} + ab + sb^{2} = \alpha^{2} + \alpha + s = 1$$

(E1)
$$a^q b + a b^q = \alpha^q + \alpha = \operatorname{Tr}_F(s+1) = 1$$

(E2)
$$a^{q^2}b + ab^{q^2} = ab + ab = 0.$$

Again $\epsilon_{i+2} = \epsilon_i$ for $i \ge 1$ and (Em) is satisfied. Hence $\operatorname{tr}(ax)^2 + \operatorname{tr}(ax)\operatorname{tr}(bx) + \operatorname{str}(bx)^2 = Q_{x+A2}^K(x)$ is, by (1.3), a form of codimension 2 radical and invariant -1.

We now consider Q_{A3}^K when 3|k. Let 3^r be the highest power of 3 dividing t so that $t=3^rt_0$ with $(3,t_0)=1$. Set $v=2^{3^r}$ so that $q=v^{t_0}$. Let δ be a root of the polynomial g_v of (2.3). Then $\delta \in GF(v^3) \subset GF(q^3) \subset K$ by (2.3)(1). Set $a=\delta^v$, $b=\delta$ and $c=\delta^v+\delta+1$. We first check that a,b,c are independent over F=GF(q). If not then 1 is in the F-span of a and b. Hence a=gb+h for some $g,h\in F$ and so $\delta^v=g\delta+h$. We plug into (2.3)(2):

(2.8)
$$\delta^{2v} + \delta^{v+1} + \delta^2 = 1$$
$$h^2 + h\delta + (1 + g + g^2)\delta^2 = 1.$$

Now $\delta \notin GF(v)$, by (2.3)(1), and so has degree 3 over GF(v). As $(3, t_0) = 1$, δ has degree 3 over $F = GF(v^{t_0})$ as well. Thus $1, \delta, \delta^2$ are independent over F. Then (2.8) gives $h^2 = 1$ and h = 0, a contradiction. Thus a, b, c are independent over F.

We compute (E0)

$$c^2 + ab = 1 + \delta^2 + \delta^{2v} + \delta^{v+1} = 0$$
 by (2.3)(2).

For the other equations, first suppose $t_0 \equiv 1 \pmod{3}$. Then $\delta^q = \delta^{v^{t_0}} = \delta^v$ as $\delta^{v^3} = \delta$. Similarly, $\delta^{q^2} = \delta^{v^{2t_0}} = \delta^{v^2}$. Then

(E1)
$$a^q b + a b^q = \delta^{v^2 + 1} + \delta^{2v} = 1 \text{ by } (2.3)(3)$$

(E2)
$$a^{q^2}b + ab^{q^2} = \delta^{v^3+1} + \delta^{v^2+v} = \delta^2 + \delta^{v^2+v} = 1 \text{ by } (2.3)(4)$$

(E3)
$$a^{q^3}b + ab^{q^3} = ab + ab = 0.$$

When $t_0 = 2 \pmod{3}$ then $\delta^q = \delta^{v^2}$ and $\delta^{q^2} = \delta^v$. Then

(E1)
$$a^{q}b + ab^{q} = \delta^{v^{3}+1} + \delta^{v^{2}+v} = 1$$

(E2)
$$a^{q^2}b + ab^{q^2} = \delta^{v^2+1} + \delta^{2v} = 1$$

(E3)
$$a^{q^3}b + ab^{q^3} = ab + ab = 0.$$

Also $\epsilon_{i+3} = \epsilon_i$ for $i \ge 1$ and if k = 2m then $a^{q^m}b \in GF(q^3) \subset GF(q^m)$ so that (Em) holds. Hence Q_{A3}^K has radical of codimension 2 and invariant 0.

Next consider Q_{x+A3}^K when 3|k and t is odd. Since t is odd we can pick s=1 as our element of F with absolute trace 1. Let v, δ , a and b be as in the previous case. We know a, b are independent over F and $\epsilon_1 = 1 = \epsilon_2$, $\epsilon_3 = 0$, $\epsilon_{i+3} = \epsilon_i$ for $i \geq 1$ and that (Em) holds. We need only check (E0):

$$a^{2} + ab + b^{2} = \delta^{2v} + \delta^{v+1} + \delta^{2} = 1$$

by (2.3)(2). Hence Q_{x+A3}^K has radical of codimension 2 and invariant -1.

Lastly, we consider Q_{x+A3} when 3|k and t is even. Then $GF(q^3) \subset K$; let γ be a primitive element of $GF(q^3)$. As t is even, 3 divides $q^2 + q + 1$. Set $\varphi = \gamma^{(q^2 + q + 1)/3}$. Then φ has order 3(q-1) so that $\varphi^{2(q-1)} + \varphi^{q-1} + 1 = 0$. Set $a = \varphi^{-2}$ and $b = \varphi^2$. They are independent over F as $(b/a)^{q-1} = \varphi^{4(q-1)} = \varphi^{q-1} \neq 1$ so that $b/a \notin F$. We compute

$$(E0) ab = 1$$

(E1)
$$a^{q}b + ab^{q} = a^{q-1} + b^{q-1} = \varphi^{q-1} + \varphi^{2(q-1)} = 1$$

(E2)
$$a^{q^2}b + ab^{q^2} = (\varphi^{q-1})^{q+1} + (\varphi^{q-1})^{q+1} = \varphi^{q-1} + \varphi^{2(q-1)} = 1$$

(E3)
$$a^{q^3}b + ab^{q^3} = ab + ab = 0.$$

Also $\epsilon_{i+3} = \epsilon_i$ for $i \geq 1$ and if k = 2m then $a^{q^m}b \in GF(q^3) \subset GF(q^m)$ so that (Em) holds. Hence Q_{x+A3}^K has radical of codimension 2 and invariant 1. \square

3. Artin-Schreier curves with many rational points.

We again consider polynomials

$$R(x) = \sum_{i=0}^{h} \epsilon_i x^{q^i},$$

with each $\epsilon_i \in GF(2) = F$ and $h = \lfloor k - 1/2 \rfloor$. The Artin-Schreier curve is

$$C_R: y^q + y = xR(x).$$

This has genus $g = \frac{1}{2}(q-1) \deg R(x)$ by [8, VI.4.1]. We consider both the curve and the quadratic form over K. The number of points in K-projective space on C_R is

$$\#C_R(K) = qN(Q_R^K) + 1 = q^k + \Lambda(Q_R^K)(q-1)\sqrt{q^{k+w}} + 1,$$

where $w = \dim \operatorname{rad}(Q_R^K)$. We will compare this to the Hasse-Weil bound

$$\#C_R(K) \le q^k + 1 + 2g\sqrt{q^k} = q^k + 1 + (q-1)q^\ell\sqrt{q^k},$$

where $\ell = \deg R(x)$. Clearly equality will hold in the Hasse-Weil bound only if k is even.

Theorem 3.1. Suppose k = 2m and the top coefficient $\epsilon_{m-1} = 1$. Then the number of points on C_R equals the Hasse-Weil bound iff one of the following holds

- (1) t is odd, R = x + A2 and 4|k,
- (2) t is even, R = x + A3 and 6|k.

Proof. Note that deg $R(x) = \ell = m-1$. The number of points on C_R equals the Hasse-Weil bound iff

$$\begin{split} \Lambda(Q_R^K)(q-1)\sqrt{q^{k+w}} &= (q-1)q^{m-1}\sqrt{q^k}\\ &\Lambda(Q_R^K)\sqrt{q^w} &= q^{m-1}\\ &w = 2(m-1) = k-2\quad\text{and}\quad \Lambda(Q_R^K) = 1. \end{split}$$

This holds, by (1.7), iff either (1) or (2) hold. \square

The restriction that $\epsilon_{m-1} = 1$ is necessary.

Example. Let k = 12 so that $\ell = 5$. Set $R = x + x^4 + x^{16}$. Then $\epsilon_4 = 1$ and $\epsilon_5 = 0$. In particular, the genus of C_R is $g = 2^{4-1} = 8$. Also dim rad $(Q_R^K) = 8$ and $\Lambda(Q_R^K) = 1$. This may be checked as follows:

Let δ satisfy $\delta^6 = \delta + 1$. Set

$$a_1 = \delta^{28}$$
 $b_1 = \delta^{56}$ $a_2 = \delta^7$ $b_2 = \delta^{35}$.

Then $a_1b_1 + a_2b_2 = 1$. Set

$$\epsilon_i = a_1^{2^i} b_1 + a_1 b_1^{2^i} + a_2^{2^i} b_2 + a_2 b_2^{2^i}.$$

Then we may compute that $\epsilon_1 = 0$, $\epsilon_2 = 1$, $\epsilon_3 = 0$, $\epsilon_4 = 1$, $\epsilon_5 = 0$, $\epsilon_6 = 0$ and $\epsilon_{i+6} = \epsilon_i$ for $i \geq 1$. Thus

$$R = \operatorname{tr}(a_1 x) \operatorname{tr}(b_1 x) + \operatorname{tr}(a_2 x) \operatorname{tr}(b_2 x)$$
$$Q_R^K \simeq x_1 x_2 + x_3 x_4,$$

giving the stated dimension of the radical and the invariant Λ .

Now

$$N(Q_R^K) = \frac{1}{2}(2^{12} + \sqrt{2^{12+8}}) = \frac{1}{2}(2^{12} + 2^{10})$$

$$\#C_R(K) = 1 + 2^{12} + 2^{10}.$$

The Hasse-Weil bound is $1 + 2^{12} + 2 \cdot 8\sqrt{2^{12}} = 1 + 2^{12} + 2^{10}$. Hence there are other Artin-Schreier curves meeting the Hasse-Weil bound besides those of (3.1).

4. Factoring linearized polynomials.

Here we will restrict to the case q=2. For $R=\sum_{j=0}^{h}\epsilon_{j}x^{2^{j}}$ define

$$R^*(x) = \sum_{j=1}^{h} \epsilon_j (x^{2^{h+j}} + x^{2^{h-j}}).$$

Then by [4,Lemma 8]

$$radQ_{R}^{K} = \{ a \in K : R^{*}(a) = 0 \}.$$

Notice that R^* is a self-reciprocal, linearized polynomial and that any self-reciprocal, linearized polynomial of degree 2^{2h} arises in this way. If S is a self-reciprocal, linearized polynomial we will say T is the associated form if T is linearized and $T^* = S$.

Proposition 4.1. Suppose k is even and 2h = k - 2. Let S be a self-reciprocal, linearized polynomial of degree 2h with associated form T. The following are equivalent:

- (1) S divides $x^{2^k} + x$.
- (2) All irreducible factors of S have degree d, where d divides k.
- (3) Either 6|k and T = A3; or 4|k and T = A2.

Proof. (1) \leftrightarrow (2) is clear. (1) implies every root of S lies in $K = GF(2^k)$. Since Q_T^K has a radical consisting of the roots of S in K, we have dim rad $Q_T^K = k-2$ and so of codimension 2. This gives (3). Conversely, (3) gives Q_T^K has codimension 2 radical and so every root of S lies in K. \square

Proposition 4.2. Let k be odd and 2h = k - 1. Let S be a self-reciprocal, linearized polynomial of degree 2h with associated form T. The following are equivalent:

- (1) S divides $(x^{2^k} + x)(x^{2^k} + x + 1)$.
- (2) Every irreducible factor of S either has degree d (where d|k), or has the form $p(x^2 + x + 1)$, where p is irreducible of degree d (where again d|k).
- (3) 3|k and T = A3.

Proof. (1) \to (2). Let q(x) be an irreducible factor of S. Then q divides q_k or $q_k + 1$, where $q_k = x^{2^k} + x$. In the first case, we have $\deg q = d$, where d|k. So suppose we are in the second case.

Now the roots of S not in K look like $a + \beta$, where $a \in K$ is a root and $\beta^2 = \beta + 1$. Namely, say $S(\alpha) = 0$ and $\alpha \notin K$. Then $q_k(\alpha) = 1$. Now

$$\beta^{2^j} = \left\{ \begin{array}{ll} \beta, & \text{if } j \text{ is even} \\ \beta^2, & \text{if } j \text{ is odd.} \end{array} \right.$$

In particular, $q_k(\beta) = 1$. Since either both $h \pm i$ are even or both are odd, we have $\beta^{h+i} + \beta^{h-i} = 0$ and so $S(\beta) = 0$. S and q_k are linearized so that their roots are additive. Hence $S(\alpha + \beta) = 0$ and $q_k(\alpha + \beta) = 0$. Thus $a = \alpha + \beta$ is a root of S in K.

Pick a root of q(x), say $a + \beta$, where $a \in K$ is also a root of S. Now $a^2 + a$ is also a root of S. Let p(x) be the irreducible polynomial of $a^2 + a$. Set $d = \deg p$; note that d|k. Set $q_0(x) = p(x^2 + x + 1)$. Now

$$q_0(a+\beta) = p(a^2 + a + \beta^2 + \beta + 1) = p(a^2 + a) = 0.$$

Thus q(x) divides $q_0(x)$. We will be done if we show deg q = 2d, the same as deg q_0 . Now deg $q = [F(a + \beta) : F]$. We have

$$(a + \beta)^2 + (a + \beta) + 1 = a^2 + a.$$

Hence

$$F \subset F(a^2 + a) \subset F(a + \beta).$$

Moreover, if $a + \beta \in F(a^2 + a)$ then $\beta \in F(a)$. But $a \in K$ and [K : F] is odd, so this is impossible. Hence

$$[F(a+\beta):F] \ge 2[F(a^2+a):F] = 2\deg p = 2d.$$

Thus $q(x) = q_0(x) = p(x^2 + x + 1)$.

 $(2) \to (1)$. Let π_1 be the product of irreducible factors of S that are of degree d, with d|k. Then $\pi_1|q_k$. Let π_2 be the product of the irreducible factors of S of type $p(x^2+x+1)$, with p irreducible of degree d, d|k. Let π_3 be the product of the p's. Then $\pi_2(x) = \pi_3(x^2+x+1)$ and $\pi_3|q_k$. Hence π_2 divides

$$q_k(x^2 + x + 1) = x^{2^{k+1}} + x^{2^k} + x^2 + x = q_k^2 + q_k = q_k(q_k + 1).$$

Moreover, no root of π_2 is in K (as each irreducible factor has even degree). Thus π_2 divides $q_k + 1$. And so $S = \pi_1 \pi_2$ divides $q_k (q_k + 1)$.

- $(1) \to (3)$. Let A denote the roots of S that are also roots of q_k and let B be the roots of S that are also roots of $q_k + 1$. As before, $S(\beta) = 0$ and $\beta \notin K$. The map $A \to B$ by $a \mapsto a + \beta$ is bijective. Hence $|A| = 2^{k-2}$. Now rad $Q_T^K = A$ and so has codimension 2. Apply the main theorem (2.4) to get (3).
- $(3) \to (1)$. We have that the codimension of $\operatorname{rad} Q_T^K$ is 2 so that 2^{k-2} roots of S lie in K. The other roots of S are $a+\beta$, for $a\in K$ a root of S. Now each root $a\in K$ is a root of g_k . And for each $a+\beta$ we have

$$(a+\beta)^{2^k} + (a+\beta) + 1 = (a^{2^k} + a) + (\beta^{2^k} + \beta + 1) = 0,$$

as k is odd. So S divides $q_k(q_k+1)$. \square

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