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The Stable Manifold Theorem for Stochastic Systems with Memory (Probability Seminar, Université Henri Poincaré Nancy 1)

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THE STABLE MANIFOLD THEOREM

FOR

STOCHASTIC SYSTEMS WITH MEMORY

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Outline

- Smooth cocycles in Hilbert space. Stationary trajectories.
- Linearization of a cocycle along a stationary trajectory.
- Ergodic theory of cocycles in Hilbert space.
- Hyperbolicity of stationary trajectories. Lyapunov exponents.
- Cocycles generated by stochastic systems with memory. Via random diffeomorphism groups.
- The *Local Stable Manifold Theorem* for stochastic differential equations with memory (SFDE's): Existence of smooth stable and unstable manifolds in a neighborhood of a hyperbolic stationary trajectory.
- Proofs based on Ruelle-Oseledec (non-linear) multiplicative ergodic theory and perfection techniques.

The Cocycle

$(\Omega, \mathcal{F}, P) :=$ complete probability space.

$\theta : \mathbf{R}^+ \times \Omega \to \Omega$ a P-preserving (ergodic) semigroup on $(\Omega, \mathcal{F}, P)$.

$E :=$ real (separable) Hilbert space, norm $\|\cdot\|$, Borel σ-algebra.

Definition.

Let k be a non-negative integer and $\epsilon \in (0, 1]$. A $C^{k,\epsilon}$ *perfect cocycle* (X, θ) on E is a measurable random field $X : \mathbf{R}^+ \times E \times \Omega \to E$ such that:

(i) For each $\omega \in \Omega$, the map $\mathbf{R}^+ \times E \ni (t, x) \mapsto X(t, x, \omega) \in E$ is continuous; for fixed $(t, \omega) \in \mathbf{R}^+ \times \Omega$, the map $E \ni x \mapsto X(t, x, \omega) \in E$ is $C^{k,\epsilon}$.

(ii) $X(t+s, \cdot, \omega) = X(t, \cdot, \theta(s, \omega)) \circ X(s, \cdot, \omega)$ for all $s, t \in \mathbf{R}^+$ and all $\omega \in \Omega$.

(iii) $X(0, x, \omega) = x$ for all $x \in E, \omega \in \Omega$.

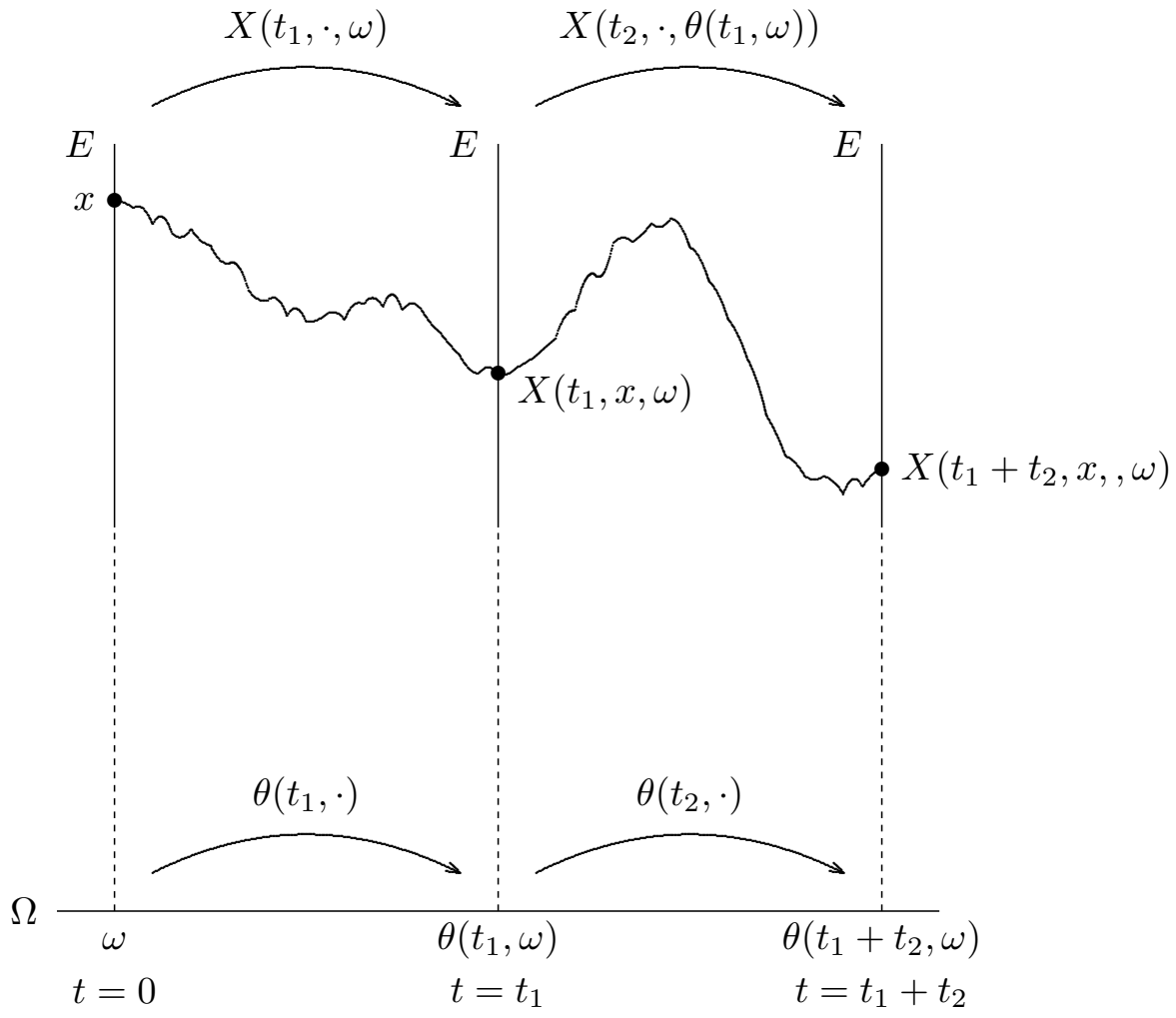


Figure illustrates the cocycle property. Vertical solid lines represent random fibers consisting of copies of E. (X, θ) is a "vector-bundle morphism".

Definition

The cocycle (X, θ) has a *stationary point* if there exists a random variable $Y : \Omega \to E$ such that

$$X(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{1}$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $X(t, Y) = Y(\theta(t))$.

Linearization. Hyperbolicity.

Linearize a $C^{k,\epsilon}$ cocycle (X, θ) along a stationary random point Y: Get an $L(E)$-valued cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega))$. (Follows from cocycle property of X and chain rule.)

Theorem. *(Oseledec-Ruelle)*

Let $T : \mathbf{R}^+ \times \Omega \to L(E)$ be strongly measurable, such that (T, θ) is an $L(E)$-valued cocycle, with each $T(t, \omega)$ compact. Suppose that

$$E \sup_{0 \le t \le 1} \log^+ \|T(t, \cdot)\|_{L(E)} < \infty, \quad E \sup_{0 \le t \le 1} \log^+ \|T(1-t, \theta(t, \cdot))\|_{L(E)} < \infty.$$

Then there is a sure event $\Omega_0 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \in \mathbf{R}^+$, and for each $\omega \in \Omega_0$, the limit

$$\lim_{n \to \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. $\Lambda(\omega)$ is self-adjoint with a non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots$$

where the λ_i's are distinct. Each e^{λ_i} has a fixed finite non-random multiplicity m_i and eigen-space $F_i(\omega)$, with $m_i := dim F_i(\omega)$. Define

$$E_1(\omega) := E, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \ i > 1.$$

Then

$$\cdots \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$\lim_{t\to\infty} \frac{1}{t} \log \|T(t,\omega)x\| = \lambda_i(\omega) \quad \textit{if} \quad x \in E_i(\omega) \backslash E_{i+1}(\omega),$$

and

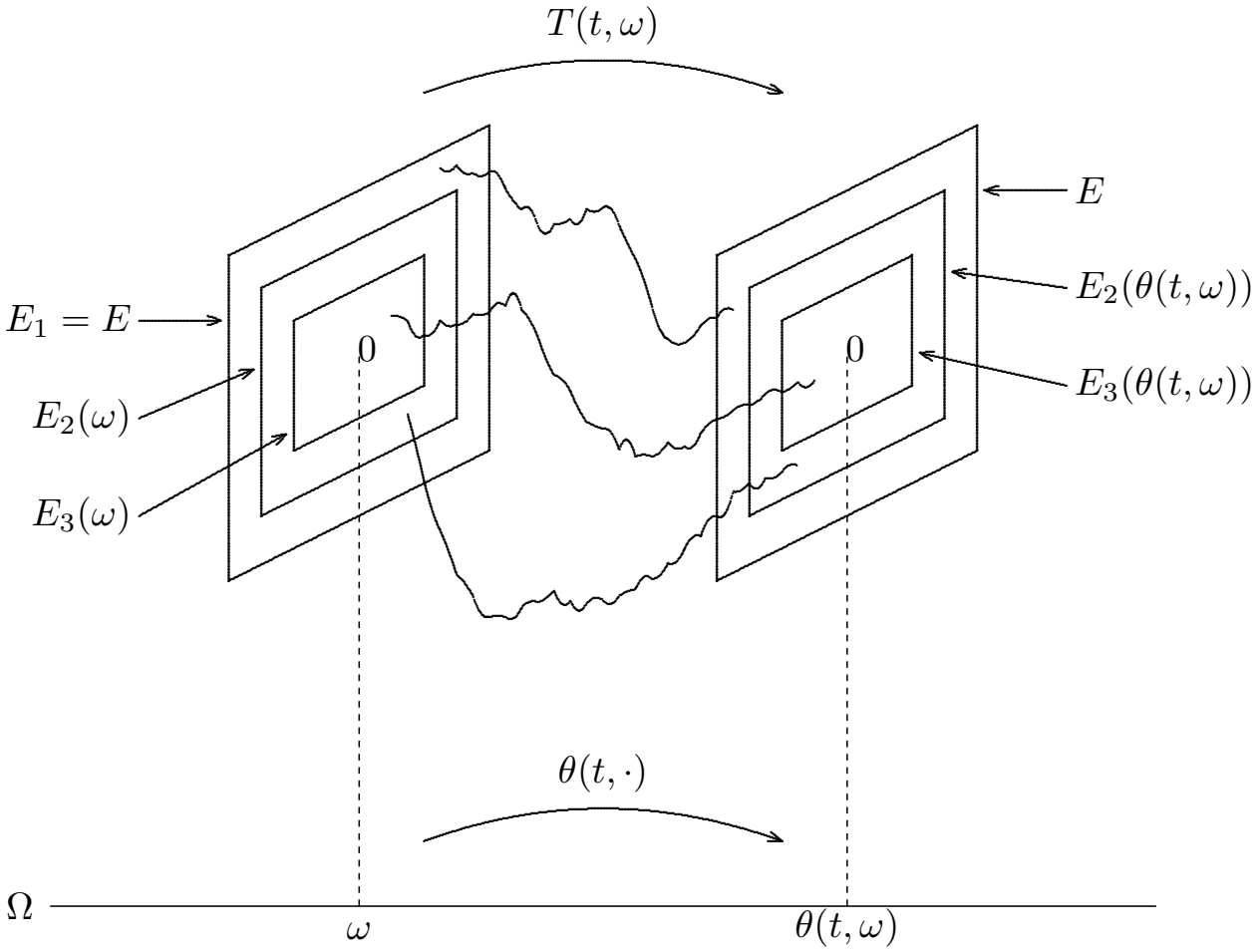
$$T(t,\omega)(E_i(\omega)) \subseteq E_i(\theta(t,\omega))$$

for all $t \geq 0, \ i \geq 1$.

Proof.

Based on the discrete version of Oseledec's multiplicative ergodic theorem and the perfect ergodic theorem. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]). □

Spectral Theorem



Definition

A stationary point $Y(\omega)$ of (I) is said to be *hyperbolic* if the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega))$ has a non-

vanishing Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, viz. $\lambda_i \neq 0$ for all $i \geq 1$.

Let $i_0 > 1$ be such that $\lambda_{i_0} < 0 < \lambda_{i_0-1}$.

Suppose

$$E \log^+ \sup_{0 \leq t_1, t_2 \leq r} \|D_2 X(t_2, Y(\theta(t_1)), \theta(t_1))\|_{L(M_2)}) < \infty.$$

By Oseledec-Ruelle Theorem, there is a sequence of closed finite-codimen-sional (Oseledec) spaces

$$\cdots E_{i-1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$E_i(\omega) = \{(v, \eta) \in M_2 : \lim_{t \to \infty} \frac{1}{t} \log \|DX(t, Y(\omega), \omega)(v, \eta)\| \leq \lambda_i\}, \quad i \geq 1,$$

for all $\omega \in \Omega^*$, a sure event in $\mathcal{F}$ satisfying $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$.

Denote by $\{U(\omega), S(\omega) : \omega \in \Omega^*\}$ the unstable and stable subspaces associated with the linearized cocycle (DX, θ) as given by ([Mo.1], Theorem 4, Corollary 2) and ([M-S.1], Theorem 5.3). Then get a measurable invariant splitting

$$E = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \Omega^*,$$

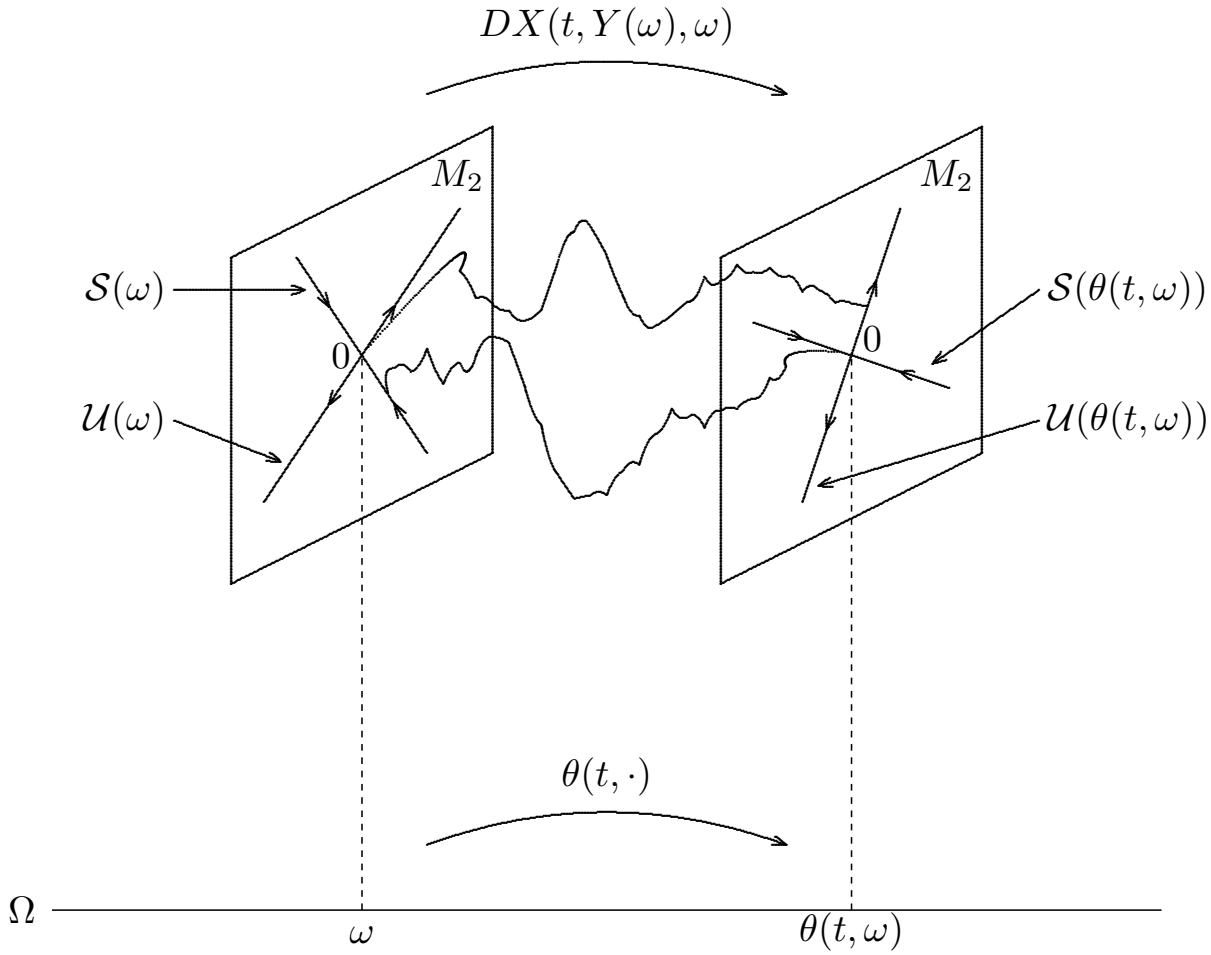
$$DX(t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)),\ DX(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)),$$

for all $t \geq 0$, together with the exponential dichotomies

$$\|DX(t, Y(\omega), \omega)(x)\| \geq \|x\| e^{\delta_1 t} \quad \text{for all} \quad t \geq \tau_1^*, x \in \mathcal{U}(\omega),$$

$$\|DX(t, Y(\omega), \omega)(x)\| \leq \|x\| e^{-\delta_2 t} \quad \text{for all} \quad t \geq \tau_2^*, x \in \mathcal{S}(\omega),$$

where $\tau_i^* = \tau_i^*(x, \omega) > 0, i = 1, 2,$ are random times and $\delta_i > 0, i = 1, 2,$ are fixed.


$DX(t, Y(\omega), \omega)$
M_2
M_2
$\mathcal{S}(\omega)$
$\mathcal{S}(\theta(t,\omega))$
0
0
$\mathcal{U}(\omega)$
$\mathcal{U}(\theta(t,\omega))$
$\theta(t,\cdot)$
Ω
ω
$\theta(t,\omega)$

Stochastic Systems with Memory

"Regular" Itô SFDE with finite memory:

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t)\,dt + \sum_{i=1}^{m} G_i(x(t))\,dW_i(t), \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (I)$$

Solution segment $x_t(s) := x(t+s),\ t \geq 0, s \in [-r, 0]$.

m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$, $W(0) = 0$.

Ergodic Brownian shift θ on Wiener space $(\Omega, \mathcal{F}, P)$. $\bar{\mathcal{F}} := P-$completion of $\mathcal{F}$.

State space M_2, Hilbert with usual norm $\|\cdot\|$.

Can allow for "smooth memory" in diffusion coefficient.

$H : M_2 \to \mathbf{R}^d$ of class $C^{k,\delta}$, globally bounded.

$G : \mathbf{R}^d \to L(\mathbf{R}^p, \mathbf{R}^d)$ is of class $C_b^{k+1,\delta}$.

$B((v, \eta), \rho)$ open ball of radius ρ and center $(v, \eta) \in M_2$;

$\bar{B}((v,\eta),\rho)$ corresponding closed ball.

Then (I) has a stochastic semiflow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ with $X(t,(v,\eta),\cdot) = (x(t), x_t)$. X is of class $C^{k,\epsilon}$ for any $\epsilon \in (0,\delta)$, takes bounded sets into relatively compact sets in M_2. (X,θ) is a perfect cocycle on M_2 ([M-S.4]).

Theorem. *([M-S], 1999) (Local Stable and Unstable Manifolds)*

Assume smoothness hypotheses on H and G. Let $Y : \Omega \to M_2$ be a hyperbolic stationary point of the SFDE (I) such that $E(\|Y(\cdot)\|^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$

Suppose the linearized cocycle $(DX(t,Y(\omega),\omega),\theta(t,\omega),t \geq 0)$ of (I) has a Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. Define $\lambda_{i_0} := max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all finite λ_i are positive, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized semiflow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,

(ii) *$\mathcal{F}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0,1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:*

There are $C^{k,\epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds $\tilde{\mathcal{S}}(\omega)$, $\tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) $\tilde{\mathcal{S}}(\omega)$ is the set of all $(v, \eta) \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$\|X(n, (v, \eta), \omega) - Y(\theta(n, \omega))\| \leq \beta_1(\omega)\, e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega) - Y(\theta(t, \omega))\| \leq \lambda_{i_0}$$

for all $(v, \eta) \in \tilde{\mathcal{S}}(\omega)$. Each stable subspace $\mathcal{S}(\omega)$ of the linearized semiflow DX is tangent at $Y(\omega)$ to the submanifold $\tilde{\mathcal{S}}(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. In particular, codim $\tilde{\mathcal{S}}(\omega)$ = codim $\mathcal{S}(\omega)$, is fixed and finite.

(b) $\limsup_{t \to \infty} \frac{1}{t} \log \left[\sup \left\{ \frac{\|X(t, (v_1, \eta_1), \omega) - X(t, (v_2, \eta_2), \omega)\|}{\|(v_1, \eta_1) - (v_2, \eta_2)\|} : (v_1, \eta_1) \neq (v_2, \eta_2),\ (v_1, \eta_1),\ (v_2, \eta_2) \in \tilde{\mathcal{S}}(\omega) \right\} \right] \leq \lambda_{i_0}$.

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ *such that*

$$X(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega))$$

for all $t \geq \tau_1(\omega)$*. Also*

$$DX(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)), \quad t \geq 0.$$

(d) $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $(v, \eta) \in \bar{B}(Y(\omega), \rho_2(\omega))$ *with the property that there is a unique "history" process* $y(\cdot, \omega) : \{-nr : n \geq 0\} \to M_2$ *such that* $y(0, \omega) = (v, \eta)$ *and for each integer* $n \geq 1$*, one has* $X(r, y(-nr, \omega), \theta(-nr, \omega)) = y(-(n-1)r, \omega)$ *and*

$$\|y(-nr, \omega) - Y(\theta(-nr, \omega))\|_{M_2} \leq \beta_2(\omega) e^{-(\lambda_{i_0-1} - \epsilon_2)nr}.$$

Furthermore, for each $(v, \eta) \in \tilde{\mathcal{U}}(\omega)$*, there is a unique continuous-time "history" process also denoted by* $y(\cdot, \omega) : (-\infty, 0] \to M_2$ *such that* $y(0, \omega) = (v, \eta)$*,* $X(t, y(s, \omega), \theta(s, \omega)) = y(t + s, \omega)$ *for all* $s \leq 0, 0 \leq t \leq -s$*, and*

$$\limsup_{t \to \infty} \frac{1}{t} \log \|y(-t, \omega) - Y(\theta(-t, \omega))\| \leq -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}(\omega)$ of the linearized semiflow DX is tangent at $Y(\omega)$ to $\tilde{\mathcal{U}}(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. In particular, $\dim \tilde{\mathcal{U}}(\omega)$ is finite and non-random.

(e) Let $y(\cdot, (v_i, \eta_i), \omega), i = 1, 2$, be the history processes associated with $(v_i, \eta_i) = y(0, (v_i, \eta_i), \omega) \in \tilde{\mathcal{U}}(\omega)$, $i = 1, 2$. Then

$$\limsup_{t\to\infty} \frac{1}{t} \log\left[\sup\left\{ \frac{\|y(-t, (v_1, \eta_1), \omega) - y(-t, (v_2, \eta_2), \omega)\|}{\|(v_1, \eta_1) - (v_2, \eta_2)\|} : \right.\right.$$
$$\left.\left. (v_1, \eta_1) \neq (v_2, \eta_2),\ (v_i, \eta_i) \in \tilde{\mathcal{U}}(\omega), i = 1, 2 \right\}\right] \leq -\lambda_{i_0 - 1}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ such that

$$\tilde{\mathcal{U}}(\omega) \subseteq X(t, \cdot, \theta(-t, \omega))(\tilde{\mathcal{U}}(\theta(-t, \omega)))$$

for all $t \geq \tau_2(\omega)$. Also

$$DX(t, \cdot, \theta(-t, \omega))(\mathcal{U}(\theta(-t, \omega))) = \mathcal{U}(\omega), \quad t \geq 0;$$

and the restriction

$$DX(t, \cdot, \theta(-t, \omega))|\mathcal{U}(\theta(-t, \omega)) : \mathcal{U}(\theta(-t, \omega)) \to \mathcal{U}(\omega), \quad t \geq 0,$$

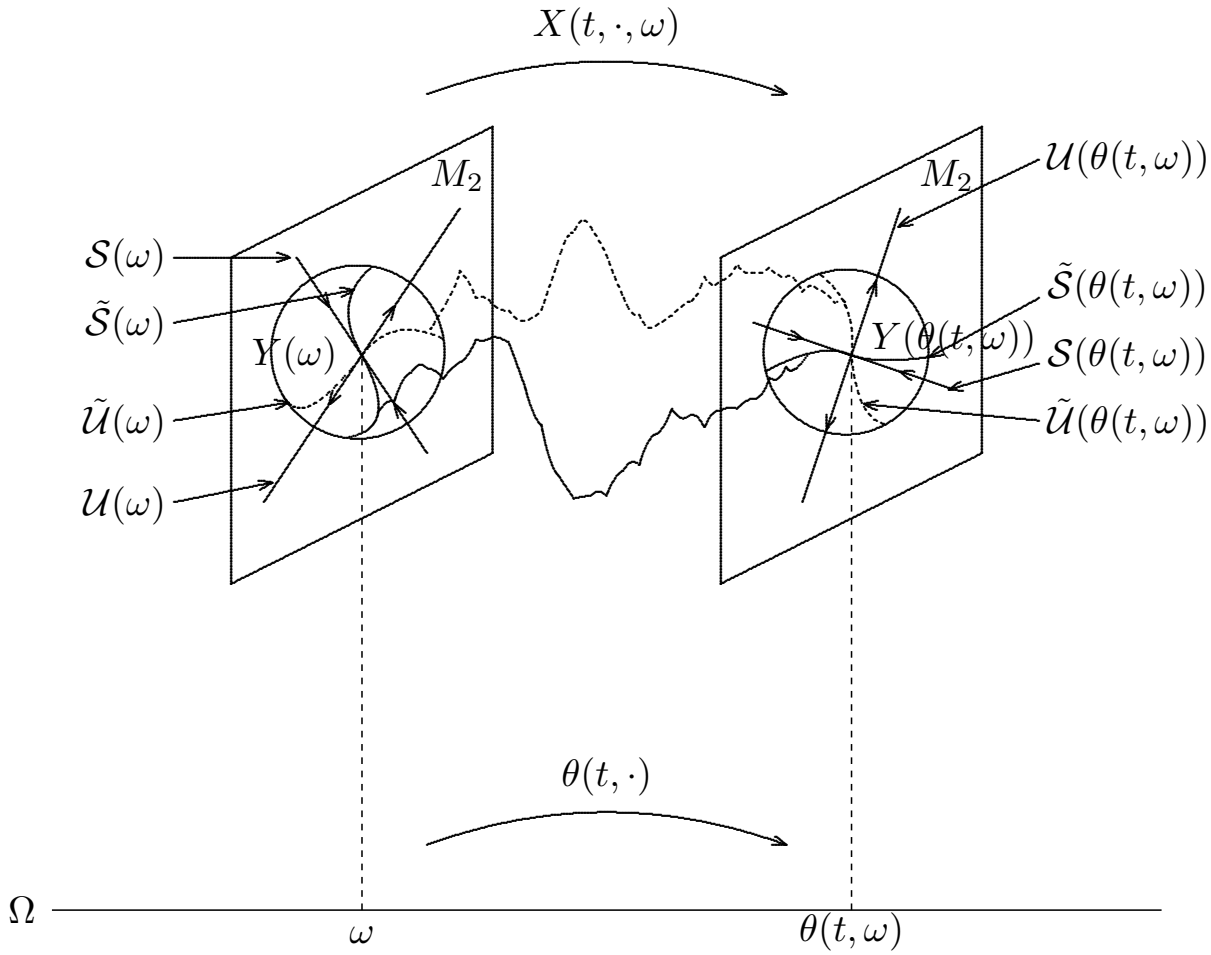
is a linear homeomorphism onto.

(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ *and* $\tilde{\mathcal{S}}(\omega)$ *are transversal, viz.*

$$M_2 = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

Assume, in addition, that H, G *are* C_b^∞*. Then the local stable and unstable manifolds* $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ *are* C^∞*.*

Figure summarizes essential features of Stable Manifold Theorem:



$t > \tau_1(\omega)$

A picture is worth a 1000 words!

Example

Consider the affine linear sfde

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t)\,dt + G\,dW(t), \quad t > 0 \\ x(0) &= v \in \mathbf{R}^d, \quad x_0 = \eta \in L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (I'')$$

where $H : M_2 \to \mathbf{R}^d$ is a continuous linear map, G is a fixed $(d \times p)$-matrix, and W is p-dimensional Brownian motion. Assume that the linear deterministic $(d \times d)$-matrix-valued FDE

$$dy(t) = H \circ (y(t), y_t)\,dt$$

has a semiflow

$$T_t : L(\mathbf{R}^d) \times L^2([-r, 0], L(\mathbf{R}^d)) \to L(\mathbf{R}^d) \times L^2([-r, 0], L(\mathbf{R}^d)), t \geq 0,$$

which is uniformly asymptotically stable. Set

$$Y := \int_{-\infty}^{0} T_{-u}(I, 0) G\,dW(u) \qquad (2)$$

where I is the identity $(d \times d)$-matrix. Integration by parts and

$$W(t, \theta(t_1, \omega)) = W(t + t_1, \omega) - W(t_1, \omega), \quad t, t_1 \in \mathbf{R}, \qquad (3)$$

imply that Y has a measurable version satisfying (1). Y is Gaussian and thus has finite moments of all orders. See

([Mo.1], Theorem 4.2, Corollary 4.2.1, pp. 208-217.) More generally, when H is hyperbolic, one can show that a stationary point of (I'') exists ([Mo.1]).

In the general white-noise case an invariant measure on M_2 for the one-point motion gives rise to a stationary point provided we suitably enlarge the underlying probability space. Conversely, let $Y : \Omega \to M_2$ be a stationary random point independent of the Brownian motion $W(t), t \geq 0$. Let $\rho := P \circ Y^{-1}$ be the distribution of Y. By independence of Y and W, ρ is an invariant measure for the one-point motion

Outline of Proof

- By definition, a *stationary* random point $Y(\omega) \in M_2$ is invariant under the semiflow X; viz $X(t, Y) = Y(\theta(t, \cdot))$ for all times t.
- We linearize the semiflow X along the stationary point $Y(\omega)$ in M_2. In view of the stationarity of Y and the cocycle property of X, this gives a linear perfect cocycle $(DX(t, Y), \theta(t, \cdot))$ in $L(M_2)$, where D denotes spatial (Fréchet) derivatives.
- Ergodicity of θ allows for the notion of *hyperbolicity* of a stationary solution of (I) via Oseledec-Ruelle theorem: Use local compactness of the semiflow for times greater than the delay r ([M-S.4]), and apply multiplicative ergodic theorem in order to yield a discrete non-random Lyapunov spectrum $\{\lambda_i : i \geq 1\}$ for the linearized cocycle. Y is *hyperbolic* if $\lambda_i \neq 0$ for every i.
- Assuming that $\|Y\|^{\epsilon_0}$ is integrable (for small ϵ_0) and using the variational method of construction of the semiflow, we show that the linearized cocycle satisfies the hypotheses for "perfect versions" of ergodic theorem and Kingman's subadditive ergodic theorem. These refined versions yield invariance of the Oseledec

spaces under the continuous-time linearized cocycle. In particular, the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear semiflow X.

- We establish continuous-time integrability estimates on the spatial derivatives of the non-linear cocycle X in a neighborhood of the stationary point Y. These esitmates follow from the variational construction of the stochastic semiflow coupled with known global spatial estimates for finite-dimensional stochastic flows.
- We introduce the auxiliary perfect cocycle

$$Z(t,\cdot,\omega) := X(t,(\cdot)+Y(\omega),\omega) - Y(\theta(t,\omega)), \quad t \in \mathbf{R}^+, \omega \in \Omega.$$

By refining the arguments in ([Ru.2], Theorems 5.1 and 6.1), we construct local stable/unstable manifolds for the discrete cocycle $(Z(nr,\cdot,\omega),\theta(nr,\omega))$ near 0 and hence (by translation) for $X(nr,\cdot,\omega)$ near $Y(\omega)$ for all ω sampled from a $\theta(t,\cdot)$-invariant sure event in Ω. This is possible because of the continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between delay periods of length r and further refining the arguments in [Ru.2], we then

show that the above manifolds also serve as local stable/unstable manifolds for the *continuous-time* semiflow X near Y.

- The final key step is to establish the asymptotic invariance of the local stable manifolds under the stochastic semiflow X. This is achieved by appealing to the arguments underlying the proofs of Theorems 4.1 and 5.1 in [Ru.2] and some difficult estimates using the continuous-time integrability properties, and the perfect subadditive ergodic theorem. The asymptotic invariance of the local unstable manifolds follows by employing the concept of a *stochastic history process* for X coupled with similar arguments to the above. The existence of the history process compensates for the lack of invertibility of the semiflow.