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Abstract

This study develops an index of network polarization. This index seeks to overcome systematic problems with the extant measures of polarization in political science, sociology, and economics. The Network Polarization Index (NPI) encompasses the structure of relations among nodes in the network, endogenizes the grouping of nodes into any of the types of groups used in network analysis (cliques, blocks, clusters, communities), and incorporates information about each of these groups (their cohesion and sizes), as well as relations between pairs of groups. NPI satisfies all of axioms of polarization posited by Duclos, Esteban, and Ray (2004), as well as two additional axioms I specify here. The Duclos *et al.* (2004) measure of polarization does not generally satisfy the bipolarization axiom. A Monte Carlo simulation examines the properties of NPI. This is followed by an empirical analysis of the relationship between alliance, trade polarization and international conflict. Alliance and trade NPI have robust and significant effects on the magnitude, severity, and duration of international conflict over the 1870-2001 period.

Keywords: Polarization, Networks, Groups, International Conflict, International Trade.

1. Introduction

Network analytic measures have been used to identify various properties of entire networks (e.g., density, transitivity, group centralization), of sub-networks, of groups of actors (e.g., components, cliques, blocks, communities), of dyads (e.g., structural and role equivalence), and of individual nodes (e.g., centrality). Yet, existing network indices do not describe an important property of networks: polarization. Conceptually, polarization is a structural feature of a system that tells us how nodes and groups are positioned vis-à-vis each other. Polarization is an important focus of research in the social sciences. Economists, sociologists, and political scientists, are all interested in different dimensions of polarization.

The concept of polarization is used in a wide range of contexts and is measured in different ways. Many of the existing measures of polarization describe a given population on some attribute (e.g., income, military capabilities). Other measures focus on the grouping of a set of units (states, parties, social groups, individuals). Very few measures, however, combine the two dimensions of polarization: the relations among units and their attributes. This paper develops a general index of network polarization that can fit a wide range of theoretical and empirical interests, all of which can be captured by network analysis. The next section discusses some of the existing measures of polarization. The third section reviews one of the most significant measures of polarization created by Duclos, Esteban, and Ray (2004). Section 4 specifies the basic Network Polarization Index (NPI). Section 5 extends this index by incorporating additional information about the attributes of the groups and nodes. Section 6 examines the mathematical properties of NPI and compares them to the Duclos *et al.* (2004) index. Section 7 explores the properties of this measure and its relations to other measures of network structure via a Monte Carlo simulation. Section 8 examines the effects of polarization of international alliance and trade networks on the level of international conflict.

2. Measures of Polarization in the Social Sciences

A cursory search of the social and humanities citation indexes over the 1975-2009 period revealed 1,927 articles in which the word polarization appeared either in the title or in the abstract of the article. The term "social polarization" appeared in 650 of these, "political polarization" in 410, and "economic polarization" in additional 395. The term "polarity" appeared in 998 titles and/or abstracts. It is hard to say whether these numbers suggest a frequent use of these terms, but they do suggest that this concept and its variations capture a fair amount of interest across the social sciences.

Some examples demonstrate the range of conceptions of polarization in the literature. Woo (2003: 403) measures social polarization as the Gini index of income inequality. Keefer and Knack (2002), use a variety of ethnic, income, and land clustering measures across specific strata of a population. Esteban and Ray (2005) review a large number of indices of economic polarization that are based on income groups.

Political polarization has had an important role in the study of cabinet stability in multiparty democracies. The most common measure of polarization used in this literature is the proportion of seats in the parliament controlled by extremist parties (Warwick, 1994; Biglaiser and Brown, 2003). Evans (2003), following DiMaggio, Evans, and Bryson (1996) measured political polarization with survey data using dispersion indices, bimodality (measured as kurtosis of response distributions), and consolidation of opinions (measured as difference of means across variables between groups).

In international politics, the concepts of polarity and polarization have a central place in theories of peace and war and of international economic processes. Consequently, there exist quite a few measures of polarity and polarization (Wayman and Morgan 1991). One measurement strategy focuses on the distribution of national capabilities (e.g., Singer and Ray 1973; Hopf, 1991). Other measures of polarization focus on the alliance structure of the international system (e.g., Wayman, 1985; Moul, 1993; Singer and Small, 1968).

The large number of polarization indices casts doubt on the marginal utility of yet another measure. Yet a new measure of polarization makes sense for several reasons.

1. Most existing measures focus either on the attributes of the units/groups, or on their relationship and organization; few combine both. For example, none of the measures of political polarization combines the ideological position of parties with their sizes. Likewise most income inequality indices do not combine the structure of social groups (e.g., in terms of religion, ethnicity, education) with their attributes (income). Furthermore, the distance between these groups on some dimension (e.g., ideology) is not reflected in most attribute-based measures (Esteban and Ray, 1994; Duclos *et al.*, 2004).
2. Most measures of polarization assume exogenous assignment of units to groups (e.g., alliances, ethnic groups, proto-coalitions). This assumption is quite tenuous in many real-life situations. In many cases, group identities are due to structures of relationships among units. In such cases, measures need to reflect a set of endogenously derived groups rather than rely on exogenous assignment of units to pre-defined ones.

3. Most measures of polarization assume that the system is divided into discrete groups. In real life, however, some units may not be in any group, while other units may be in multiple groups. The degree of overlap across group membership may be an important property of polarization (e.g., Montalvo and Reynal Querol, 2005).
4. There is a general confusion about the boundaries of polarization. Many studies fudge the substantive meaning of the endpoints of polarization; others fail to impose upper or lower bounds. Still others discuss one endpoint (either minimum or maximum) but not the other. This causes both conceptual and methodological confusion. Is a society more polarized when it is divided into two groups or into many groups? Should polarization be relatively independent of the number of groups in a society? Is a society more polarized when the groups composing it are similar on a given attribute (e.g., income, political power) or if they differ on this attribute? We need to (a) explain what we mean by minimum and maximum polarization, and (b) to justify this conception of endpoints both conceptually and mathematically.
5. Most measures are one-dimensional; they focus either on a single attribute, or on a single rule that assigns units to groups. In reality, however, a given unit may belong to different groups, each of which is defined by a different principle. For example, a given voter's "political identity" may be defined in terms of ethnicity, education, income, or residence. Virtually none of the measures of polarization allows incorporation of multiple dimensions (or what we may call, for our purposes, as multiple networks).

3. **The Duclos, Esteban, and Ray (DER) Polarization Index**

Esteban and Ray (ER 1994) and Duclos, Esteban, and Ray (DER 2004) develop one of the most sophisticated measures of polarization. Relying on an axiomatic approach, their measure overcomes various biases in measures of inequality and polarization such as the Gini index. Rehm and Reilly (2007) use this index to compare political polarization in the United States to other political systems. A special volume of the *Journal of Peace Research* (March 2008) applies the DER index to the study of civil and international conflict. This index allows elucidation of the advantages and shortcomings of existing indices of polarization and for motivating the index of network polarization proposed herein.[1]

The DER index is based on an "identity-alienation" framework. Its intuition is simple: the polarization of a given society is based on (a) the extent to which individuals identify with a given group,

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[1] See also Esteban and Ray (2005). This unpublished paper is a formal comparison of economic measures of "polarization," that are actually measures of inequality across groups.

(b) the relative sizes of these groups, and (c) the distances between these groups. Consequently polarization should reflect several properties (ER 1994, 824):

(1) *Within-group homogeneity*. Polarization should increase in the level of identification of members with their groups.

(2) *Cross-group heterogeneity*. Polarization should increase to the extent that individuals in one group feel that they share little or nothing in common with other groups.

(3) *A small number of significantly sized groups*. Negligible minorities (no matter how homogenous they are internally or how heterogeneous they are with respect to large groups) or individuals that cannot be placed within specific groups do not count.

DER (2004) offer four axioms that motivate their measure of polarization. I frame these axioms in terms that are applicable to network terminology.

Axiom 1: Suppose a population is composed of a single group, whose members are distributed along a given (set of) dimension(s). If the variance of this distribution is narrowed down (so that the mean of the distribution does not change—they call this "a single squeeze"), polarization should not increase.

Axiom 2: Suppose a population is composed of three discrete groups, distributed along some dimension. It the variance of the two extreme groups is "squeezed" towards their respective means, then polarization should not decrease.

Axiom 3: Suppose a population is composed of four discrete groups, distributed along some dimension. If the two "central" groups are pushed towards the end of the continuum, and the groups remain discrete (they don't share members), polarization must increase.

Axiom 4: Suppose a population F is more polarized than another population G. If we increase/decrease these populations by the same amount, F should continue to be more polarized than G (this is a population invariance axiom).[2]

They then prove a theorem that specifies a family of measures of polarization satisfying these axioms. These measures are described by the following formula (ER 1994: 834).

$$ERPOL = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} p_i^{1+\alpha} p_j \mid y_i - y_j \mid \quad [1]$$

where p_i and p_j are the proportions of groups i and j in the population, respectively, α is an index of group identification (α ∈ [.25, 1]) and y_i, y_j are the attribute characteristics (e.g., mean income) of

2 Esteban and Ray (2005) add four more axioms but I do not discuss these here because they complicate the measures of polarization significantly without adding much substance.

groups *i* and *j*. DER (2004) generalize this index to take into account groups that are not described by a single attribute, but rather by a distribution of a given attribute (such as income). Accordingly, the index they propose has the following form:

$$DERPOL = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} p_i^{1+\alpha} p_j \mid y_j - y_i \mid d_j d_i \qquad [2]$$

where d_i, and d_j denote group densities.

This measure of polarization overcomes two of the problems that characterize other measures of polarization: (1) It combines the organization of units into groups with the attributes of the group (cohesion or homogeneity), (2) It incorporates relationships between pairs of groups (inter-group distances), thus allowing, in principle, a way of dealing with overlapping groups.[3] This measure has also a clear lower boundary condition: minimum polarization is observed when the population is bunched into a single group (regardless of density—axiom 1). Finally, DERPOL offers a conception of polarization that is different from inequality-based measures such as the Gini index.[4]

However, some of the problems of traditional polarization measures remain. First, DERPOL assumes exogenous assignment of group memberships; it is based on single-dimensionality of group assignments; and—strictly speaking—it assumes discrete groups. Second, it contains a number of arbitrary features that are not well justified theoretically and are also not mathematically necessary. For example, they exclude individuals who are not part of identifiable groups (ER 1994, 824; DER 2004, 1740). Likewise, the restriction to a "small number of significantly sized groups" is both arbitrary and vague. It is arbitrary because it eliminates areas of interest where there exist large numbers of groups and when unaffiliated individuals may matter for the level of social polarization. It is vague because DER do not specify what they mean by "a small number of significantly sized groups."

Third, the assumption of a uniform group identification parameter (α) is not defended at all. Why would individuals in one group identify with the group at exactly the same rate as individuals in another group? Moreover, the choice of the precise value of this parameter is not explained or defended. When it comes to practical applications of their index, DER make ad-hoc assumptions about the size of this parameter (e.g., Esteban and Ray 2008).[5]

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3 I demonstrate this below, but it must be noted that I am taking some creative liberty with the interpretation of the distance $\lfloor y_i - y_j \rfloor$ parameter in Equation [2].

4 When the homogeneity parameter α is zero, the DER measure converges to the Gini coefficient (DER 2004, 1746).

5 The DER index offers a significant advance over other measures of polarization (e.g., Wang and Tsui, 2000; Wolfson 1997). See Esteban and Ray (2005) for a comparative analysis of these measures. It follows that the li-

I illustrate some of the problems associated with attempts to generalize such measures of polarization to broader social contexts. Assume a set of N = [1,.., n] political parties denoted by their political positions Y = [y_1,…, y_n] distributed along a single left-right dimension in the [0,1] range. Parties' seat proportions are denoted by P = [p_1, …, p_n] (where $\Sigma_p = 1$, $p_i < .5\ \forall\ i \in N$). For convenience, ideal points match parties' index numbers from left to right ($y_1 < y_2 < \ldots < y_n$).

Since no party controls a majority of seats, two or more of them must form a coalition in order to establish a government. Again, to simplify, we assume that coalitions are possible only between ideologically adjacent parties (no leapfrogging). For this illustration, I restrict these coalitions to minimum winning coalition (M_w) such that $m_i \in M_w$ *iff* $\Sigma_i p_i > .5$ [$p_i \in c_m$] and $\Sigma p_i - p_k < .5\ \forall k \in i$.

Now, how do we conceptualize polarization in this setting? Figure 1.1 provides a graphic illustration of a number of possible minimum winning coalitions in this system.

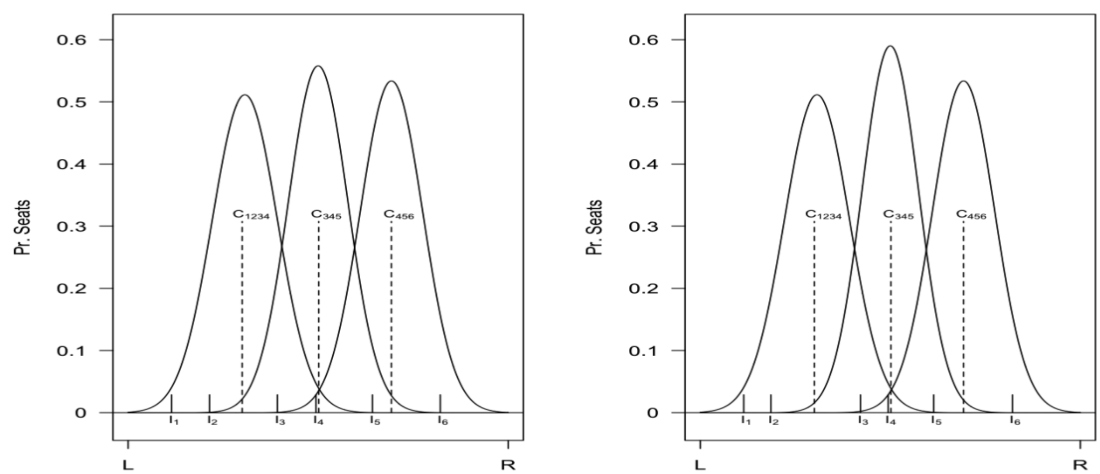

Figure 1: Possible Winning Coalitions in a Six-Party System

Each of the coalitions has some membership overlap with the other coalitions. Coalitions C_{1234} and C_{345} share members 3 and 4, coalitions C_{345} and C_{456} share members 4 and 5, and so forth. How do we measure inter-group distance when groups are not discrete?

Moreover, the variances of these coalitions differ. If we think of the identification of an individual (in our case a specific party) with the group (coalition) in terms of the distance between the party's ideal point and the coalition's ideal point, then the notion of group homogeneity as a constant is not

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mitations of these measures with respect to the broader aspects of social and political polarization are quite significant.

very meaningful. Party 4 is clearly happier with coalition C_{345} than with all other coalitions of which it is a member. It is impossible to tell whether party 3 is happier with coalition C_{1234} than with coalition C_{345}.

In Figure 1.2, two things change. First, parties 3 and 5 moved towards the center, so that the variance of coalition C_{345} has shrunk (this is a "squeeze" in the DER terminology). Second, party 2 moved left, but given the move of party 3 to the center, the range and variance (as well as the ideal point) of coalition C_{1234} has not changed. Let us examine the DER axioms in this context. First, ignoring for the moment the move of party 2 to the left, it is intuitively clear that the polarization of the political system in Figure 1.2 should be higher than that represented by Figure 1.1. This is inconsistent with axiom 1 in DER.

If we focus on the left-shift of party 2 and assume that everything else remains the same, should polarization increase in Figure 1.2 compared to 1.1? It is not clear. If we extrapolate from axiom 3, then polarization should increase. But given that the mean and variance of coalition C_{1234} have not changed, intuition would suggest that if the only difference between Figures 1.1 and 1.2 were the location of party 2 then polarization in both cases should remain the same.

Now, these figures represent a simple case because they cover very few groups and there are no isolate parties. But how do we treat a highly diversified population in which some individuals do not belong to specific groups, or in which there are multiple groups, all of relatively small sizes, or where individuals belong to multiple groups? This discussion suggests at least four desired attributes of polarization indices.

1. The relationship between the number of groups and polarization is nonlinear: *ceteris paribus*, one group reflects zero polarization; maximum polarization can occur only if the population is divided into two groups; a division of the population to more than two groups decreases polarization.
2. *Ceteris paribus,* as the cohesion of groups declines so does polarization.
3. Polarization should decline with the degree of inter-group overlap, and increase with the degree of inter-group distance.
4. Polarization should increase with the degree of equality among group sizes, or in some attribute of groups (e.g., power, number of seats in parliament, income).

Given these desiderata, I suggest that polarization should not:

(a) ignore individuals with no group identification,

(b) be limited to a small number of groups,

(c) assume a uniform group identification index.

Consequently, I accept the four axioms offered by DER, but suggest two additional axioms.

Axiom 5: Individual Defection. Suppose a population of size *n* is composed of a single group. If a single node eliminates a single tie that connects it to the group, polarization must increase.

Axiom 5 seems intuitive. A cohesive population in the sense that all individuals identify with a single group is completely nonpolarized. Under such conditions, suffice it for a single person to be alienated from the group for polarization to increase.

Axiom 6: Bipolarity (maximum polarization). Maximum polarization can be accomplished if and only if society is: (a) divided into two groups, (b) these groups share the same amount of an attribute (or the same number of people when no attribute is defined), and (c) these groups are discrete (or are at a maximum distance from each other in some space). The intuition here is that measures of polarization need to have well-defined upper and lower bounds. Axioms 1 and 5 establish the lower bound. Axiom 6 establishes the substantive meaning of maximum polarization. Any deviation from one of the conditions specified in axiom 6 reduces polarization.

I now turn to a discussion of the Network Polarization Index.

4. The Network Polarization Index (NPI): A Basic Introduction

The Network Polarization Index has a number of components. These components correspond to the elements of the DER measures, but each is defined differently. We can interpret these components in terms of an identity-alienation framework. I discuss each of these components in turn, starting with the simplest formulation of the index, and expanding it to accommodate more complex formulations.

Note that most network indices are standardized on the unit [0,1] interval to control for network size. This is obtained by the following general structure:

$$Network\ Index = \frac{Actual\ value \mid Network_T}{Maximum value \mid Network_T} \qquad [3]$$

Where the subscript *T* indicates a specific property of the network. For example, the density of a binary network is the number of ties within a network made up of *n* nodes, divided by the maximum density of a network of size *n* ([$n(n-1)$]) (Wasserman and Faust, 1997: 164).[6]

I start with basic notation. Let $\boldsymbol{X}$ be a Sociomatrix of order *n*. Let $G = [g_1, g_2, ..., g_k]$ be the set of groups derived from the network.[7] A group g_j is an endogenously-derived subset (of size 1,2,…, *n*).

6 In valued networks, density is the sum of all values divided by $max(v_{ij})n(n-1)$ where $max(v_{ij})$ is the maximum value that a dyadic relationship can assume.

Let S = [$s_1, s_2,..., s_j, \ldots, s_k$] be the number of nodes in groups $g_1, g_2,..., g_j, \ldots, g_k$, respectively, and let P = [$p_1 = s_1/n, \ldots p_k = s_k/n$] be the group members proportions. The set G has the following properties.

1. The size of each element of G [$g_j \in G$] can vary from s_j=1, 2,..., n.
2. A given group can be either a discrete subset of the network, or a non-discrete subset. A partition of a network into a set of disjoint subsets (e.g., blocks, clusters) implies that a node can be in one and only one group (i.e., if $i \in g_r$, then $i \notin g_k$, and $m_k \cap m_r = \varnothing \ \forall k \neq r; k, r \in G$—where m_k and m_r are the set of nodes in groups k and r, respectively). In contrast, a partition of the network into a set of non-discrete groups (e.g., cliques, communities) suggests that a given node can be a member of one or more groups, and that any two groups may overlap in terms of membership up to two nodes). The only constraint is that a given group cannot be equal to, or a proper subset of, another group. Formally, given any two groups g_k and g_r there must exist at least one node $i \in g_k$ and $i \notin g_r$ and at least another $j \notin g_k$ and $j \in g_r$, $\forall g \in G$.

The NPI is a product of two elements: group polarization and group overlap. Group polarization measures the relationship between members of groups and nonmembers. Group overlap measures the relationships between groups. The most basic version of the NPI assumes minimal (relational) data. Extensions of this measure incorporate additional information about the attributes of units or the structure of groups.

4.1. *Group polarization*

Assume a network ***X.*** This can be a binary or valued network; it could be symmetric ($x_{ij} = x_{ji}$ $\forall i,j \in N$) or directional ($x_{ij} \neq x_{ji}$ for at least one pair $i,j \in N$). However, for the derivation of some types of groups (e.g., cliques, communities), valued and/or directional Sociomatrices need to be binarized and symmetrized by some exogenous rule, for reasons that become apparent below.[8] This suggests a considerable loss of information about the direction and strength of relationships. However, in the next section I show how the network polarization index can be expanded to recapture most of the lost information.

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7 I show below that the measure of polarization applies to any and all types of partitions of a network into (joint or disjoint) subsets. These include the following types: cliques, blocks, clusters, and communities. Any partition of the network into a set of subsets can be used.

8 See Wasserman and Faust (1997: 273-278) for clique extraction strategies in directional/valued networks. Communities require binarization of valued networks but not symmetrization (Leicht and Newman 2008).

We extract from ***X*** all groups,[9] and derive the group affiliation matrix (**GA). GA** is an $n \times k$ matrix, with rows representing nodes and its columns representing groups. Entry ga_{ij} equals one if unit *i* is affiliated with group *j* and zero otherwise. Note that, for some cases, isolates may be is in a single-member group. To illustrate this, consider Figure 2, displaying a hypothetical network.

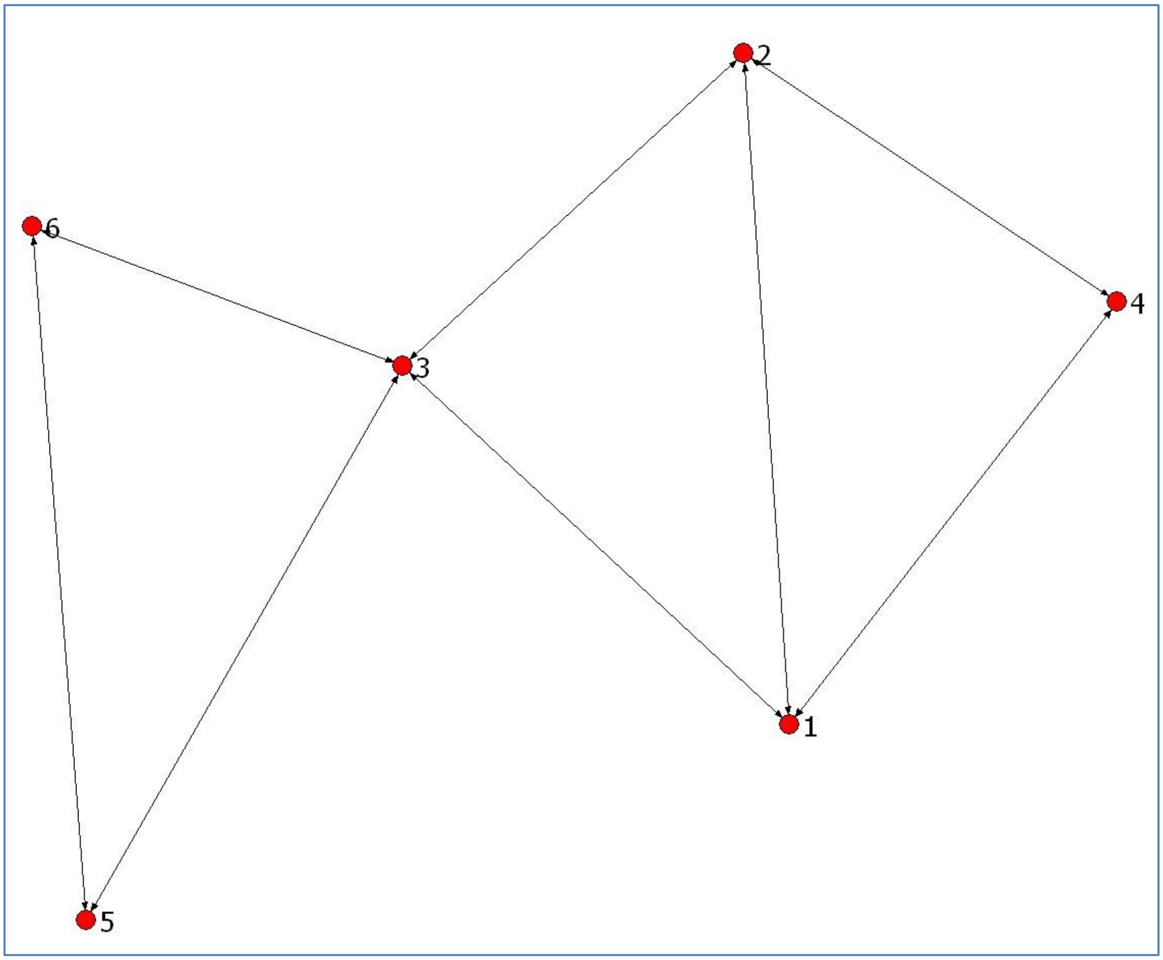

Figure 2: A Hypothetical Six-Node Network

The Sociomatrix corresponding to this figure is given in Table 2.1 and the **GA** matrix (in this case the group concept used is that of cliques) is given by Table 2.2. This network collapses into three cliques. The bottom rows reflect the clique sizes in terms of the number of members in each (s_i), the proportion of members (p_i).

Table 2: Clique Structure for Figure 2

9 There are multiple algorithms for group extraction. These vary both for the extraction of a given type of group (e.g., blocks) and across group types. Typically different algorithms produce different partitions of groups, but this is not of concern here. The index discussed here can be applied to any group extraction algorithm as long as it produces group affiliation and group density matrices as discussed below.

Table 2.1: Sociomatrix

	1	2	3	4	5	6
1	0	1	1	1	0	0
2	1	0	1	1	0	0
3	1	1	0	0	1	1
4	1	1	0	0	0	0
5	0	0	1	0	0	1
6	0	0	1	0	1	0

Table 2.2: Group Affiliation Matrix

	q1	q2	q3	Row sum
1	0	1	1	2
2	0	1	1	2
3	1	0	1	2
4	0	1	0	1
5	1	0	0	1
6	1	0	0	1
S_i	3	3	3	
p_i	0.5	0.5	0.5	

Table 2.4: Group Density

	q1	q2	q3
q1	1	0	0.33
q2	0	1	0.67
q3	0.33	0.67	1

Group polarization measures the distance between the nodes that are members of a given group and nodes that are not. The first step requires specification of the maximum possible polarization between members of a given group and the remaining nodes. This yields a set of pairwise comparisons across all dyads making up the network. For a pair of cells *i* and *m* in column *j*, this yields $d_{imj}=(ga_{ij}-ga_{mj})^2$. For all pairs of cells in this column, this yields $d_j=\sum_{i=1}^{n-1}\sum_{m=i+1}^{n}(ca_{ij}-ca_{mj})^2$. Maximum polarization for each column of **CA** is when half of the nodes in the network are members of a group and half are not. Stated in terms of proportions, the maximum polarization of group vs. non group members as $d_{jmax} = max[p_j(1\text{-}p_j)] = 0.5(1–0.5) = 0.25$. For a group affiliation matrix of size *k*, $D_{max} = 0.25k$.

For a group affiliation matrix, ***GA*** of size $n \times k$, the Group Polarization index (*CPOL*) is given by:

$$GPOL = \frac{actual\ group\ polarization | Network_T}{maximum\ group\ polarization | Network_T} = \frac{\sum_{j=1}^{k} p_j(1-p_j)}{GPOL_{\max|k}} \qquad [4]$$

Given that $p_j = s_j/n$, and $D_{\max|k} = 0.25k$, we substitute these terms in (4)

$$GPOL = \frac{\sum_{j=1}^{k} \frac{s_j}{n}(1-\frac{s_j}{n})}{0.25k} = \frac{4\sum_{j=1}^{k} s_j(n-s_j)}{kn^2} \qquad [5]$$

This index varies from zero when **GA** is an $n \times 1$ matrix with all nodes in one group $(q_1 = n)$, to one when **GA** is an $n \times 2$ matrix with exactly half (in the case of an even n), or $(n-1)/2$ nodes (in case of an odd n) are in one group and the remaining nodes are in the other group. A disconnected network (where all nodes are isolates), yields $k = n$ groups of size 1. GPOL for such networks is given by,

$$GPOL_0 = \frac{4(n-1)}{n^2} \qquad [6]$$

This means that the larger number of nodes without any ties to other nodes, the lower the group polarization index. Third, as the number of groups exceeds 2, GPOL decreases, just as we suggested above. I now turn to the second element of the network polarization index.

4.2. Group Density Index (GDI)

The group polarization measure may be extremely misleading if used in isolation. Consider the GA matrix in Table 2.2. Group polarization is one because each group has half of the nodes as members and half as nonmembers. But this network is less than maximally polarized. Groups *g*2 and *g*3 overlap considerably in terms of membership, and groups *g*1 and *g*3 overlap with respect to one member. Thus, the degree of polarization in this network requires us to consider both group polarization and the extent to which any two groups share the same members. The *Group Density Index* (GDI) measures both the within and between-group density. This is given in the Group Density (**GD**) matrix. **GD** is a $k \times k$ matrix. Element gd_{ij} is the proportion actual links to the possible links between nodes in group g_i and those of g_j. The main diagonal gd_{ii} indicates the within group density of g_i.[10]

Accordingly, the group density index is the ratio of ties across groups to the maximum possible ties between the nodes belonging to these groups. The **GD** matrix is given in Table 2.3.

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10 As in the general case of density $0 \leq gd_{ij} \leq 1$.

The GDI measures the extent to which members in one group are linked to members of another group. Our notion of polarization requires that group be not only discrete because a given extraction algorithm produced them as such, but that members of one group have few or no ties with members of another group. Thus, the lower the cross-group density, the higher the degree of polarization in the network. Accordingly, the group density index is given by

The COI provides information on the extent of polarization of the groups in the network with respect to each other.[11] Note that $0 \leq COI < 1$ because maximal convergence between two cliques q_i, q_j (with $s_i \leq s_j$) is $max(coc_{ij})=coc_{jj}-1$. A strict variation of COI that varies from zero to one is given by

$$GDI = \frac{\sum_{i=1}^{k}\sum_{j=1}^{k} gd_{ij} - \sum_{j=1}^{k} gd_{jj}}{k(k-1)} \qquad [7]$$

Note that we ignore within group density. This is so because we are interested only in the difference between groups. The characteristics of groups will come into play in the more complex versions of NPI.

4.3. The Network Polarization Index (NPI)

Network polarization depends both on the way units are grouped and on the relationship (overlap) among groups. This yields the simple version of the NPI. NPI is the product of the clique polarization index and the complement of the clique overlap index, namely $NPI = CPOL \times (1- COI)$, specifically:

$$NPI = GPOL \times (1-GDI) = \frac{4\sum_{i=1}^{k} s_j(n-s_j)}{kn^2} \times \left(1 - \frac{\sum_{i=1}^{k}\sum_{j=1}^{k} gd_{ij} - \sum_{j=1}^{k} gd_{jj}}{k(k-1)}\right) =$$

$$= \frac{\left[4\sum_{i=1}^{k} s_j(n-s_j)\right]\left[k(k-1) - \left(\sum_{i=1}^{k}\sum_{j=1}^{k} gd_{ij} - \sum_{j=1}^{k} gd_{jj}\right)\right]}{n^2k^2(k-1)} \qquad [9]$$

I discuss the mathematical properties of this index below. For now, however, it is important to review some of the limitations of the measure as specified above.

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11 When the network converges into a single clique, COI is set to zero by definition.

1. When the group type used is the clique, *NPI* is limited to symmetric relations. Directional networks must be symmetrized. This requires imposing an exogenous structure on the network that goes beyond the information provided by the original network data.
2. Group affiliation is a binarized reduction of the information contained in valued networks. Some group extraction algorithms (e.g., those used to extract blocks or clusters) are based on the original valued nature of the network. Other algorithms (e.g., clique extraction algorithms or community extraction algorighms) require binarization. This causes significant loss of information about the strength of ties between nodes.
3. This version of NPI ignores the attributes of the groups and of their members. In particular it ignores elements of polarization as one may reasonably conceive it: group cohesion and group size.
4. A question arises whether this concept of NPI is applicable to multiplexes.
5. Inter-group distance is measured in terms of convergence or discreteness of groups. GDI does not allow measurement of distance between two discrete groups; it treats all discrete groups as having the same distance between them. However, when groups are distributed along a meaningful scale (e.g., income, left-right ideological continuum), distance between discrete groups matters.

Let us discuss these issues in some detail. First, network polarization must reflect the relative homogeneity or heterogeneity of groups (DER 2004). Two networks may be composed of two discrete groups of equal size in which half the population is in one group and the remaining half is in the other. However, the homogeneity of the groups (the variance of the attributes of their members) in one network is roughly equal, whereas in the other network one of the two groups is much more homogeneous than the other. This is illustrated in Figures 1.1 and 1.2. The squeeze of the [345] coalition in Figure 1.2 suggests that the system characterized by this figure should be more polarized than the one given in Figure 1.1.

Second, the current NPI measures group size in terms of the number of members (states in the international system, political parties in a multiparty parliamentary system, individuals in a social class). But this assumes that all nodes in the network are of an identical weight. In many cases this is an unreasonable assumption. Node—and thus group—sizes may be determined by some attribute (e.g., the military capability of states, proportion of seats of political parties in the legislature, the income of individuals in a social group). In such cases, polarization increases with the extent of the relative equality of the attribute-based sizes of groups. A political system is more polarized if the discrete proto-coalitions are of equal size than when one of the coalitions controls a majority of the seats. An international system is more polarized if its two discrete alliance blocks are equal in capabilities, not necessarily in terms of the number of states in each.

Third, polarization may reflect systems in which group membership is determined along multiple dimensions. Consider a political system in which ideology is not uni-dimensional. A set of coalitions that forms on the basis of the economic platforms of parties is not necessarily equivalent to the set of coalitions that may be formed on the basis of their foreign policy positions. Similarly, a society may be divided along income groups, religious groups, or geographic groups. A person's identity may be defined along multiple dimensions. To measure polarization in such cases we need to take account of multiple dimensions.

Finally, NPI must incorporate a measure of inter-group distance that allows distinction between discrete groups when there exists a well-defined dimension that defines the distribution of cliques. The DER index does not know how to treat distances between overlapping groups. At the same time, GDI treats all discrete cliques as having maximal inter-clique distance. A more general way of measuring distance between discrete cliques is in order.

These desired properties of polarization require additional information about the attributes of units in the network or about the attributes of the system (e.g., the rule(s) by which units are spread within the network). If such information is available, we can develop a progressively richer measure of NPI. These extensions rely on the basic structure of NPI, but modify its components to accommodate the attributes of units and to generalize for cliques along multiple dimensions.

5. Extensions of NPI

I offer two modifications in the *NPI* that allow us to capture (a) group cohesion, and (b) group size. I then extend the measure to deal with multiple networks and scale-based inter-clique distances.

Group Cohesion. Group cohesion reflects the extent to which members of a clique are similar in terms of some exogenously defined attribute (e.g., wealth, ideological distance, etc.), or in terms of the pattern of ties between clique members. In the latter case, clique cohesion can be derived endogenously from the original Sociomatrix **X**.

Clique cohesion offers two improvements on the simple measure of network polarization. First, when we use profiles of ties between group members and all other units in the network, we recapture the information that was lost due to binarization and symmetrization of directional and/or valued networks. Second, this extension allows us to incorporate the characteristics of individual groups, thus distinguishing between groups that have the same number of members but differ in terms of the relations among (or similarity of) members.

Let us first deal with cohesion defined by relations among network members. In such cases, the cohesion of a group is a function of the degree of similarity or affinity among group members. We consider a group to be highly cohesive if members have similar or identical patterns of ties with all nodes (not only with other members of the clique).[12] Network analysts use measures of structural equivalence to reflect the similarity between any two nodes (Wasserman and Faust, 1997: 366-375). One such measure is based on Euclidean distance of the ties between node *i* and any other node in the network and the ties between node *j* and any other node in the network. I measure dyadic cohesion as standardized structural equivalence index defined on the unit interval [0,1].

$$se_{ij} = 1 - \frac{\sum_{k=1}^{n} | x_{ik} - x_{jk} | + \sum_{k=1}^{n} | x_{ki} - x_{kj} |}{2n\,\mathbf{max}\left(| x_{ik} - x_{jk} |, | x_{ki} - x_{kj} | \right)} \qquad [10]$$

Where se_{ij} is the structural equivalence score of nodes *i* and *j* and $\boldsymbol{max}(|\mathbf{x_{ik}}-\mathbf{x_{jk}}|, |\mathbf{x_{ki}}-\mathbf{x_{kj}}|)$ is the largest possible pairwise distance in the network. Performing this calculation on any dyad in the original Sociomatrix **X** (without binarizing or symmetrizing it) allows us to recapture the information entailed in valued or directional networks. The result is a structural equivalence matrix **SE** of order *n*, where each entry se_{ij} denotes the standardized structural equivalence of nodes *i,j*. The higher the *se* score of two nodes, the closer se_{ij} is to unity.

We consider a group to be cohesive to the extent that the *se* scores of all the element dyads approach unity. Therefore group cohesion is defined as the average se_{ij} score for all dyads in a given clique, that is:

$$c_j = \frac{2\sum_{i \in j}^{s_j - 1} \sum_{j=i+1}^{s_j} se_{ij}}{s_j (s_j - 1)} \qquad [11]$$

Note that, by definition, $0 < c_j \leq 1$. Alternatively, cohesion can be based on an exogenously defined attribute. In such a case, we can substitute **SE** by another cohesion matrix as theoretically appropriate.

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12 A useful example in political science is one in which the political similarity among states is defined by the similarity of their alliance or trade portfolios (Bueno de Mesquita, 1981; Signorino and Ritter, 1998; Maoz *et al.*, 2006).

Group Size. We incorporate the size of each group in terms of some attribute of theoretical importance. For example, if we want to measure the polarization of the international system in terms of alliance structures, the capabilities of the states comprising an alliance clique is a better measure of its size than the number of states in that clique. Likewise, in parliamentary systems, cliques represent proto-coalitions and their sizes are defined by the proportion of seats they control.

This suggests the following operation. Define an attribute vector **A** of order n. Each entry a_i reflects a size-related attribute of node i measured as a proportion of the network's size. Thus, if we were to measure capabilities of alliances then the attribute vector would reflect the share of the system's capabilities of each state in the system. Likewise, the power of parties is measured by the vector of seat shares.

To incorporate size into the measurement of *NPI* we multiply **GA** elementwise by **A,** to derive a weighted group affiliation matrix $\mathbf{GA}_p$ (where the subscript p suggests that the entries in **GA** are proportions of size-related attributes of clique members).

Inter-Clique Distance. In many situations, groups in societies are lined up on multiple dimensions. Each dimension of the space may be characterized by some real-valued vector (e.g., ideology, income, religion). In such cases we define a $k \times m$ matrix **GC** of group characteristics with elements gc_{ij} denoting the value of clique i on the scale m (standardized on a [0,1] unit square). For a single dimension **GC** is an vector of length k. We now define the **GD** matrix such that its entries are:

$$gd_{ij} = 1 - \sqrt{\frac{1}{m}\sum_{k=1}^{m}(gc_{ik} - gc_{jk})^2} \qquad [12]$$

The GDI index remains the same and so does the NPI.[13]

Recalculating NPI with cohesion and size: We now have three versions of *NPI*. The choice of the specific version depends on the theoretical purpose of the measure and on the amount of data available to the researcher. When we have cohesion scores (that are derived either endogenously via the **SE** matrix or exogenously via some other measure of dyadic affinity), but not size attribute data beyond the number of group members, we discount the group polarization by its cohesion.

$$GPOL_c = \frac{4\sum_{j=1}^{k} s_j(n - s_j)c_j}{kn^2} \qquad [13]$$

13 Note that for the single-dimension case $go_{ij} = 1 - |gc_{ik} - gc_{jk}|$.

CPOL is maximized if and only if the network is divided into two cliques composed of fully structurally equivalent (or perfectly cohesive) dyads. If we can incorporate both clique cohesion and clique size into the measurement of *NPI*, we recalculate $CPOL_{cp}$ as follows

$$GPOL_{cp} = \frac{\sum_{j=1}^{k} p_j(1-p_j)c_j}{.25k} \qquad [14]$$

Accordingly, $NPI = GPOL_{cp} \times (1-GDI)$.

The more complex versions of *NPI* overcome most of the problems of the simpler version. First, they allow reintegration of the basic data of valued or directional networks into the cohesion scores. Second, they incorporate size attributes when these are deemed important. Third, they are flexible with respect to possible inclusion of exogenous data on nodes into the measurement of group cohesion, group size, or both. Fourth, they allow uni- or multi-dimensional exogenous measures of inter-group distance.

Polarization in multiple networks: When a system is characterized by multiple relations, we are confronted by a set of networks. Polarization may differ from one network to another. A society may be highly divided in terms of religious groups, but it may be fairly homogeneous politically. An international system may be highly polarized in terms of alliance relations, but may be highly integrated in terms of patterns of trade. One way to assess polarization along multiple networks is to measure the polarization index of each specific network and somehow integrate these separate indexes across these networks (e.g., to take some weighted mean of the polarization indexes across networks).

There are several ways of extracting groups from multiplexes. In some cases, e.g., block extraction, the approach is simple (Wasserman and Faust, 1997: 367-370). In other cases, e.g., community extraction, this is more complex (xxxx, 2008). In some other cases, the algorithms for group extraction from multiplexes (e.g., cliques) are under development. The general problem is reduced, however, to finding a set of groups defined by a group affiliation matrix. Once this is done, the remaining aspects of NPI become straightforward.

6. NPI—Mathematical Properties

Proposition 1: Under any of its versions, NPI satisfies axioms 1-6.

Proof. I go over each of the axioms separately.

Axiom 1: With a single group (group g_1) and a cohesion score of c_1, we get $GPOL_c = 4s_1(n-s_1)c_1/n^2 = 4n(n-n)c_1/n^2 = 0$. Reducing the variance of the distribution of nodes in g_1 would increase c_1. However, the $n(n-n)$ element of GPOL remains under any alternative formulation of this part of NPI, and since all other elements are products of this particular element, we have $GPOL_c = 0$. Consequently, $NPI = GOL_c \times (1-GDI) = 0$.

Axiom 2: It might be useful to make this axiom stricter than in DER (2004), as follows: For any number of groups, and any distribution of nodes over these groups, if the distribution of any of the groups is squeezed towards its mean (i.e., its cohesion increases), *ceteris paribus*, polarization must increase.

Consider a network with a set of $G = [g_1, g_2, \ldots, g_k]$ groups with sizes $S = [s_1, s_2, \ldots, s_k]$ and cohesion scores $C = [c_1, c_2, \ldots, c_k]$ (with $0 < c_j \leq 1 \ \forall c_j \in C$). Change $c_k^* = c_k + x$ ($x > 0$, $c_k + x \leq 1$). Thus, $GPOL_c = \frac{4\sum_{j=1}^{k} s_j(n-s_j)c_j}{kn^2} = \frac{4[s_1(n-s_1)c_1 + \ldots + s_k(n-s_k)c_k]}{kn^2}$. Changing c_k to c_k^* would change $GPOL_c$ to $GPOL_c^* = \frac{4[s_1(n-s_1)c_1 + \ldots + s_k(n-s_k)(c_k + x)]}{kn^2} > GPOL_c$. This is sufficient to prove that *NPI* satisfies axiom 2 even if *GDI* does not change. However, a squeeze of clique *k* may actually reduce GDI, because a squeeze of the distribution of group *k* towards its mean implies that the distance between g_k and any of the other groups increases (or the cross-group density declines). This results in $GDI^* \leq GDI$, and thus NPI* > NPI.

Axiom 3: The DER specification of this axiom assumes (1) four groups, (2) discrete, (3) with fixed cohesion, and (4) the two middle groups are pushed towards the respective end points of the scale (without changing their cohesion). Because this axiom involves moving discrete groups on a scale, the simple group denisty version of GDI is irrelevant here. We focus instead on the measurement of GDI where groups are distributed along a single dimension.

Assume two networks $\mathbf{X_1}$ and $\mathbf{X_2}$, each with four discrete groups, identical group sizes, and group cohesions. Set group positions on a [0,1] scale. Let $G_1 = [g_{11}, \ldots, g_{14}]$, G_2 $[g_{21}, \ldots, g_{24}]$ be the groups sets of $\mathbf{X_1}$ and $\mathbf{X_2}$, respectively. Let $C_1 = [c_{11}, \ldots, c_{14}]$ and $C_2 = [c_{21}, \ldots, c_{24}]$ be the scale scores of the groups in G_1 and G_2, respectively. Let $c_{11} = c_{21}$ and $c_{14} = c_{24}$, and $|c_{11} - c_{12}| < |c_{21} - c_{22}|$, $|c_{11} - c_{13}| > |c_{21} - c_{23}|$, $|c_{12} - c_{13}| < |c_{22} - c_{23}|$, $|c_{12} - c_{14}| < |c_{22} - c_{24}|$, $|c_{13} - c_{14}| < |c_{23} - c_{24}|$. Let $c_{12} - c_{22} = a$ and $c_{13} - c_{23} = b$. Accordingly, we substitute these values in C_2 which now becomes $[c_{11}, c_{12}-a, c_{13}+b, c_{14}]$. The two matrices GD_1 and GD_2 are shown in Table 3.

Table 3: GD Matrices in Axiom 3

$\mathbf{GD_1}$	g_{11}	g_{12}	g_{13}	g_{14}
g_{11}	$1-\|c_{11}-c_{11}\|=1$	$1-\|c_{11}-c_{12}\|$	$1-\|c_{11}-c_{13}\|$	$1-\|c_{11}-c_{14}\|$

g_{12}	$1-\lvert c_{11}-c_{12}\rvert$	1	$1-\lvert c_{12}-c_{13}\rvert$	$1-\lvert c_{12}-c_{14}\rvert$
g_{13}	$1-\lvert c_{11}-c_{13}\rvert$	$1-\lvert c_{12}-c_{13}\rvert$	1	$1-\lvert c_{13}-c_{14}\rvert$
g_{14}	$1-\lvert c_{11}-c_{14}\rvert$	$1-\lvert c_{12}-c_{14}\rvert$	$1-\lvert c_{13}-c_{14}\rvert$	1

GD$_2$	g_{21}	g_{22}	g_{23}	g_{24}
g_{21}	1	$1-\lvert c_{11}-c_{12}\rvert-a$	$1-\lvert c_{11}-c_{13}\rvert+b$	$1-\lvert c_{11}-c_{14}\rvert$
g_{22}	$1-\lvert c_{11}-c_{12}\rvert-a$	1	$1-\lvert c_{12}-c_{13}\rvert+a+b$	$1-\lvert c_{12}-c_{14}\rvert+a$
g_{23}	$1-\lvert c_{11}-c_{13}\rvert+b$	$1-\lvert c_{12}-c_{13}\rvert+a+b$	1	$1-\lvert c_{13}-c_{14}\rvert-b$
g_{24}	$1-\lvert c_{11}-c_{14}\rvert$	$1-\lvert c_{12}-c_{14}\rvert+a$	$1-\lvert c_{13}-c_{14}\rvert-b$	1

To prove that NPI satisfies axiom 3, we need to prove that $GDI_2 < GDI_1$. We first define the values of each GDI. With a single dimension, Equation [12] reduces to:

$$gd_{ij} = 1 - |c_{ik} - c_{jk}| \quad [15]$$

And *COI* is defined by

$$GDI_1 = \frac{\sum_{i=1}^{k}\sum_{j=1}^{k}\left(1-|c_{1i}-c_{1j}|\right) - \sum_{j=1}^{k}(1-cd_{jj})}{k(k-1)} \quad [16]$$

Given that $|c_{1i} - c_{1j}| = 0$ if $j = i$, we have $\Sigma_j(1\text{-}cd_{jj}) = 0$. So Equation [16] becomes:

$$GDI_1 = \frac{k(k-1) - 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}|c_{1i}-c_{1j}|}{k(k-1)} = 1 - \frac{2\sum_{i=1}^{k-1}\sum_{j=1+1}^{k}|c_{1i}-c_{1j}|}{k(k-1)} \quad [17]$$

Likewise, GDI_2 is given by:

$$GDI_2 = 1 - \frac{2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}|c_{2i}-c_{2j}|}{k(k-1)} \quad [18]$$

Subtracting GDI_2 from GDI_1 yields,

$$GDI_1 - GDI_2 = 1 - \frac{2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}|c_{1i}-c_{1j}|}{k(k-1)} - 1 + \frac{2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}|c_{2i}-c_{2j}|}{k(k-1)} = \frac{2\left(\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}|c_{2i}-c_{2j}| - \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}|c_{1i}-c_{1j}|\right)}{k(k-1)} \quad [19]$$

Substituting $c_{12} - a$ for c_{22} and $c_{13} + b$ for c_{23} converts [19] into,

$$GDI_1 - GDI_2 = \\ = \frac{2\left[\left(|c_{11}-c_{12}|-a+|c_{11}-c_{13}|+b+|c_{11}-c_{14}|+|c_{12}-c_{13}|+a+b+|c_{12}-c_{14}|+a+|c_{13}-c_{14}|-b\right)}{k(k-1)} - \\ \frac{\left(|c_{11}-c_{12}|+|c_{11}-c_{13}|+|c_{11}-c_{14}|+|c_{12}-c_{13}|+|c_{12}-c_{14}|+|c_{13}-c_{14}|\right)\right]}{k(k-1)} = \\ = \frac{2\left(-a+b+a+b+a-b\right)}{k(k-1)} = \frac{2(a+b)}{k(k-1)} \quad [20]$$

Since a and b are positive (with range of [0,1]), $GDI_1 - GDI_2 > 0$. Given that the only thing that changes is the location of c_{22} and c_{23}, $GPOL_1 = GPOL_2$. Thus $[NPI_1 = GPOL_1 \times (1–GDI_1)] < [NPI_2 = GPOL_2 \times (1–GDI_2)]$.

Axiom 4: Population Invariance. NPI actually satisfies a stricter version of this axiom, that can be restated as follows: If a given network $\mathbf{X_1}$ is more polarized than another network, $\mathbf{X_2}$ and the networks are scaled up or down by the same amount, leaving all (relative) distribution unchanged, then the difference in polarization between $\mathbf{X_1}$ and $\mathbf{X_2}$ remains the same**.** Formally let $\mathbf{X_1}$ and $\mathbf{X_2}$ be two networks of size n. Let $NPI_1 – NPI_2 = r$. Assume that we add to (or subtract from) both networks a set of nodes t such that both new networks $\mathbf{X_1}$***** and $\mathbf{X_2}$***** are now of size $n+t$ (or $n–t$). Since the distributions do not change, this implies that $G_1 = G_1^*$ and $G_2 = G_2^*$; $P_1 = P_1^*$, $P_2 = P_2^*$; and $C_1 = C_1^*$, $C_2 = C_2^*$.[14] It is therefore easy to show that $NPI_1 = NPI_1^*$ and $NPI_2 = NPI_2^*$.

Axiom 5: Assume a fully connected network $\mathbf{X_1}$. Thus $G_1 = g_{11}$, $S_1 = N$, GPOL = 0, and NPI = 0. Consider two scenarios. The first entails a node eliminating all of its ties with the other nodes. The new network, denoted by $\mathbf{X_2}$**,** yields $G_2 = [g_{21}, g_{22}]$ and $S_2 = [s_{21}=(n–1), s_{22}=1]$. Since the pattern of ties within cliques does not change, the cohesion vector of $\mathbf{X_2}$ is given by $C_2 = [c_{21}=1, c_{22}=1]$. It follows that the matrix of group density of the second network GD_2 becomes a 2 × 2 matrix with the following structure $CO_2 = [1, 0; 0, 1]$. This means $GDI_2 = 0$. Thus NPI for the new network $\mathbf{X_2}$ depends strictly on CPOL. In this case, we get

$$GPOL_2 = \frac{4\sum_{i=1}^{k} s_{2i}(n-s_{2i})c_{2i}}{kn^2} = \frac{4\left[(n-1)\times 1 + 1\times(n-1)\right]}{2n^2} = \frac{4(n-1)}{n^2} > 0 \quad [21]$$

This implies $GPOL_2 > GPOL_1$.

The second scenario entails a case where a single node eliminates a single tie. In this case, the new network $\mathbf{X_2}$ yields two groups with $s_{21} = s_{22} = n–1$ and $c_{21} = c_{22} < 1$. *CPOL* is therefore,

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14 This can happen if and only if the addition or subtraction of nodes does not change the group structure. If nodes are added, the new nodes must have identical relational patterns to existing nodes across all cliques. Alternatively, subtracted nodes are reduced equally from all cliques. Addition or subtraction of nodes that change clique structures violates this axiom for both NPI and DERPOL.

$$GPOL_2 = \frac{4\sum_{i=1}^{k} s_{2i}(n-s_{2i})c_{2i}}{kn^2} = \frac{4\left[(n-1)c_{21}+(n-1)c_{22})\right]}{2n^2} = \frac{2\left[(n-1)(c_{21}+c_{22})\right]}{n^2} > 0 \quad [22]$$

GD in this case is = [1, *(n–1)/n*; *(n–1)/n*, 1]. Thus COI is given by:

$$GDI_2 = \frac{1+(n-1)/n+(n-1)/n+1-2}{k(k-1)} = \frac{2(n-1)/n}{2} = \frac{n-1}{n} \quad [23]$$

Thus, we get

$$\begin{aligned} NPI_2 &= \frac{2(n-1)(c_1+c_2)}{n^2} \times \left(1-\frac{n-1}{n}\right) = \frac{2(n-1)(c_1+c_2)}{n^2} \times \frac{1}{n} \\ &= \frac{2(n-1)(c_1+c_2)}{n^3} > 0 \end{aligned} \quad [24]$$

Axiom 6: *Bipolarization*. In order to prove that the NPI satisfies this axiom, I show that violation of the any of the three conditions stipulated in this axiom induces NPI < 1. These conditions are: (a) two groups, (b) of equal size, and (c) no inter-group overlap. Suppose a network divided into G=3 groups that satisfies conditions (b) and (c). In order to meet these conditions, it must be the case that S = [$n/3, n/3, n/3$], and $GDI = 0$. (since there is no interclique overlap, it follows that $c_j = 1 \ \forall q_j \in Q$.) This yields,

$$GPOL = \frac{4\times\frac{3n}{3}\left(n-\frac{n}{3}\right)}{3n^2} = \frac{4n\left(\frac{2n}{3}\right)}{3n^2} = \frac{8n^2}{9n^2} = \frac{8}{9}. \quad [25]$$

Generalizing this to any number of cliques where $k > 2$ and $s_j = n/k$,

$$GPOL = \frac{4\times\frac{kn}{k}\left(n-\frac{n}{k}\right)}{kn^2} = \frac{4n\left(\frac{kn-n)}{k}\right)}{kn^2} = \frac{\frac{4n^2(k-1)}{k}}{kn^2} = \frac{4(k-1)}{k^2}, \quad [26]$$

which can equal to 1 iff $k=2$.

To prove that violation of condition (b) yields NPI <1, suppose we have k=2 discrete groups (of maximum cohesion each) but one is larger than the other by one node, so that $Q = 2$, $S = [s_1 = (n+1)/2, s_2 = (n-1)/2]$, and GDI =0. Again, NPI = GPOL, and[15]

15 Here too I assume that $c_j = 1 \ \forall j \in Q$. Any $c_j < 1$ will, by definition, bring NPI<1.

$$GPOL = \frac{4\left[\frac{n+1}{2}\left(n-\frac{n+1}{2}\right)+\frac{n-1}{2}\left(n-\frac{n-1}{2}\right)\right]}{2n^2} =$$
$$= \frac{2\left[\left(\frac{n+1}{2}\right)\left(\frac{2n-n-1}{2}\right)+\left(\frac{n-1}{2}\right)\left(\frac{2n-n+1}{2}\right)\right]}{n^2} \qquad [27]$$
$$= \frac{2\left[\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)+\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\right]}{n^2} = \frac{n^2-1}{n^2} < 1.$$

The proof that a violation of condition (c) regarding intergroup overlap induces NPI < 1 is trivial. With any non-diagonal element of the GD matrix $gd_{ij} > 0$, we get $GDI > 0$. When GPOL = 1 and GDI>0, NPI = 1 × (1 − GDI) < 1. Q.E.D.

It is instructive to explore the boundary relations between NPI and some of the other commonly used measures of network structure. Table 4 provides this analysis. Proofs are trivial so they are not discussed here. The table suggests that there are clear boundary relations between NPI and most structural indices of network structure. However, the relationships across the range of NPI's values are not all that simple or evident. The Monte Carlo simulation in the next section provides evidence of the relationship between NPI and other measures of system structure.

Table 4: Relationships between NPI and other Network Attributes

Network Attribute	NPI = 0	NPI = 1	Relationship	Comment
Density	1.0	$\frac{1}{n-1}$	Linear-negative	
Transitivity (Clustering Coefficient)	1.0	$\frac{n-4}{4(n-1)}$	U-shaped	
No. Components*	1.0	2.0	Unclear	
Giant Component/n**	1.0	0.5	Unclear	
Group Degree Centralization***	0.0	0.0	Inverse U-shaped	

Notes: * No. of Components. A component is a closed subset of reachable nodes.
** G/N. The proportion of network nodes in the largest component.
*** Group Degree Centralization (Wasserman and Faust, 1997) is a measure of centralization based on degree centrality.

It is measured as $DGC = \frac{\sum_{i=1}^{n}(DC(\mathbf{max}) - DC_i)}{(n-1)(n-2)}$

where *DC*(*max*) is the highest degree centrality score in a network of size *n*.

5. NPI: Statistical Properties and Empirical Implications

In order to provide a more general sense of the propoerties of NPI, a Monte Carlo simulation was performed. The key features of the simulation are as follows.

1. The simulation is based on 10,000 random networks of sizes ranging from 10 to 170 (increasing at an interval of ten, i.e., 10, 20, …. 170).
2. In each run of the simulation, the program calculated the network characteristics listed in Table 4. I have also calculated other characteristics of the network that are relevant for the assessment of the properties of NPI. These include the number of cliques in the network, the average size of the cliques (s_i), the average number of clique memberships per node, *GPOL, GPOL$_c$,* and *GDI.* I also calculated a number of NPI versions, starting with the simple version, and continuing with the NPI$_c$ values. The latter incorporate cohesion indices using the **SE** matrices derived from each of the sample **X** Sociomatrices.
3. I calculate DERPOL as follows. I use a group-identification α value of 1.0, to make it maximally equivalent to GPOL without cohesion. Second, I interpret the inter-group distance as $|y_i - y_j| = 1 - gd_{ij}$. Finally, $d_i = c_i \ \forall i \in Q$. I interpret the density of a clique to be equal to the cohesion score of that clique as defined above. Consequently, DERPOL is now defined as:

$$DERPOL = \sum_{i=1}^{k=1} \sum_{j=i+1}^{k} p_i^2 p_j (1 - gd_{ij}) c_i c_j \qquad [28]$$

4. The simulation was run on three different group types: cliques, blocks (generated via CONCOR—Wasserman and Faust 1997), and clusters. A run based on community structures (Leicht and Newman 2008) in incomplete at this time; results will be reported at a later time.

The descriptive statistics of this simulation are given in Table 5. The correlations between these structural network indices are shown in Table 6.

The results of Table 6 suggest that the NPI based on cliques is negatively correlated with the number of cliques. The correlation between NPI and the number of blocks is not very meaningful as the number of blocks is determined by the depth of the block generation procedure (the number of clusters is fixed exogenously). The correlations between NPI scores based on different group types are moderately-high. The correlations between NPI scores and ERPOL are moderately-high to very high. The correlations between network density, network transitivity, and NPI scores are negative and low. Finally, the correlations between group centralization and NPI scores are moderate to moderately-high.

Overall, these results suggest that the NPI offers a generally distinct measure of network structure. This index differs across the type of grouping of nodes, but generally speaking, a given network shows similar degrees of polarization across group types.

Table 5: Basic Statistics of the Monte Carlo Simulation of Network Polarization

Variable	**N**	**Mean**	**Std. Dev.**	**Min**	**Max**
Network ID	10000	5000.500	2886.896	1.000	10000.000
No. of Nodes	10000	90.260	51.667	10.000	170.000
No. of Cliques	10000	35909.130	21488.570	3.000	50000.000
No. of Clusters	10000	4.000	0.000	4.000	4.000
No. of Blocks	10000	3.997	0.163	3.000	6.000
Avg. Clique Size	10000	11.110	2.857	3.615	15.777
Avg. Cluster Size	10000	22.911	12.732	2.500	42.500
Avg. Block Size	10000	22.571	12.908	1.667	42.500
Avg. No. of Clique Mem. Per Node	10000	4147.403	2481.817	2.000	7853.857
Avg. No. of Cluster Mem. Per Node	10000	1.000	0.000	1.000	1.000
Avg. No. of Block Mem. Per Node	10000	1.000	0.000	1.000	1.000
No. of Components--Cliques	10000	1.000	0.000	1.000	1.000
No. of Components--Clusters	10000	4.000	0.000	4.000	4.000
No. of Components--Blocks	10000	3.997	0.163	3.000	6.000
Simple GPOL--Cliques	10000	0.503	0.182	0.295	1.000
Simple GPOL--Clusters	10000	0.150	0.173	0.035	0.740
Simple GPOL--Blocks	10000	0.745	0.021	0.547	0.880
Simple NPI--Cliques	10000	0.421	0.097	0.274	0.678
Simple NPI--Clusters	10000	0.057	0.127	0.001	0.608
Simple NPI--Blocks	10000	0.567	0.015	0.484	0.645
GOI--Cliques	10000	0.391	0.114	0.188	0.878
GOI--Clusters	10000	0.459	0.125	0.115	0.921
GOI--Blocks	10000	0.482	0.119	0.120	0.936
GPOL-Cohesion--Cliques	10000	0.249	0.085	0.148	0.480
GPOL-Cohesion--Clusters	10000	0.105	0.122	0.026	0.525
GPOL-Cohesion--Blocks	10000	0.393	0.026	0.358	0.619
NPI-Cohesion--Cliques	10000	0.163	0.059	0.083	0.270
NPI-Cohesion--Clusters	10000	0.105	0.122	0.026	0.525
NPI-Cohesion--Blocks	10000	0.393	0.026	0.358	0.619
Density	10000	0.500	0.019	0.311	0.700
Transitivity	10000	0.499	0.024	0.255	0.717
Network Interndependence	10000	0.060	0.009	0.009	0.259

Degree Group Centralization	10000	0.157	0.076	0.069	0.722
ERPOL	10000	0.060	0.054	0.017	0.207

Key: No. Cliques: Average number of cliques. Clique Size: Average number of nodes per clique. Clique Memb.: Average number of clique memberships per node. GPOL: simple group polarization index. GOI: group overlap index. Siimple NPI = GPOL × (1-GOI). GPOL-Coh: group polarization index with cohesion. NPI-Coh = GPOL-Coh × (1-GOI).

Table 6: Correlation Matrix of Network Attributes in the Monte-Carlo Simulation

Variable	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1. N	1.000															
2. Clq. Size	0.960	1.000														
3. Clst. Size	.	0.960	1.000													
4. Blk Size	.	0.960	1.000	1.000												
5. GPOL Clq.	-0.923	-0.959	-0.923	-0.923	1.000											
6. GPOL Clst.	-0.762	-0.858	-0.762	-0.762	**0.901**	1.000										
7. GPOL Blk	0.159	0.201	0.170	0.167	**-0.178**	**-0.148**	1.000									
8. GPOL Ch. Clq	-0.935	-0.957	-0.935	-0.934	0.997	0.877	-0.206	1.000								
9. GPOL Ch. Cls	-0.759	-0.859	-0.759	-0.758	0.906	0.992	-0.198	**0.884**	1.000							
10. Gpol Ch. Blk	-0.636	-0.726	-0.621	-0.633	0.771	0.821	0.173	**0.857**	**0.758**	1.000						
11. NPI Ch. Clq	-0.948	-0.897	-0.948	-0.948	0.846	0.624	-0.159	0.870	0.604	0.493	1.000					
12. Npi Ch. Cls.	-0.759	-0.859	-0.759	-0.758	0.906	0.992	-0.198	0.880	1.000	0.852	**0.619**	1.000				
13. NPI Ch. Blk	-0.636	-0.726	-0.621	-0.633	0.771	0.821	0.173	0.753	0.852	1.000	**0.509**	**0.858**	1.000			
14. Density	-0.004	0.165	0.009	-0.004	-0.063	0.011	0.040	-0.064	-0.006	-0.015	**-0.100**	**-0.006**	**-0.015**	1.000		
15. Transitivity	0.082	0.341	0.085	0.082	-0.278	-0.122	0.067	-0.242	-0.130	-0.123	**-0.170**	**-0.130**	**-0.123**	0.819	1.000	
16. ERPOL	-0.827	-0.909	-0.827	-0.827	0.942	0.978	-0.195	0.929	0.963	0.775	**0.742**	**0.961**	**0.779**	0.009	-0.107	1.000
17. Grp. Deg. Cent	-0.708	-0.755	-0.688	-0.707	0.779	0.792	-0.161	0.776	0.817	0.730	**0.610**	**0.817**	**0.730**	-0.016	-0.008	0.783

Note: All correlations are statistically significant due to large n.

6. Network Polarization and International Conflict

International relations theorists have long debated the effects of systemic polarization on international stability (Waltz, 1979; Deutsch and Singer 1964; Bueno de Mesquita 1978; Moul 1993; Mearsheimer 1990). Empirical tests of this relationship have yielded mixed results (Bueno de Mesquita and Lalman 1988; Wayman and Morgan, 1991). Moreover, most polarization indices that have been used to test this relationship had the problems enumerated above. I apply the NPI framework to retest this relationship on two networks: security networks—i.e., alliances—and economic—i.e., international trade—networks.

Maoz (2006) reviewed key themes in the literature on polarization and conflict in the international system. He argues that the theoretical IR literature does not have clear expectations about the way in which network polarization affects the level of systemic conflict. However, based on the results of that study, I adopt the hypotheses relating alliance and trade polarization to international conflict.

H1. The higher the degree of alliance polarization, the higher the amount of conflict in the international system.

H2. The higher the degree of trade polarization, the higher the amount of conflict in the international system.

The intuition behind these hypotheses is discussed elsewhere (Maoz 2006, 2010).

Dependent Variables.

International Conflict. I used the dyadic MID dataset (Maoz, 2005) that consists of all Militarized Interstate Disputes (MIDs) and interstate wars over the period 1816-2001. A MID is "a set of interactions between or among states involving the threat, display, or use of force in short temporal intervals. To be included, these interactions must be overt, non-accidental, government-sponsored and government-directed" (Gochman and Maoz, 1984: 586). I use a number of indices of systemic conflict.

No. of MIDs/Wars. Number of dyadic MIDs/wars underway in the system in a given year.

Conflict Severity. Each MID is assigned a severity score in the [0,100] range that reflects the highest level of hostility reached by any of the parties. This score is based on the Maoz (1982: 217-231) severity scale. Severity scores are averaged over all MIDs for a given year.

Duration. Duration in days is aggregated over all MIDs for a given year.

Independent Variables. These include alliance NPI/DERPOL and trade NPI/DERPOL indices. I discuss each measure separately.

Alliance NPI. The polarization index is composed of two variables. First, I use the Alliance Treaties and Obligations Project (ATOP) data on formal international alliances (Leeds, 2005). I operationalize alliance commitments between two states as follows (Maoz 2010):

$$COMMIT_{ij} = \begin{cases} 0 & \text{if no alliance} \\ 0.3 & \text{if consultation treaty} \\ 0.4 & \text{if neutrality} \\ 0.5 & \text{if nonaggression} \\ 0.65 & \text{if offensive pact} \\ 0.75 & \text{if defense pact} \end{cases}$$

The relative commitment variable is defined as $RELCOMMIT_{ij} = \frac{\sum_{a=1}^{5} Commit_{aij}}{2.6}$. Since two states may have multiple alliance commitments in a given year, their relative commitment is the ratio of the sum of their specific commitment scores to the sum of all possible commitments. This variable is used to generate the alliance commitment Sociomatrices (**AC**), with entries $ac_{ij}=0$ if states *ij* had no formal alliance in a given year, and $ac_{ij}=relcommit_{ij}$ otherwise. The second variable is the Combined Index of National Capability (COW 2003; Singer 1990) [CINC]. This measures a state's share of the system's military capabilities. This variable is used to measure the capabilities (sizes) of the alliance cliques.

For each year *t*, I generate an alliance commitment network $\mathbf{AC_t}$ of size n_t (independent states). Entry ac_{ijt} reflects the alliance commitment score of state *i* to state *j* in year *t*. The attribute vector $\mathbf{CINC_t}$ represents the capabilities of the *n* states in year *t*. The clique affiliation matrix in year *t* reflects the closed subsets of states that have any type of alliance with each other in that year. This matrix is multiplied elementwise by $\mathbf{CINC_t}$ to give us the proportion of the system's capabilities owned by a given alliance clique, such that $p_i = \Sigma_i cinc_i$. The cohesion scores of the clique are obtained as the average Euclidean distance between the alliance profiles of any two states using the original alliance commitment scores of the Sociomatrix. DERPOL is measured as in Equation [36] above.

Trade Polarization. I use the Oneal-Russett (2005) dataset that updates and extends Gleditsch's (2002) trade dataset. This dataset covers the years 1870-2001. For each year, I define a matrix $\mathbf{E_t}$ with entries e_{ij} in the trade matrix representing the proportion of state *i*'s exports directed to state *j* (thus e_{ji} reflects the value of goods and services imported by state *j* from state *i*).

Trade networks tend to have extremely high connectivity and complexity scores, resulting in a huge number of cliques (when any level of trade is included, the number of cliques for 1999 is over 42,000). In order to simplify this, I used a clique membership cutoff of $e_{ij}/\Sigma e_i = 0.001$ (two states are in the same clique if at least one state exports to the other state one tenth of one percent of its total exports). The weight of trade cliques was measured by a $\mathbf{GDP}_{it}$ vector with entries gdp_{it} representing state's i share of the world's GDP for that year. Thus, the p_i scores in the trade clique affiliation matrix are the share of the world's GDP "owned" by members of clique i. The cohesion scores are defined as the complement of the average Euclidean distance scores of the trade profiles for all dyads making up a given trade clique.[16]

Control Variables. I use the same control variables employed by Maoz (2006). These are documented in more detail in that article, and include the following:

Capability Concentration. Singer and Ray's (1973) measure of capability concentration.[17]

Number of Major Powers. Number of states designated as major powers by the Correlates of War Project (see Maoz 2010: Ch. 7).

Results

I start with a description of alliance and trade network series. Figure 4 shows how the three versions of alliance NPI differ over time. The simple version of NPI (incorporating only relational information without clique cohesion scores) and the NPI_c version that incorporates cohesion scores are almost perfectly correlated. This correlation is significantly higher than the correlation between these two versions of NPI in the Monte Carlo simulation. On the other hand, once the relative sizes of alliance cliques are re-interpreted in terms of their aggregate capabilities, substantial differences arise between the versions of NPI. The correlations decrease (but are still statistically significant). Note that the different versions of alliance NPI begin to diverge after World War I, suggesting that power played a major role in the polarization of the international system throughout most of the twentieth century.

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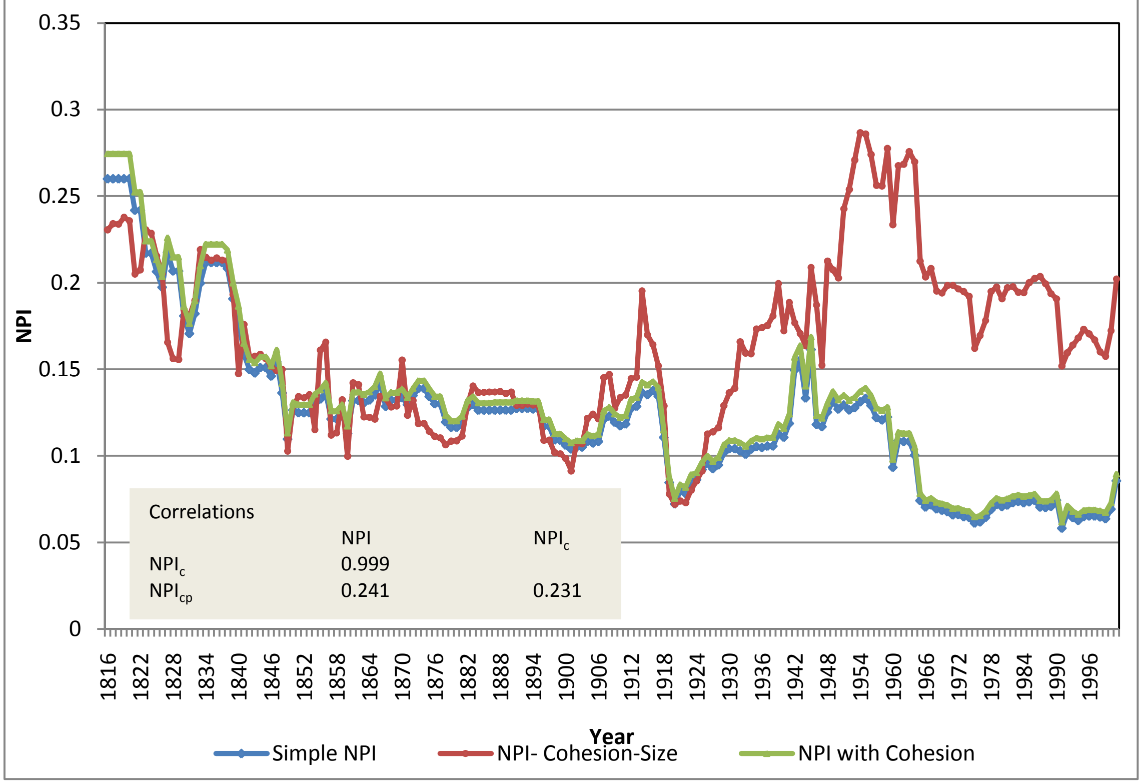
16 This is a somewhat questionable procedure. Low se_{ij} scores indicate that members of a dyad have a very similar set of trading partners. This may mean that they actually compete over the same markets, rather than having similar trading preferences. Nevertheless, without information on the commodities that make up these trading profiles, it is reasonable to assume that cohesion scores within a clique increase to the extent that states making up this clique have similar pattern of trading partnerships.

17 This is one of the measures of polarization often used by international relations theorists. It is a measure of the concentration of CINC scores.

Figure 5 shows the alliance and trade NPIs. As can be seen, these series are inversely correlated at a moderate level. While the alliance series displays significant fluctuations over time, trade polarization suggests a generally declining trend, suggesting growing levels of trade interdependence over time.

Figure 4: Alliance Polarization, 1816-2001: Three Versions of NPI



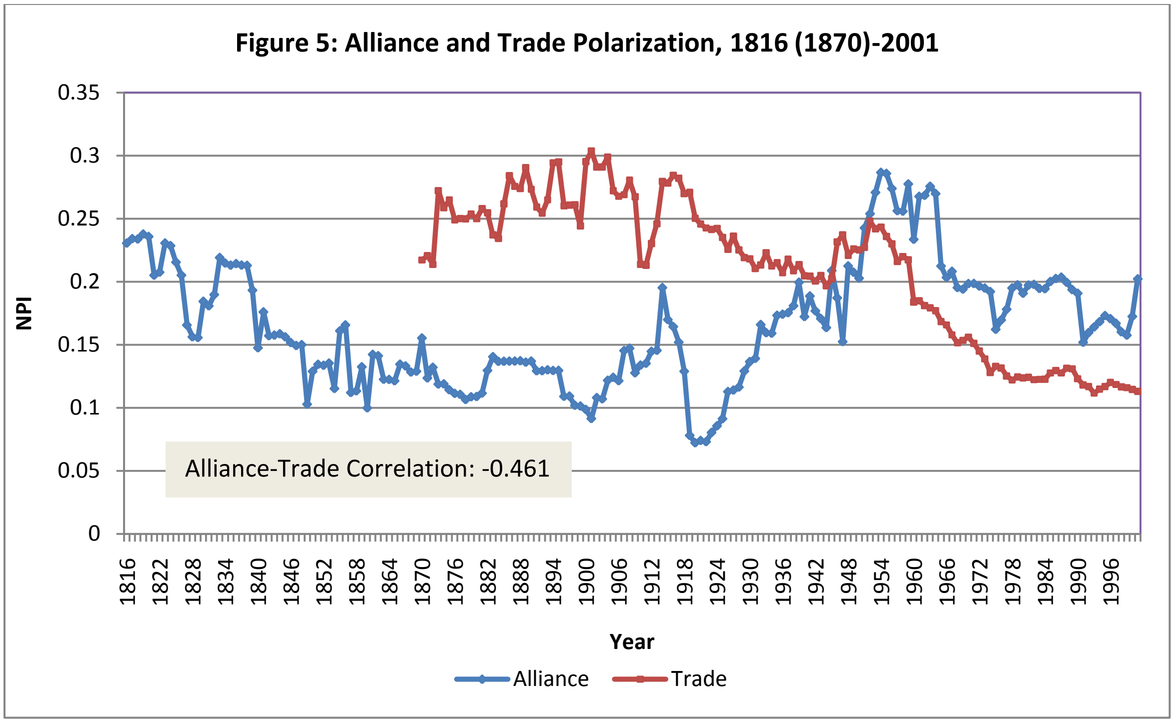


Table 7 examines the effects of alliance and trade network polarization on international conflict. The results suggest that alliance polarization had a robust positive effect on conflict, while trade polarization has a negative effect on conflict. These results replicate and extend those of Maoz (2006) who focused only on the simple version of NPI. The NPI versions shown here are the NPI_{cs} measures, but the findings hold when the simpler versions of alliance and trade NPI are used. The table does not show the effects of polarization measured in terms of alliance or trade DERPOL on measures of conflict because the raw DERPOL measures are highly correlated with time (which is also correlated with the number of MIDs and duration but not with war or relative hostility). Therefore they do not display a consistently significant effect on conflict. These results suggest that as strategic polarization—measured by alliance structure and by the distribution of capabilities across alliances—increases, the level of conflict in the system also rises. Trade polarization has a consistently dampening effect on conflict.

6. **Conclusion**

This study developed a measure of network polarization. It showed that this measure satisfies all of the axioms put forth by DER (2004), as well as two additional axioms. The DER polarization index satisfies the first five axioms, but fails to satisfy the bipolarization axiom, except under a very limited case ($\alpha = 1$). The Monte Carlo simulation and the empirical application of the measures of polarization on international alliance and trade data reveal some important empirical properties of these measures. These analyses also demonstrate that these measures have important implications for social networks analysis, in general, and for the analysis of international networks, in particular.

Table 7: Effects of Network Polarization on International Conflict, 1870-2001

	Dependent Variable			
Independent Variable	**No. MIDs**$^{+}$	**No. Wars**$^{+}$	**Relative Hostility**$^{++}$	**Duration**$^{+}$
Alliance NPI	8.949**	10.970**	41.277*	10.666**
	(0.999)	(2.272)	(22.718)	(1.314)
Trade NPI	-10.782**	-12.897**	-114.143**	-11.345**
	(1.713)	(3.404)	(37.349)	(2.140)
Capability Concentration	2.616**	14.720**	165.490**	6.677**
	(1.515)	(2.716)	(37.276)	(1.786)
No. Major Powers	0.279**	0.664**	3.808**	0.406**
	(0.058)	(0.119)	(1.222)	(0.074)
Ar(1) (Rho)	0.787**	0.571**	0.288	0.863**
	(0.062)	(0.070)		(0.059)
Constant	1.494**	-6.473**	10.839	4.201**
	(0.555)	(1.249)	(13.074)	(0.713)
Root MSE	0.412	0.977	7.932	0.557
Durbin-Watson Statistic			2.054	
N	130	130	130	130
F	70.85**	28.8**	5.89**	65.18**
Adj. R^2	0.730	0.519	0.132	0.713

Notes: Numbers in parentheses are robust standard errors
$^{+}$ Autoregressive Poisson Regression
$^{++}$ Time-series Regression with Cochrane-Orcutt correction for autocorrelation
* $p < 0.05$ ** $p < 0.01$

Clearly, this is a first cut into a complex subject, and as such the results should be seen as tentative. Additional applications of this index examining the effect of polarization on cabinet duration in parliamentary systems (Maoz and Somer-Topcu 2010) and to a broader set of international networks (Maoz 2010) have shown that this index helps illuminate a number of political processes. More simulation-based and empirical research would enable us to better assess the empirical utility of this family of measures. Yet, these results appear sufficiently encouraging to warrant further exploration of these issues.

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