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Dynamics of Stochastic Systems with Memory (Mathematics and Statistics Colloquium, Wright State University)

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DYNAMICS

OF

STOCHASTIC SYSTEMS

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Deterministic ODE's: Stable Manifolds

ODE on $\mathbf{R}^d$:

$$dx(t) = h(x(t))\,dt \qquad (ODE)$$

driven by a vector field $h : \mathbf{R}^d \to \mathbf{R}^d$, C_b^k; viz. all derivatives $D^j h, 1 \le j \le k$, continuous and globally bounded.

Assume *hyperbolic equilibrium* at 0: $h(0) = 0$; $Dh(0) \in L(\mathbf{R}^d)$ has all eigenvalues off imaginary axis.

Then (ODE) has a C_b^k flow $\phi : \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$ s.t.

(i) $\phi(\cdot, x) =$ unique solution of (ODE) through $x \in \mathbf{R}^d$.

(ii) $\phi(t, 0) = 0, t \in \mathbf{R}$.

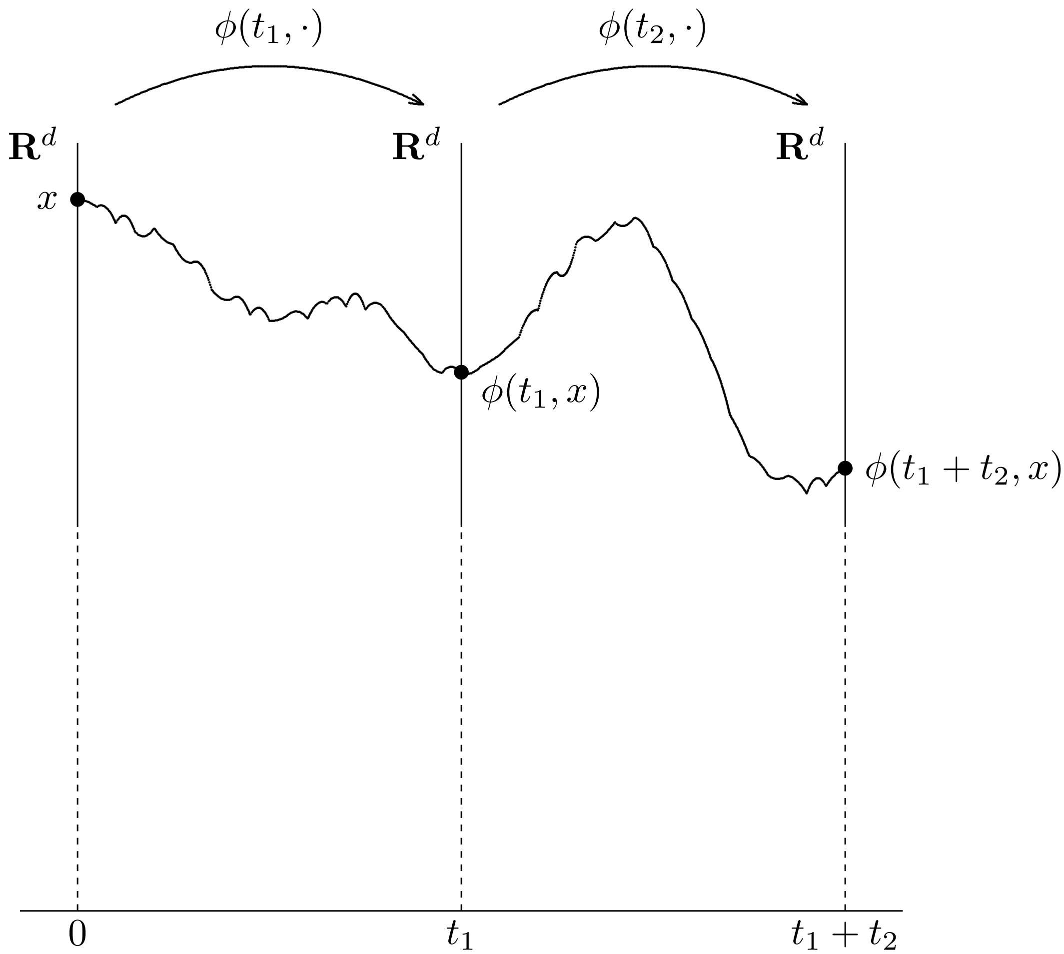
(iii) Group property:

$$\phi(t_1 + t_2, \cdot) = \phi(t_2, \cdot) \circ \phi(t_1, \cdot), \quad t_1, t_2 \in \mathbf{R}$$

(iv) Local flow-invariant stable/unstable C^k manifolds in a neighborhood of 0.

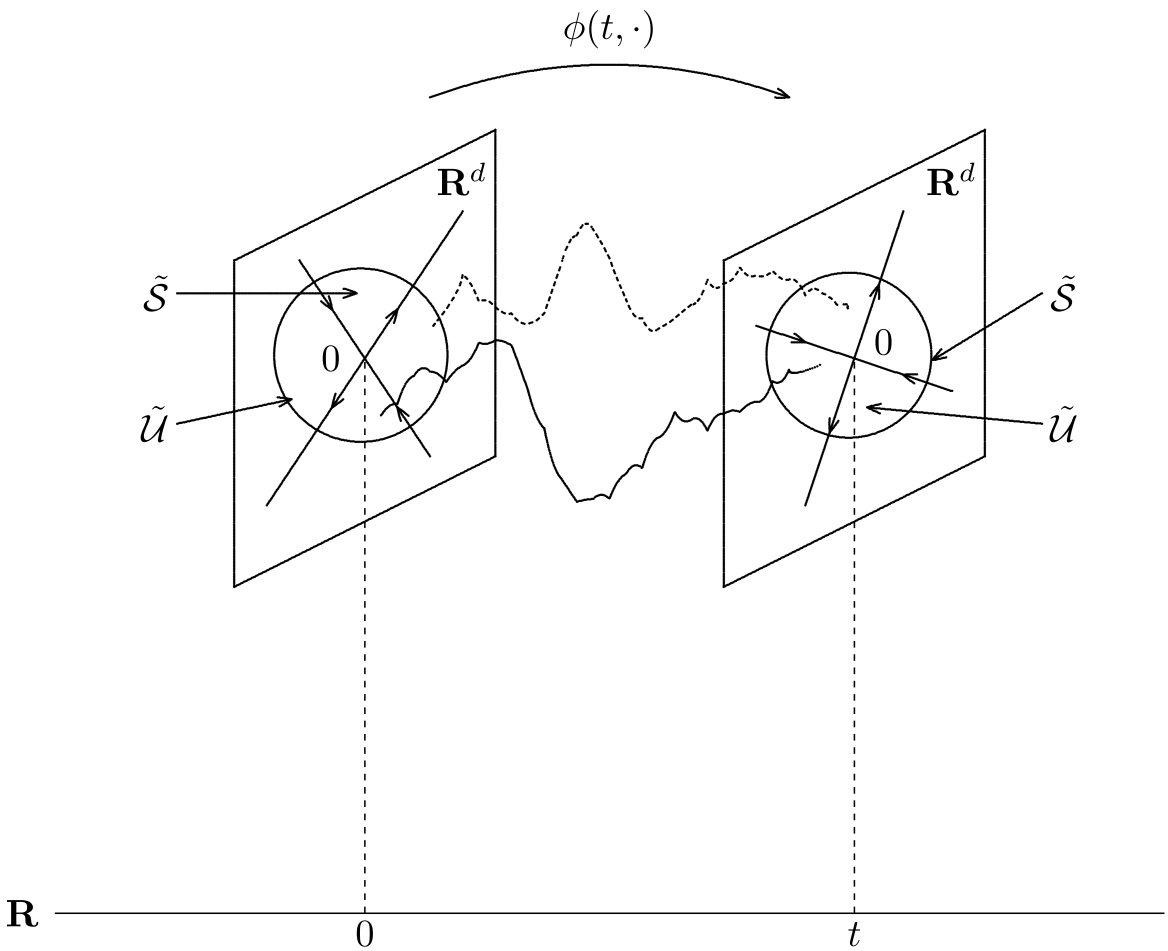
Properties (i)-(iv) are "generic" among all vector fields.

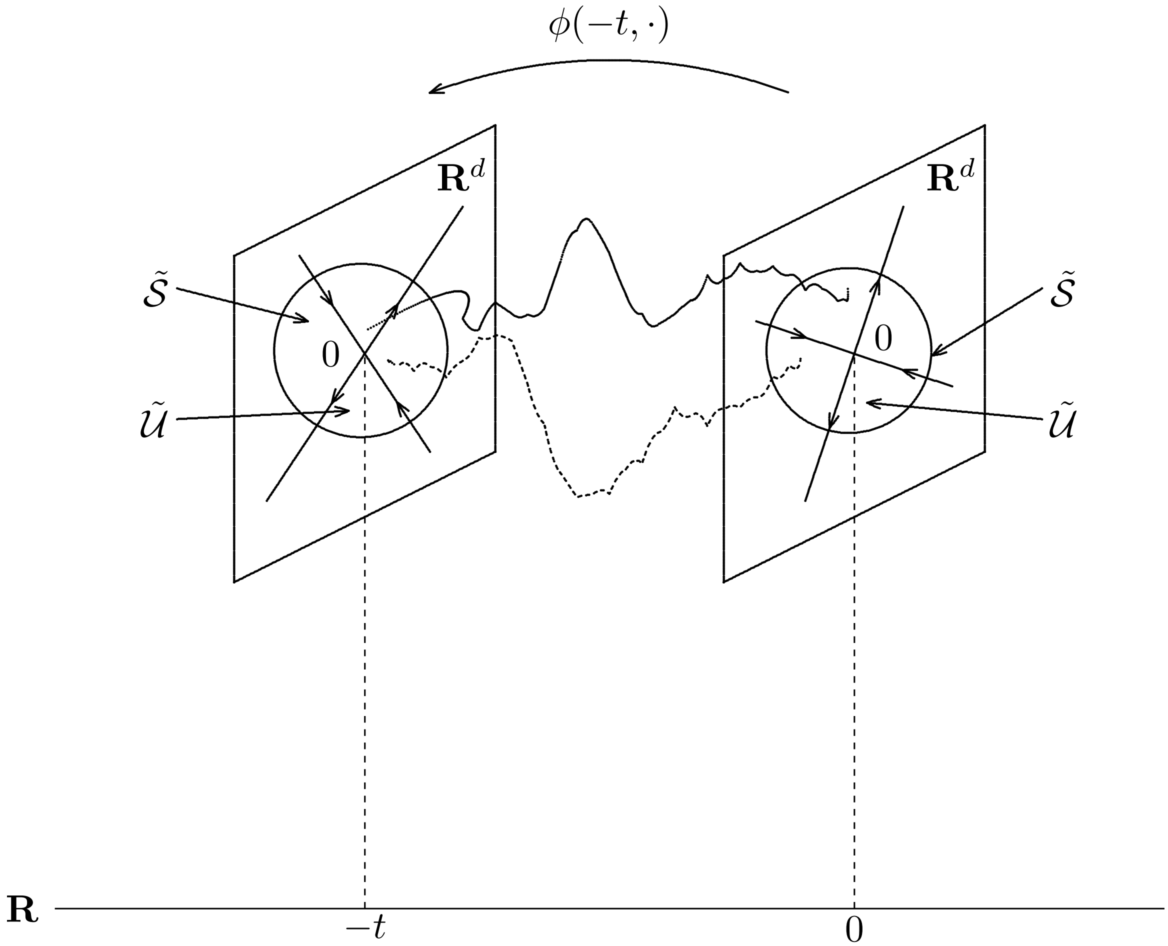
The Flow



$\phi(t_1, \cdot)$
$\phi(t_2, \cdot)$
$\mathbf{R}^d$
$\mathbf{R}^d$
$\mathbf{R}^d$
x
$\phi(t_1, x)$
$\phi(t_1 + t_2, x)$
0
t_1
$t_1 + t_2$

Local Stable/Unstable Manifolds





$\phi(-t,\cdot)$
$\mathbf{R}^d$
$\mathbf{R}^d$
$\tilde{\mathcal{S}}$
$\tilde{\mathcal{S}}$
0
0
$\tilde{\mathcal{U}}$
$\tilde{\mathcal{U}}$
$\mathbf{R}$
$-t$
0

What happens

if vector field

is noisy??

CASE STUDY:
Stochastic Systems with Memory
Stable Manifolds

Outline

- Smooth cocycles in Hilbert space. Stationary trajectories.
- Linearization of a cocycle along a stationary trajectory.
- Ergodic theory of cocycles in Hilbert space.
- Hyperbolicity of stationary trajectories. Lyapunov exponents.
- Cocycles generated by stochastic systems with memory. Via random diffeomorphism groups.
- *Local Stable Manifold Theorem* for stochastic differential equations with memory (SFDE's): Existence of smooth stable and unstable manifolds in a neighborhood of a hyperbolic stationary trajectory.
- Proof: Ruelle-Oseledec multiplicative ergodic theory+ perfection techniques.

The Cocycle

$(\Omega, \mathcal{F}, P) :=$ complete probability space.

$\theta : \mathbf{R}^+ \times \Omega \to \Omega$ a P-preserving (ergodic) semigroup on $(\Omega, \mathcal{F}, P)$.

$E :=$ real (separable) Hilbert space, norm $\|\cdot\|$, Borel σ-algebra.

Definition.

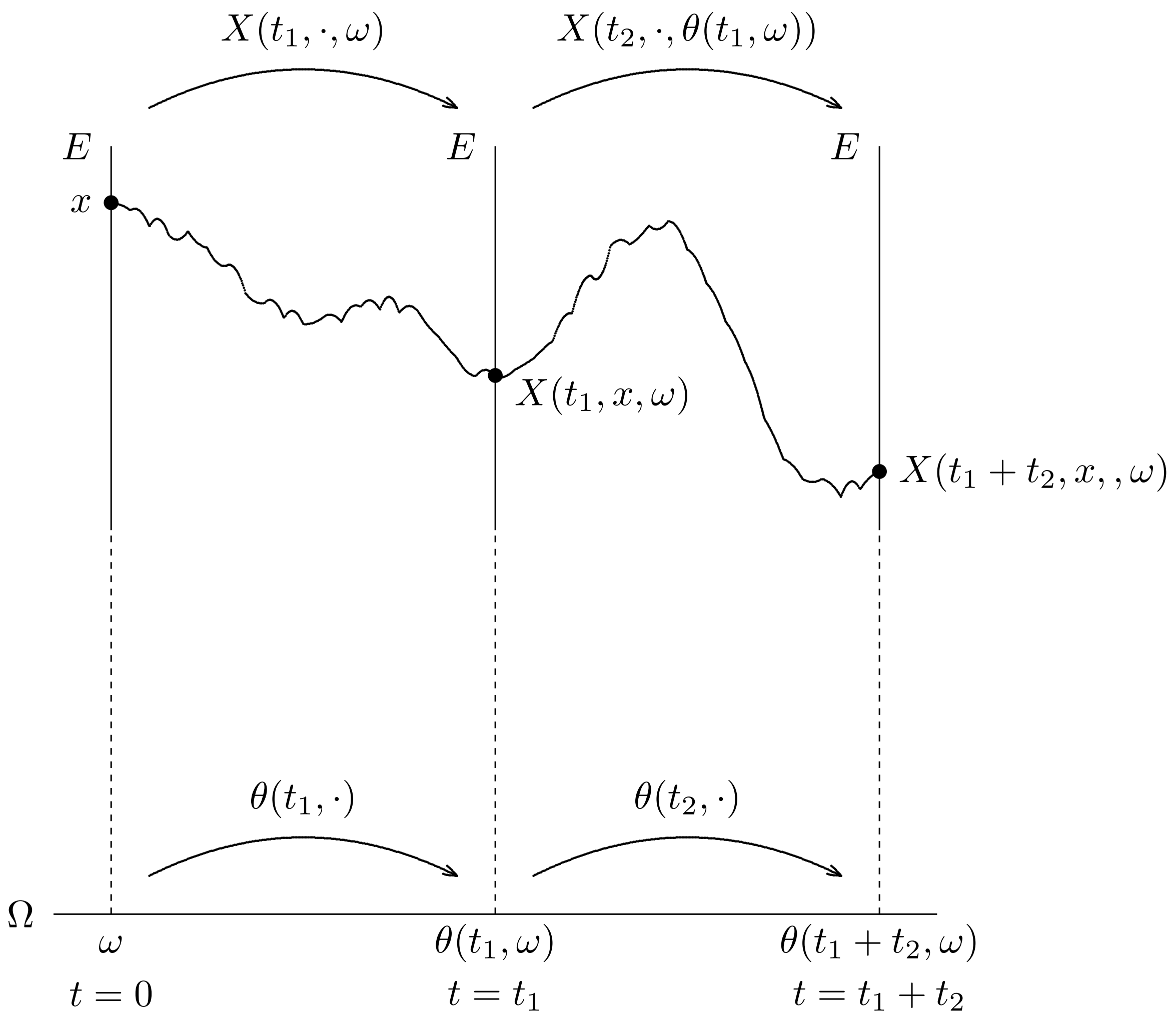
$k =$ non-negative integer, $\epsilon \in (0,1]$. A $C^{k,\epsilon}$ *perfect cocycle* (X, θ) on E is a measurable random field $X : \mathbf{R}^+ \times E \times \Omega \to E$ such that:

(i) For each $\omega \in \Omega$, the map $\mathbf{R}^+ \times E \ni (t,x) \mapsto X(t,x,\omega) \in E$ is continuous; for fixed $(t,\omega) \in \mathbf{R}^+ \times \Omega$, the map $E \ni x \mapsto X(t,x,\omega) \in E$ is $C^{k,\epsilon}$ $\big(D^k X(t,x,\omega)$ is C^ϵ in $x\big)$.

(ii) $X(t_1+t_2, \cdot, \omega) = X(t_2, \cdot, \theta(t_1,\omega)) \circ X(t_1, \cdot, \omega)$ for all $t_1, t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.

(iii) $X(0,x,\omega) = x$ for all $x \in E, \omega \in \Omega$.

Cocycle Property



Vertical solid lines represent random fibers: copies of E. (X, θ) is a "vector-bundle morphism".

Definition

A random variable $Y : \Omega \to E$ is a *stationary point* for the cocycle (X, θ) if

$$X(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{1}$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $X(t, Y) = Y(\theta(t))$.

Linearization. Hyperbolicity.

Linearize a $C^{k,\epsilon}$ cocycle (X, θ) along a stationary random point Y: Get an $L(E)$-valued cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega))$. (Follows from cocycle property of X and chain rule.)

Theorem. *(Oseledec-Ruelle)*

Let $T : \mathbf{R}^+ \times \Omega \to L(E)$ be strongly measurable, such that (T, θ) is an $L(E)$-valued cocycle, with each $T(t, \omega)$ compact. Suppose that

$$E \sup_{0 \le t \le 1} \log^+ \|T(t, \cdot)\|_{L(E)} < \infty, \quad E \sup_{0 \le t \le 1} \log^+ \|T(1-t, \theta(t, \cdot))\|_{L(E)} < \infty.$$

Then there is a sure event $\Omega_0 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \in \mathbf{R}^+$, and for each $\omega \in \Omega_0$,

$$\lim_{n \to \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. $\Lambda(\omega)$ is self-adjoint with a non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots$$

where the λ_i's are distinct. Each e^{λ_i} has a fixed finite non-random multiplicity m_i and eigen-space $F_i(\omega)$, with $m_i := dim F_i(\omega)$. Define

$$E_1(\omega) := E, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^\perp, \ i > 1.$$

Then

$$\cdots \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$\lim_{t\to\infty} \frac{1}{t} \log \|T(t,\omega)x\| = \lambda_i \quad \textit{if} \quad x \in E_i(\omega)\backslash E_{i+1}(\omega),$$

and

$$T(t,\omega)(E_i(\omega)) \subseteq E_i(\theta(t,\omega))$$

for all $t \geq 0, \ i \geq 1$.

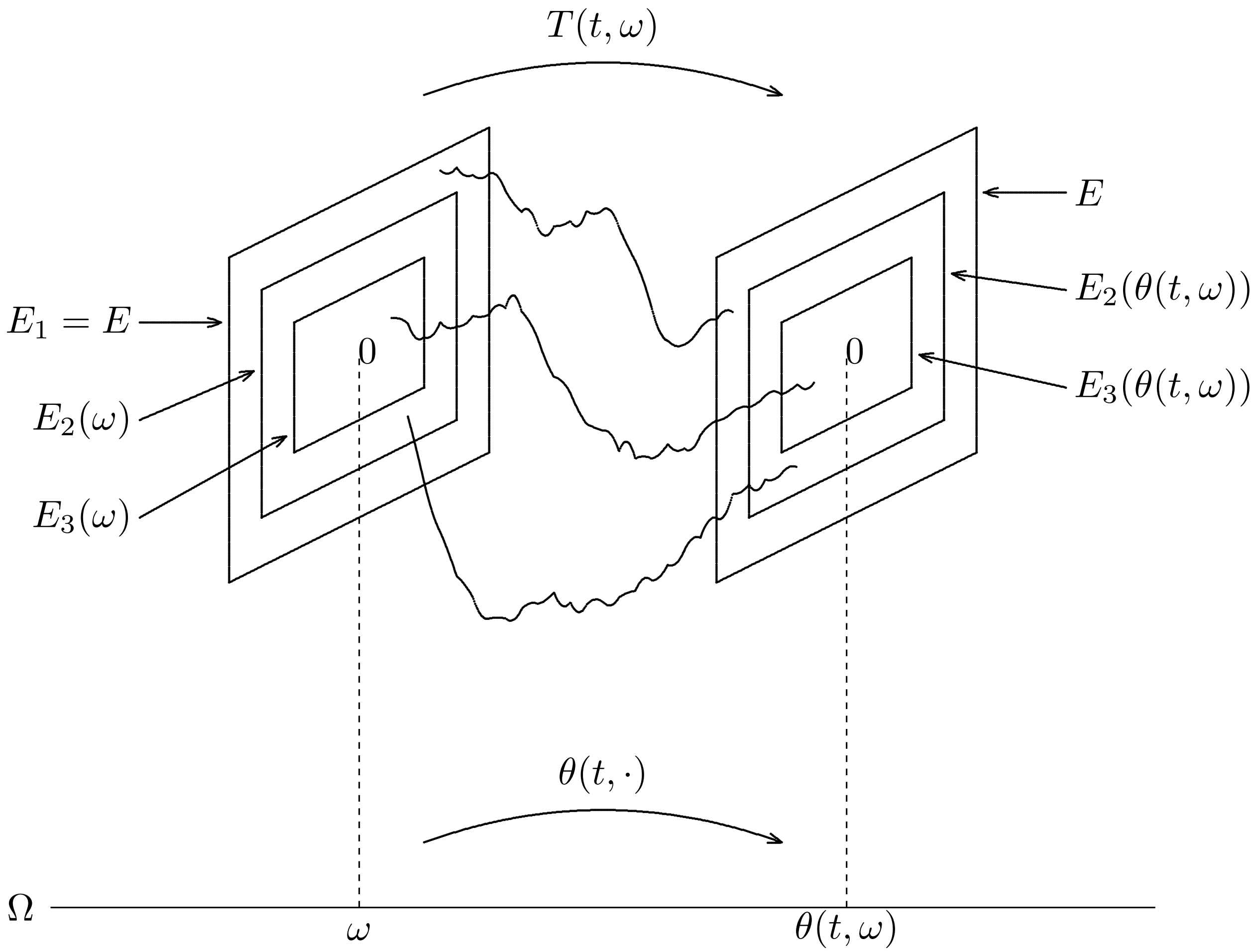
Proof.

Based on discrete version of Oseledec's multiplicative ergodic theorem and the perfect ergodic theorem. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]). □

Lyapunov Spectrum:

$\{\lambda_1, \lambda_2, \lambda_3, \cdots\} :=$ *Lyapunov spectrum* of (T, θ).

Spectral Theorem



Definition

A stationary point $Y(\omega)$ of (X, θ) is *hyperbolic* if the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega))$ has a non-vanishing Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, viz. $\lambda_i \neq 0$ for all $i \geq 1$.

Let $i_0 > 1$ be s.t. $\lambda_{i_0} < 0 < \lambda_{i_0 - 1}$.

Suppose

$$E \log^+ \sup_{0 \le t_1, t_2 \le r} \|D_2 X(t_2, Y(\theta(t_1)), \theta(t_1))\|_{L(E)}) < \infty.$$

By Oseledec-Ruelle Theorem, there is a sequence of closed finite-codimensional (Oseledec) spaces

$$\cdots E_{i-1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$E_i(\omega) = \{x \in E : \lim_{t \to \infty} \frac{1}{t} \log \|DX(t, Y(\omega), \omega)(x)\| \le \lambda_i\}, \quad i \ge 1,$$

for all $\omega \in \Omega^*$, a sure event in $\mathcal{F}$ satisfying $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$.

Let $\{U(\omega), S(\omega) : \omega \in \Omega^*\}$ be the unstable and stable subspaces associated with the linearized cocycle (DX, θ) ([Mo.1], Theorem 4, Corollary 2; [M-S.1], Theorem 5.3). Then get a measurable invariant splitting

$$E = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \Omega^*,$$

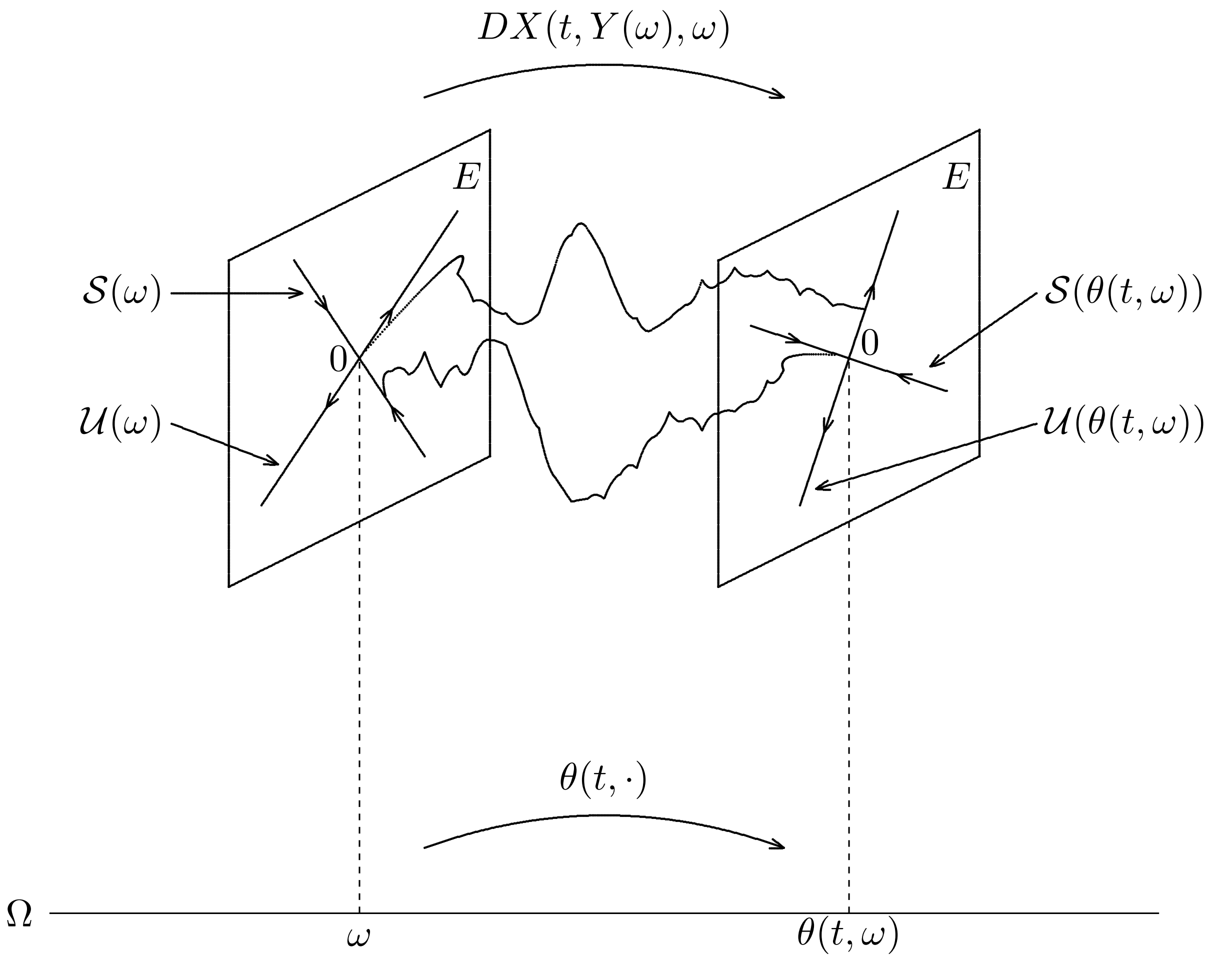
$$DX(t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)), \ DX(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)),$$

for all $t \ge 0$, with *exponential dichotomies*

$$\|DX(t, Y(\omega), \omega)(x)\| \ge \|x\| e^{\delta_1 t} \quad \text{for all} \quad t \ge \tau_1^*, x \in \mathcal{U}(\omega),$$

$$\|DX(t, Y(\omega), \omega)(x)\| \leq \|x\| e^{-\delta_2 t} \quad \text{for all} \quad t \geq \tau_2^*, x \in \mathcal{S}(\omega),$$

with $\tau_i^* = \tau_i^*(x, \omega) > 0, i = 1, 2,$ random times and $\delta_i > 0, i = 1, 2,$ fixed.



Stochastic Systems with Memory

"Regular" Itô SFDE with finite memory:

$$\left.\begin{aligned} dx(t) = H(x(t), x_t)\, dt + \sum_{i=1}^{m} G_i(x(t))\, dW_i(t), \\ (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (I)$$

Solution segment $x_t(s) := x(t+s),\ t \geq 0, s \in [-r, 0]$.

m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$, $W(0) = 0$.

Ergodic Brownian shift θ on Wiener space $(\Omega, \mathcal{F}, P)$. $\bar{\mathcal{F}} := P-$completion of $\mathcal{F}$.

State space M_2, Hilbert with usual norm $\|\cdot\|$.

Can allow for "smooth memory" in diffusion coefficient.

$H : M_2 \to \mathbf{R}^d$, $C^{k,\delta}$, globally bounded.

$G : \mathbf{R}^d \to L(\mathbf{R}^m, \mathbf{R}^d),\ C_b^{k+1,\delta};\ G := (G_1, \cdots, G_m)$.

$B((v, \eta), \rho)$ open ball, radius ρ, center $(v, \eta) \in M_2$;

$\bar{B}((v, \eta), \rho)$ closed ball.

Then (I) has a stochastic semiflow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ with $X(t, (v, \eta), \cdot) = (x(t), x_t)$. X is $C^{k,\epsilon}$ for any $\epsilon \in (0, \delta)$, takes bounded sets into relatively compact sets in M_2. (X, θ) is a perfect cocycle on M_2 ([M-S.4]).

Theorem. *([M-S], 1999) (Local Stable and Unstable Manifolds)*

Assume smoothness hypotheses on H and G. Let $Y : \Omega \to M_2$ be a hyperbolic stationary point of the SFDE (I) such that $E(\|Y(\cdot)\|^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$

Suppose the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ of (I) has a Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. Define $\lambda_{i_0} :=$ max$\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all finite λ_i are positive, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized semiflow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,

(ii) $\bar{\mathcal{F}}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k,\epsilon}$ $(\epsilon \in (0, \delta))$ submanifolds $\tilde{\mathcal{S}}(\omega)$, $\tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) $\tilde{\mathcal{S}}(\omega)$ is the set of all $(v, \eta) \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$\|X(nr, (v, \eta), \omega) - Y(\theta(nr, \omega))\| \leq \beta_1(\omega)\, e^{(\lambda_{i_0} + \epsilon_1)nr}$$

for all integers $n \geq 0$. *Furthermore,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega) - Y(\theta(t,\omega))\| \leq \lambda_{i_0}$$

for all $(v,\eta) \in \tilde{\mathcal{S}}(\omega)$. *Each stable subspace* $\mathcal{S}(\omega)$ *of the linearized semiflow* DX *is tangent at* $Y(\omega)$ *to the submanifold* $\tilde{\mathcal{S}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. *In particular, codim* $\tilde{\mathcal{S}}(\omega) =$ *codim* $\mathcal{S}(\omega)$, *is fixed and finite.*

(b) $\limsup_{t\to\infty} \frac{1}{t} \log\left[\sup\left\{ \frac{\|X(t,(v_1,\eta_1),\omega) - X(t,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1)-(v_2,\eta_2)\|} : (v_1,\eta_1) \neq (v_2,\eta_2),\ (v_1,\eta_1),\ (v_2,\eta_2) \in \tilde{\mathcal{S}}(\omega)\right\}\right] \leq \lambda_{i_0}$.

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ *such that*

$$X(t,\cdot,\omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t,\omega))$$

for all $t \geq \tau_1(\omega)$. *Also*

$$DX(t,Y(\omega),\omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)), \quad t \geq 0.$$

(d) $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $(v,\eta) \in \bar{B}(Y(\omega),\rho_2(\omega))$ *with the property that there is a unique "history" process* $y(\cdot,\omega) : \{-nr : n \geq$

$0\} \to M_2$ *such that* $y(0,\omega) = (v,\eta)$ *and for each integer* $n \geq 1$, *one has* $X(r, y(-nr,\omega), \theta(-nr,\omega)) = y(-(n-1)r,\omega)$ *and*

$$\|y(-nr,\omega) - Y(\theta(-nr,\omega))\|_{M_2} \leq \beta_2(\omega) e^{-(\lambda_{i_0-1}-\epsilon_2)nr}.$$

Furthermore, for each $(v,\eta) \in \tilde{\mathcal{U}}(\omega)$, *there is a unique continuous-time "history" process also denoted by* $y(\cdot,\omega) : (-\infty, 0] \to M_2$ *such that* $y(0,\omega) = (v,\eta)$, $X(t, y(s,\omega), \theta(s,\omega)) = y(t+s,\omega)$ *for all* $s \leq 0, 0 \leq t \leq -s$, *and*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|y(-t,\omega) - Y(\theta(-t,\omega))\| \leq -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}(\omega)$ *of the linearized semiflow* DX *is tangent at* $Y(\omega)$ *to* $\tilde{\mathcal{U}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. *In particular,* $\dim \tilde{\mathcal{U}}(\omega)$ *is finite and non-random.*

(e) Let $y(\cdot,(v_i,\eta_i),\omega), i = 1,2$, *be the history processes associated with* $(v_i,\eta_i) = y(0,(v_i,\eta_i),\omega) \in \tilde{\mathcal{U}}(\omega)$, $i = 1,2$. *Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log\left[\sup\left\{ \frac{\|y(-t,(v_1,\eta_1),\omega) - y(-t,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1) - (v_2,\eta_2)\|} : (v_1,\eta_1) \neq (v_2,\eta_2),\ (v_i,\eta_i) \in \tilde{\mathcal{U}}(\omega), i = 1,2 \right\}\right] \leq -\lambda_{i_0-1}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ *such that*

$$\tilde{\mathcal{U}}(\omega) \subseteq X(t,\cdot,\theta(-t,\omega))(\tilde{\mathcal{U}}(\theta(-t,\omega)))$$

for all $t \geq \tau_2(\omega)$. *Also*

$$DX(t, \cdot, \theta(-t, \omega))(\mathcal{U}(\theta(-t, \omega))) = \mathcal{U}(\omega), \quad t \geq 0;$$

and the restriction

$$DX(t, \cdot, \theta(-t, \omega))|\mathcal{U}(\theta(-t, \omega)) : \mathcal{U}(\theta(-t, \omega)) \to \mathcal{U}(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto.

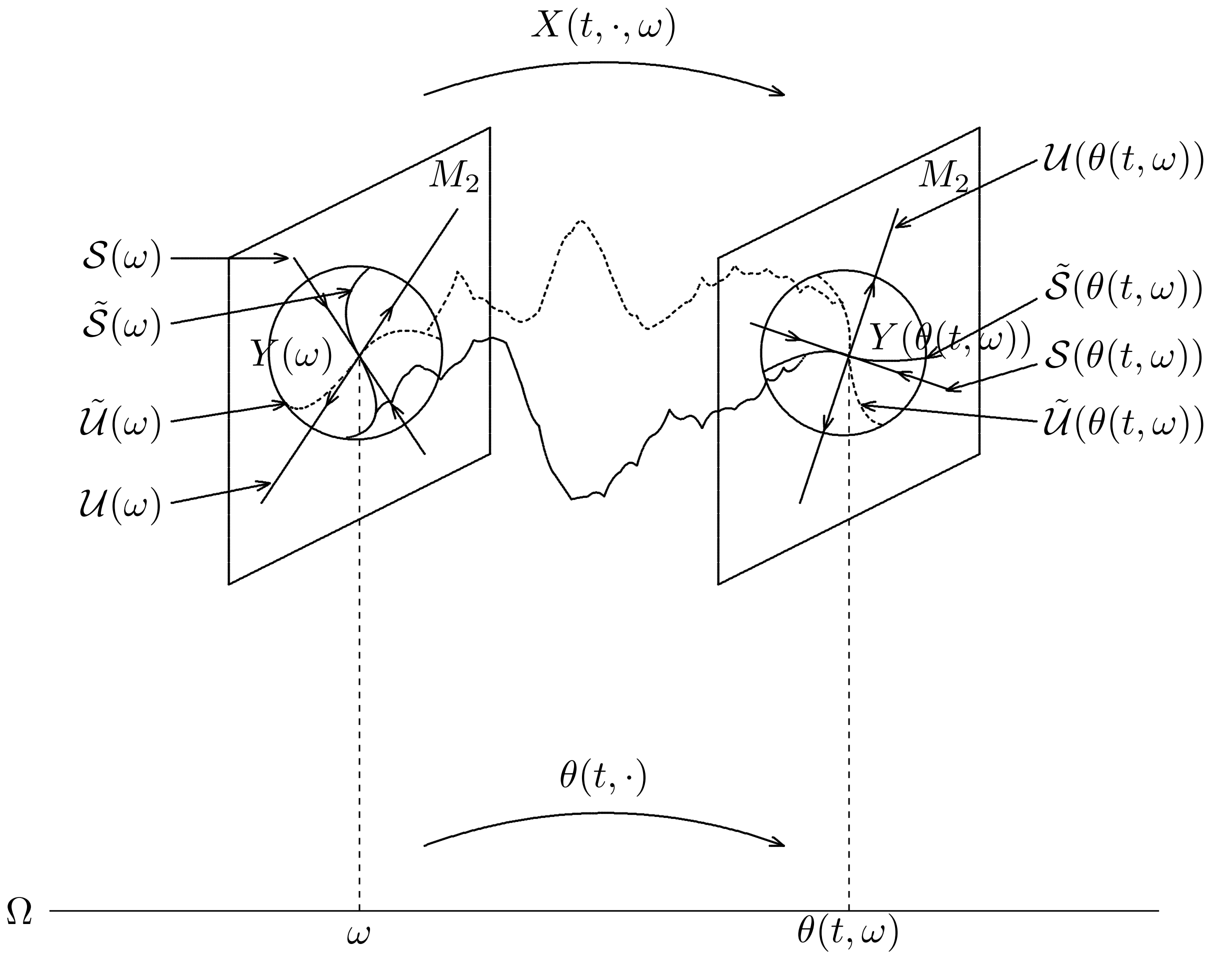
(g) *The submanifolds* $\tilde{\mathcal{U}}(\omega)$ *and* $\tilde{\mathcal{S}}(\omega)$ *are transversal, viz.*

$$M_2 = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

Assume, in addition, that H, G *are* C_b^∞. *Then the local stable and unstable manifolds* $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ *are* C^∞.

Figure summarizes essential features of Stable Manifold Theorem:

Stable Manifold Theorem



$t > \tau_1(\omega)$

A picture is worth a 1000 words!

Example

Affine linear sfde:

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t)\,dt + G\,dW(t), \quad t > 0 \\ x(0) &= v \in \mathbf{R}^d, \quad x_0 = \eta \in L^2([-r,0], \mathbf{R}^d) \end{aligned}\right\} \qquad (I'')$$

$H : M_2 \to \mathbf{R}^d$ continuous linear map, G a fixed $(d \times p)$-matrix, and W p-dimensional Brownian motion. Assume that the $(d \times d)$-matrix-valued FDE

$$dy(t) = H \circ (y(t), y_t)\,dt$$

has a semiflow

$$T_t : L(\mathbf{R}^d) \times L^2([-r,0], L(\mathbf{R}^d)) \to L(\mathbf{R}^d) \times L^2([-r,0], L(\mathbf{R}^d)), t \geq 0,$$

which is uniformly asymptotically stable. Set

$$Y := \int_{-\infty}^0 T_{-u}(I, 0) G\,dW(u) \qquad (2)$$

where I is the identity $(d \times d)$-matrix. Integration by parts and

$$W(t, \theta(t_1, \omega)) = W(t + t_1, \omega) - W(t_1, \omega), \quad t, t_1 \in \mathbf{R}, \qquad (3)$$

imply that Y has a measurable version satisfying (1). Y is Gaussian and thus has finite moments of all orders. See

([Mo.1], Theorem 4.2, Corollary 4.2.1, pp. 208-217.) More generally, when H is hyperbolic, one can show that a stationary point of (I'') exists ([Mo.1]).

For general white-noise case, get an invariant measure on M_2 for the one-point motion by enlarging probability space. Conversely, let $Y : \Omega \to M_2$ be a stationary random point independent of the Brownian motion $W(t)$, $t \geq 0$. Then $\rho := P \circ Y^{-1}$ (distribution of Y) is an invariant measure for the one-point motion (by independence of Y and W).

Outline of Proof

- By definition, a *stationary* random point $Y(\omega) \in M_2$ is invariant under the semiflow X; viz $X(t, Y) = Y(\theta(t, \cdot))$ for all times t.
- Linearize the semiflow X along the stationary point $Y(\omega)$ in M_2. By stationarity of Y and the cocycle property of X, this gives a linear perfect cocycle $(DX(t, Y), \theta(t, \cdot))$ in $L(M_2)$, where $D =$ spatial (Fréchet) derivatives.
- Ergodicity of θ allows for the notion of *hyperbolicity* of a stationary solution of (I) via Oseledec-Ruelle theorem: Use local compactness of the semiflow for times greater than the delay r ([M-S.4]), and apply multiplicative ergodic theorem to get a discrete nonrandom Lyapunov spectrum $\{\lambda_i : i \geq 1\}$ for the linearized cocycle. Y is *hyperbolic* if $\lambda_i \neq 0$ for every i.
- Assume that $\|Y\|^{\epsilon_0}$ is integrable (for small ϵ_0). Variational method of construction of the semiflow shows that the linearized cocycle satisfies hypotheses of "perfect versions" of ergodic theorem and Kingman's subadditive ergodic theorem. These refined versions give

invariance of the Oseledec spaces under the continuous-time linearized cocycle. Thus the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear semiflow X.

- Establish continuous-time integrability estimates on the spatial derivatives of the non-linear cocycle X in a neighborhood of the stationary point Y. Estimates follow from the variational construction of the stochastic semiflow coupled with known global spatial estimates for finite-dimensional stochastic flows.
- Introduce the auxiliary perfect cocycle

$$Z(t,\cdot,\omega) := X(t,(\cdot)+Y(\omega),\omega) - Y(\theta(t,\omega)), \quad t \in \mathbf{R}^+, \omega \in \Omega.$$

Refine arguments in ([Ru.2], Theorems 5.1 and 6.1) to construct local stable/unstable manifolds for the discrete cocycle $(Z(nr,\cdot,\omega), \theta(nr,\omega))$ near 0 and hence (by translation) for $X(nr,\cdot,\omega)$ near $Y(\omega)$ for all ω sampled from a $\theta(t,\cdot)$-invariant sure event in Ω. This is possible because of the continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between delay periods of length r and further refining the arguments in [Ru.2], show that the above manifolds also serve as local stable/unstable manifolds for the *continuous-time* semiflow X near Y.

- Final key step: Establish the asymptotic invariance of the local stable manifolds under the stochastic semiflow X. Use arguments underlying the proofs of Theorems 4.1 and 5.1 in [Ru.2] and some difficult estimates using the continuous-time integrability properties, and the perfect subadditive ergodic theorem. Asymptotic invariance of the local unstable manifolds follows by employing the concept of a *stochastic history process* for X coupled with similar arguments to the above. Existence of history process compensates for the lack of invertibility of the semiflow.

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