8-1998

On the Dynamics of Stochastic Differential Systems (The Seventh Vilnius Conference on Probability Theory and Mathematical Statistics)

Salah-Eldin A. Mohammed
Southern Illinois University Carbondale, salah@sfde.math.siu.edu

The Seventh Vilnius Conference on Probability Theory and Mathematical Statistics; August 12-18, 1998; Vilnius, Lithuania

ON THE DYNAMICS OF STOCHASTIC DIFFERENTIAL SYSTEMS

Vilnius : August 13, 1998

Salah-Eldin A. Mohammed

Southern Illinois University

Carbondale, IL 62901–4408 USA

Web site: **http://salah.math.siu.edu**

Outline

- Formulate a *Local Stable Manifold Theorem* for stochastic differential equations with and without memory (SFDE's and SODE's).
- Spatial Kunita-type semimartingales noise, with stationary ergodic increments.
- Start with the existence of a stochastic semiflow for SDE.
- Concept of a hyperbolic stationary trajectory. For Stratonovich SODE, stationary trajectory is a solution of the forward /backward anticipating equation for all time.
- Existence of a stationary random family of asymptotically invariant stable and unstable manifolds within a stationary neighborhood of the hyperbolic stationary solution.
- For Stratonovich SODE, stable and unstable manifolds are dynamically characterized using forward and backward solutions of anticipating versions of the equation.

- Proofs based on Ruelle-Oseledec (non-linear) multiplicative ergodic theory and anticipating stochastic calculus.

Formulation of the Theorem

I. SODE Case:

Stratonovich SODE

$$dx(t) = h(x(t))\,dt + \sum_{i=1}^{m} g_i(x(t)) \circ dW_i(t), \qquad (I)$$

on $\mathbf{R}^d$ driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$.

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}}, P) :=$ canonical filtered Wiener space.

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R} \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F} :=$ Borel σ-field of Ω;

$\mathcal{F}_t :=$ sub-σ-field of $\mathcal{F}$ generated by the evaluations $\omega \to \omega(u),\;\; u \leq t, \quad t \in \mathbf{R}$.

$P :=$ Wiener measure on Ω.

$h, g_i : \mathbf{R}^d \to \mathbf{R}^d, 1 \leq i \leq m$, vector fields on $\mathbf{R}^d$. For some $k \geq 1, \delta \in (0,1)$, h is $C_b^{k,\delta}$, viz. h has all derivatives $D^j h, 1 \leq j \leq k$, continuous and globally bounded, $D^k h$ Hölder continuous with exponent δ. g_i, $1 \leq i \leq m$, globally bounded and $C_b^{k+1,\delta}$.

$\theta : \mathbf{R} \times \Omega \to \Omega$ is the (ergodic) Brownian shift

$$\theta(t, \omega)(s) := \omega(t+s) - \omega(t), \quad t, s \in \mathbf{R}, \, \omega \in \Omega.$$

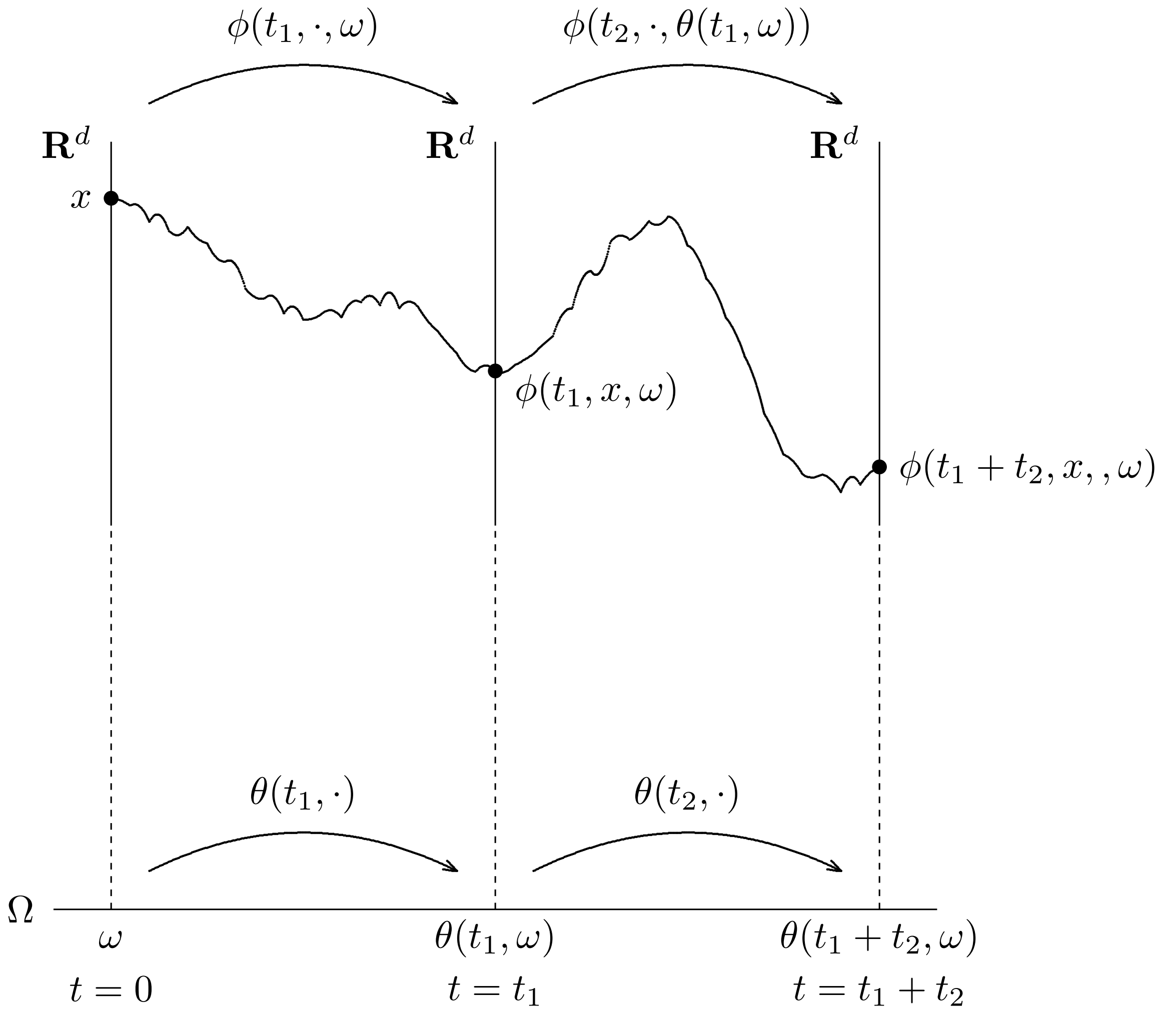
Let $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ be the stochastic flow of (I) $\big(\phi(t, \cdot, \omega) = [\phi(-t, \cdot, \theta(t, \omega))]^{-1}, t < 0\big)$. Then ϕ is a perfect $C^{k,\epsilon}$ cocycle:

$$\phi(t+s, \cdot, \omega) = \phi(t, \cdot, \theta(s, \omega)) \circ \phi(s, \cdot, \omega),$$

for all $s, t \in \mathbf{R}$ and all $\omega \in \Omega, \epsilon \in (0, \delta)$ ([I-W], [Ku], [A-S]).

Figure illustrates the cocycle property. Vertical solid lines represent random fibers consisting of copies of $\mathbf{R}^d$ (or a Banach space of paths in $\mathbf{R}^d$.) (ϕ, θ) is a "random vector-bundle morphism" over the "base" probability space Ω.

The Cocycle



$\phi(t_1, \cdot, \omega)$
$\phi(t_2, \cdot, \theta(t_1, \omega))$
$\mathbf{R}^d$
$\mathbf{R}^d$
$\mathbf{R}^d$
x
$\phi(t_1, x, \omega)$
$\phi(t_1 + t_2, x, , \omega)$
$\theta(t_1, \cdot)$
$\theta(t_2, \cdot)$
Ω
ω
$\theta(t_1, \omega)$
$\theta(t_1 + t_2, \omega)$
$t = 0$
$t = t_1$
$t = t_1 + t_2$

Definition

SODE (I) has a *stationary point* if there exists an $\mathcal{F}$-measurable random variable $Y : \Omega \to \mathbf{R}^d$ such that

$$\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{1}$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $\phi(t, Y) = Y(\theta(t))$.

Examples of Stationary Points

1. Fixed points:

$$d\phi(t) = h(\phi(t))\, dt + \sum_{i=1}^{m} g_i(\phi(t)) \circ dW_i(t)$$

$$h(x_0) = g_i(x_0) = 0, \quad 1 \le i \le m$$

Take $Y(\omega) = x_0$ for all $\omega \in \Omega$.

2. Linear affine case $d = 1$:

$$d\phi(t) = \lambda\phi(t)\, dt + dW(t)$$

$\lambda > 0$ fixed, $W(t) \in \mathbf{R}$. Take

$$Y(\omega) := -\int_0^\infty e^{-\lambda u}\, dW(u),$$

$$\theta(t,\omega)(s) = \omega(t+s) - \omega(t).$$

Check that $\phi(t, Y(\omega), \omega) = Y(\theta(t,\omega))$, using integration by parts and variation of parameters.

3. Affine linear SODE in $d = 2$:

$$d\phi(t) = A\phi(t)\,dt + GdW(t)$$

with A a fixed hyperbolic 2×2-diagonal matrix; G a constant matrix.

4. Generate a large class of stationary points as follows: Let ρ be an invariant probability measure for the one-point motion in $\mathbf{R}^d$. Then ρ gives rise to a stationary point by suitably enlarging the underlying probability space: If $P_t : C_b(\mathbf{R}^d, \mathbf{R}) \to C_b(\mathbf{R}^d, \mathbf{R}), t \geq 0$, is the Markov semigroup associated with the SODE, then

$$\int_{\mathbf{R}^d} (P_t f)(x) d\rho(x) = \int_{\mathbf{R}^d} f(x)\, d\rho(x), \quad t \geq 0$$

where

$$(P_t f)(x) := E[f(\phi(t, x, \cdot))], \quad t \geq 0,\ x \in \mathbf{R}^d,$$

for all $f \in C_b(\mathbf{R}^d, \mathbf{R})$. Define

$$\tilde{\Omega} := \Omega \times \mathbf{R}^d, \quad \tilde{\mathcal{F}} := \mathcal{F} \otimes \mathcal{B}(\mathbf{R}^d), \quad \tilde{P} := P \otimes \rho, \quad \tilde{\omega} := (\omega, x) \in \tilde{\Omega},$$
$$\tilde{\theta}(t, \tilde{\omega}) := (\theta(t, \omega), \phi(t, x, \omega)),\ t \in \mathbf{R}^+,\ \omega \in \Omega, x \in \mathbf{R}^d$$
$$\tilde{\phi}(t, x', \tilde{\omega}) := \phi(t, x', \omega), \quad t \in \mathbf{R}^+,\ x' \in \mathbf{R}^d,\ \tilde{\omega} \in \tilde{\Omega}$$
$$\tilde{Y}(\tilde{\omega}) := x, \quad \tilde{\omega} = (\omega, x) \in \tilde{\Omega}.$$

The group $\tilde{\theta}(t, \cdot) : \tilde{\Omega} \to \tilde{\Omega}, t \in \mathbf{R}^+$, is $\tilde{P}$-preserving (and ergodic) (Carverhill [C]). $(\tilde{\phi}(t, \cdot, \tilde{\omega}), \tilde{\theta}(t, \tilde{\omega}))$ is a perfect cocycle on $\mathbf{R}^d$; and $\tilde{Y} : \tilde{\Omega} \to \mathbf{R}^d$ satisfies

$$\tilde{\phi}(t, \tilde{Y}(\tilde{\omega}), \tilde{\omega}) = \tilde{Y}(\tilde{\theta}(t, \tilde{\omega}))$$

for all $t \in \mathbf{R}^+, \tilde{\omega} \in \tilde{\Omega}$. Hence $\tilde{Y}$ is a stationary point for the cocycle $(\tilde{\phi}, \tilde{\theta})$, and $\rho = \tilde{P} \circ \tilde{Y}^{-1}$.

Conversely, let $Y : \Omega \to \mathbf{R}^d$ be a stationary random point satisfying the identity (1) and independent of the Brownian motion $W(t)$, $t \geq 0$. Then $\rho := P \circ Y^{-1}$ is an invariant measure for the one-point motion.

Let $\phi(t, Y)$ be a stationary solution of (I). Cocycle property of ϕ implies that the linearization

$$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$$

along the stationary solution is also a $d \times d$-matrix-valued cocycle. Using Kolmogorov's theorem, the random variables

$$\sup_{x \in \mathbf{R}^d} \frac{|D_2\phi(t,x)|}{(1+|x|^\gamma)}, \ \gamma > 0,$$

have moments of all orders. If $E \log^+ |Y| < \infty$, then $E \log^+ |D_2\phi(1,Y)| < \infty$. Apply Oseledec's Theorem to get a *non-random* finite Lyapunov spectrum:

$$\lim_{n\to\infty} \frac{1}{n} \log |D_2\phi(n, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbf{R}^d).$$

Spectrum takes finitely many values $\{\lambda_i\}_{i=1}^p$ with non-random multiplicities q_i, $1 \le i \le p$, and $\sum_{i=1}^{p} q_i = d$ ([Ru.1], Theorem I.6).

Definition

A stationary point Y is *hyperbolic* if $E \log^+ |Y(\cdot)| < \infty$, and if $(D_2\phi(n, Y(\omega), \omega), \theta(n,\omega))$ has a non-vanishing Lyapunov spectrum

$$\{\lambda_p < \cdots < \lambda_{i_0+1} < \lambda_{i_0} < 0 < \lambda_{i_0-1} < \cdots < \lambda_2 < \lambda_1\}$$

i.e. $\lambda_i \neq 0$ for all $1 \le i \le p$.

Define $\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all $\lambda_i > 0$, set $\lambda_{i_0} = -\infty$. (This implies that $\lambda_{i_0-1} := \min\{\lambda_i :$

$\lambda_i > 0\}$, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Let $\rho \in \mathbf{R}^+$, $x \in \mathbf{R}^d$.

$B(x, \rho) :=$ open ball in $\mathbf{R}^d$, center x and radius ρ;
$\bar{B}(x, \rho) :=$ corresponding closed ball;
$\mathcal{C}(\mathbf{R}^d) :=$ the class of all non-empty compact subsets of $\mathbf{R}^d$ with Hausdorff metric d^*:

$$d^*(A_1, A_2) := \sup\{d(x, A_1) : x \in A_2\} \vee \sup\{d(y, A_2) : y \in A_1\}$$

where $A_1, A_2 \in \mathcal{C}(\mathbf{R}^d)$;
$d(x, A_i) := \inf\{|x - y| : y \in A_i\}$, $x \in \mathbf{R}^d$, $i = 1, 2$;
$\mathcal{B}(\mathcal{C}(\mathbf{R}^d)) :=$ Borel σ-algebra on $\mathcal{C}(\mathbf{R}^d)$ with respect to the metric d^*.

Theorem 1 (Stable Manifold Theorem-SODE) ([M-S], 1997)

Assume that the coefficients of SODE (I) satisfy the given hypotheses. Suppose $\phi(t, Y)$ is a hyperbolic stationary trajectory of (I) with $E \log^+ |Y| < \infty$.

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

(i) *a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,*

(ii) *$\mathcal{F}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:*

There are $C^{k,\epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds $\tilde{\mathcal{S}}(\omega)$, $\tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) *$\tilde{\mathcal{S}}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that*

$$|\phi(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega)\, e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \to \infty} \frac{1}{t} \log |\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0} \tag{2}$$

for all $x \in \tilde{\mathcal{S}}(\omega)$. Each stable subspace $\mathcal{S}(\omega)$ of the linearized flow $D_2\phi$ is tangent at $Y(\omega)$ to the submanifold $\tilde{\mathcal{S}}(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. In particular, dim $\tilde{\mathcal{S}}(\omega) =$ dim $\mathcal{S}(\omega)$ and is non-random.

(b) $\limsup_{t\to\infty} \frac{1}{t} \log\left[\sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{\mathcal{S}}(\omega)}} \left\{ \frac{|\phi(t, x_1, \omega) - \phi(t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq \lambda_{i_0}$.

(c) *(Cocycle-invariance of the stable manifolds):*

There exists $\tau_1(\omega) \geq 0$ *such that*

$$\phi(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega)), \quad t \geq \tau_1(\omega). \tag{3}$$

Also

$$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t, \omega)), \quad t \geq 0. \tag{4}$$

(d) $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $x \in \bar{B}(Y(\omega), \rho_2(\omega))$ *with the property that*

$$|\phi(-n, x, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega)\, e^{(-\lambda_{i_0-1}+\epsilon_2)n} \tag{5}$$

for all integers $n \geq 0$. *Also*

$$\limsup_{t\to\infty} \frac{1}{t} \log|\phi(-t, x, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1}. \tag{6}$$

for all $x \in \tilde{\mathcal{U}}(\omega)$. *Furthermore, the unstable subspace* $\mathcal{U}(\omega)$ *of* $D_2\phi$ *is the tangent space to* $\tilde{\mathcal{U}}(\omega)$ *at* $Y(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. *In particular,* $\dim \tilde{\mathcal{U}}(\omega) = \dim \mathcal{U}(\omega)$ *and is non-random.*

(e) $\limsup_{t\to\infty} \frac{1}{t} \log\left[\sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{\mathcal{U}}(\omega)}} \left\{ \frac{|\phi(-t, x_1, \omega) - \phi(-t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq -\lambda_{i_0-1}$.

(f) *(Cocycle-invariance of the unstable manifolds):*

There exists $\tau_2(\omega) \geq 0$ *such that*

$$\phi(-t, \cdot, \omega)(\tilde{\mathcal{U}}(\omega)) \subseteq \tilde{\mathcal{U}}(\theta(-t, \omega)), \quad t \geq \tau_2(\omega). \tag{7}$$

Also

$$D_2\phi(-t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(-t, \omega)), \quad t \geq 0. \tag{8}$$

(g) *The submanifolds* $\tilde{\mathcal{U}}(\omega)$ *and* $\tilde{\mathcal{S}}(\omega)$ *are transversal, viz.*

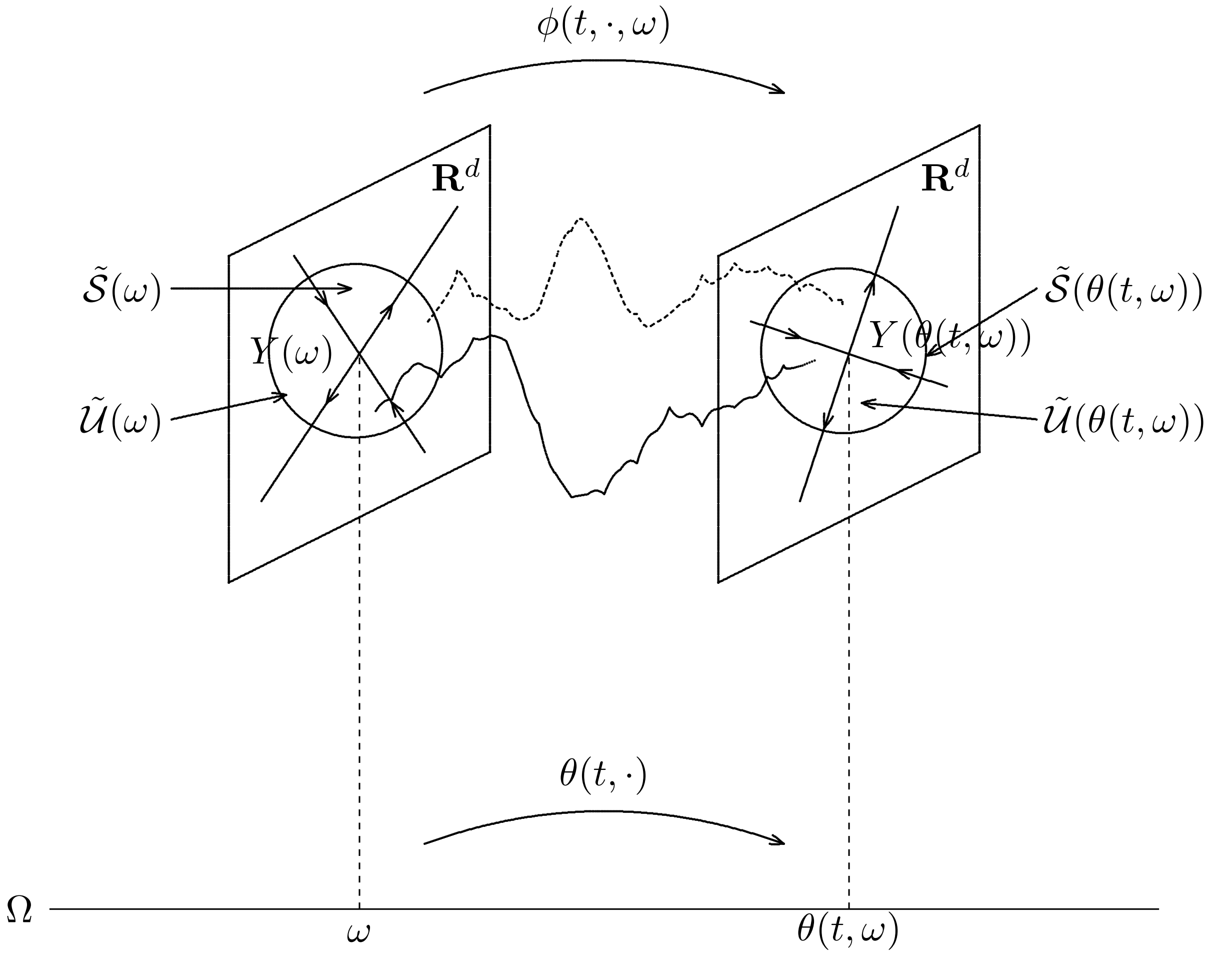
$$\mathbf{R}^d = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega). \tag{9}$$

(h) *The mappings*

$$\begin{array}{ll} \Omega \to \mathcal{C}(\mathbf{R}^d), & \qquad \Omega \to \mathcal{C}(\mathbf{R}^d), \\ \omega \mapsto \tilde{\mathcal{S}}(\omega) & \qquad \omega \mapsto \tilde{\mathcal{U}}(\omega) \end{array}$$

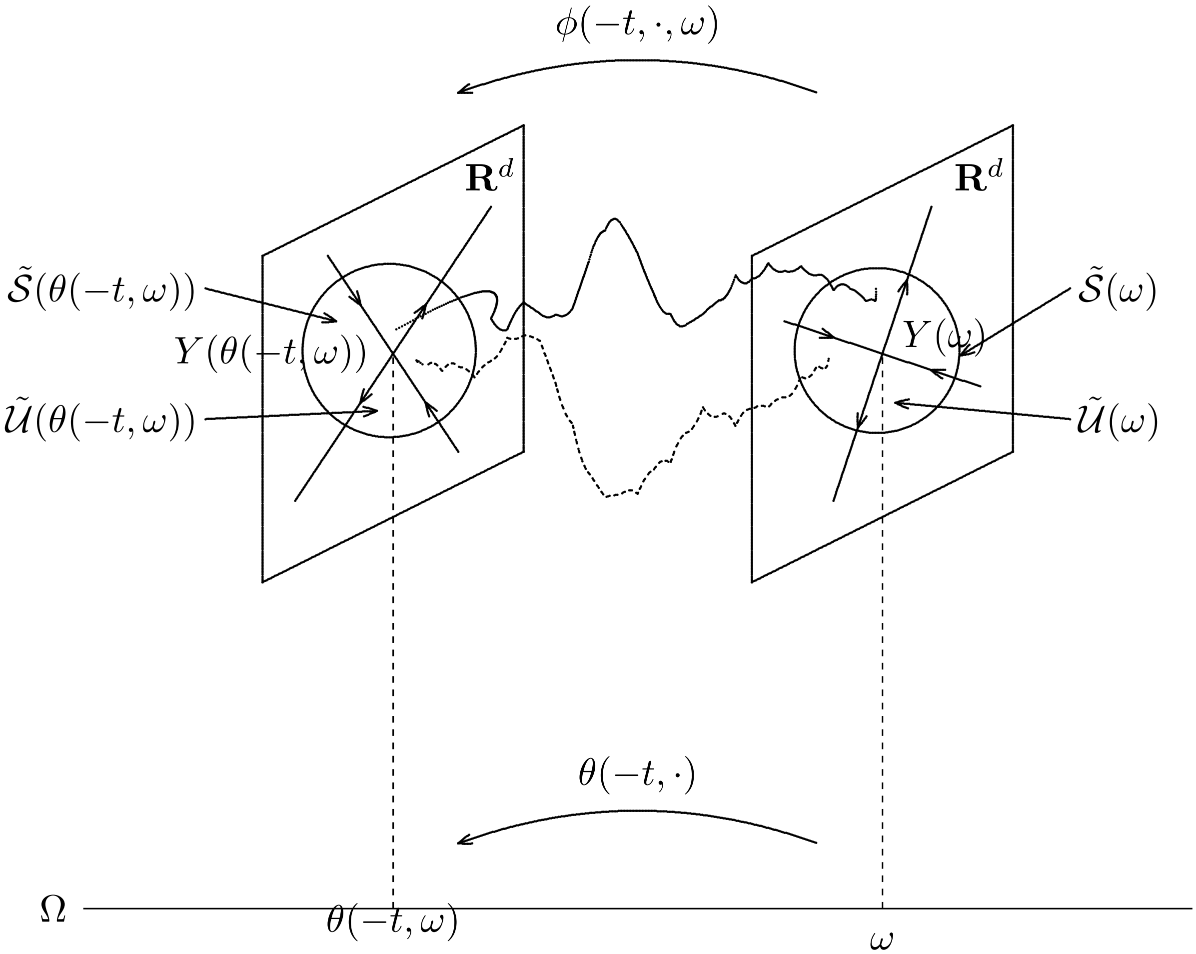
are $(\mathcal{F}, \mathcal{B}(\mathcal{C}(\mathbf{R}^d)))$*-measurable.*

Assume, further, that h, g_i, $1 \leq i \leq m$, *are* C_b^∞ *Then the local stable and unstable manifolds* $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ *are* C^∞.



$$t > \tau_1(\omega)$$

A picture is worth a 1000 words!


$\phi(-t,\cdot,\omega)$
$\mathbf{R}^d$
$\mathbf{R}^d$
$\tilde{\mathcal{S}}(\theta(-t,\omega))$
$Y(\theta(-t,\omega))$
$\tilde{\mathcal{U}}(\theta(-t,\omega))$
$\tilde{\mathcal{S}}(\omega)$
$Y(\omega)$
$\tilde{\mathcal{U}}(\omega)$
$\theta(-t,\cdot)$
Ω
$\theta(-t,\omega)$
ω

$t > \tau_2(\omega)$

II. SFDE Case:

"Regular" Itô SFDE with finite memory:

$$\left.\begin{aligned} dx(t) = H(x(t), x_t)\,dt + \sum_{i=1}^{m} G_i(x(t), g(x_t))\,dW_i(t), \\ (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (I')$$

Solution segment $x_t(s) := x(t+s),\ t \geq 0, s \in [-r, 0]$.

Smooth memory: $g : L^2([-r, 0], \mathbf{R}^d) \to \mathbf{R}^p$ is $C^{k,\delta}$ $\big(k \geq 1, \delta \in (0, 1]\big)$ with all Fréchet derivatives $D^j g, 1 \leq j \leq k$, globally bounded; $t \to g_i(x_t)$ locally of B.V., with

$$L^2([-r, T], \mathbf{R}^d) \ni x \mapsto \{t \mapsto \frac{dg(x_t)}{dt}\} \in L^2([0, T], \mathbf{R}^m)$$

globally bounded, globally Lipschitz and of class $C^{k,\delta}$.

m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$

Ergodic Brownian shift θ as before.

State space M_2, Hilbert with usual norm.

Smoothness Hypotheses:

$H : M_2 \to \mathbf{R}^d$ of class $C_b^{k,\delta}$, viz. all Fréchet derivatives $D^j H, 1 \leq j \leq k$, continuous and globally bounded, $D^k H$ Hölder continuous with exponent δ on bounded sets in M_2.

$G_i : \mathbf{R}^d \times \mathbf{R}^p \to \mathbf{R}^d, 1 \leq i \leq m$, of class $C^{k+1,\delta}$.

Then (I') has a stochastic semiflow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ with $X(t,(v,\eta),\cdot) = (x(t), x_t)$. X is of class $C^{k,\epsilon}$ for any $\epsilon \in (0,\delta)$, takes bounded sets into relatively compact sets in M_2. (X,θ) is a perfect cocycle on M_2 ([M-S]). Define *hyperbolic stationary point* $Y : \Omega \to M_2$ as in SODE case (replace ϕ by X; apply [Mo.1] to linearized cocycle).

Theorem $1'$ (Stable Manifold Theorem-SFDE) ([M-S], 1998)

Assume above smoothness Hypotheses on H, G_i, g. Let Y be a hyperbolic stationary point of (I') such that $E(\|Y(\cdot)\|^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$

Suppose $(D_2X(t,Y(\omega),\omega), \theta(t,\omega), t \geq 0)$ has a Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. Define λ_{i_0} to be the largest negative Lyapunov exponent (as before).

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,

(ii) $\mathcal{F}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0,1)$, $\beta_i > \rho_i > 0$, $i = 1,2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k,\epsilon}$ $(\epsilon \in (0,\delta))$ submanifolds $\tilde{\mathcal{S}}(\omega)$, $\tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega),\rho_1(\omega))$ and $\bar{B}(Y(\omega),\rho_2(\omega))$ (resp.) with the following properties:

(a) $\tilde{\mathcal{S}}(\omega)$ *is the set of all* $(v,\eta) \in \bar{B}(Y(\omega), \rho_1(\omega))$ *such that*

$$\|X(n,(v,\eta),\omega) - Y(\theta(n,\omega))\| \leq \beta_1(\omega)\, e^{(\lambda_{i_0}+\epsilon_1)n}$$

for all integers $n \geq 0$. *Furthermore,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t,(v,\eta),\omega) - Y(\theta(t,\omega))\| \leq \lambda_{i_0}$$

for all $(v,\eta) \in \tilde{\mathcal{S}}(\omega)$. *Each stable subspace* $\mathcal{S}(\omega)$ *of the linearized semiflow* D_2X *is tangent at* $Y(\omega)$ *to the submanifold* $\tilde{\mathcal{S}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. *In particular, codim* $\tilde{\mathcal{S}}(\omega) =$ *codim* $\mathcal{S}(\omega)$, *is fixed and finite.*

(b) $\limsup\limits_{t\to\infty} \frac{1}{t} \log\left[\sup\left\{\frac{\|X(t,(v_1,\eta_1),\omega) - X(t,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1)-(v_2,\eta_2)\|} : (v_1,\eta_1) \neq (v_2,\eta_2),\ (v_1,\eta_1),\ (v_2,\eta_2) \in \tilde{\mathcal{S}}(\omega)\right\}\right] \leq \lambda_{i_0}$.

(c) *(Cocycle-invariance of the stable manifolds):*

There exists $\tau_1(\omega) \geq 0$ *such that*

$$X(t,\cdot,\omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t,\omega)), \quad t \geq \tau_1(\omega).$$

Also

$$D_2X(t,Y(\omega),\omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t,\omega)), \quad t \geq 0.$$

(d) $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $(v,\eta) \in \bar{B}(Y(\omega), \rho_2(\omega))$ *with the property that there is a unique "history" process* $y(\cdot,\omega) : \mathbf{Z}^- \to M_2$

such that $y(0,\omega) = (v,\eta)$ *and for each integer* $n \geq 1$*, one has* $X(r, y(-nr,\omega), \theta(-nr,\omega)) = y(-(n-1)r,\omega)$ *and*

$$\|y(-nr,\omega) - Y(\theta(-nr,\omega))\|_{M_2} \leq \beta_2(\omega) e^{-(\lambda_{i_0-1}-\epsilon_2)nr}.$$

Furthermore, for each $(v,\eta) \in \tilde{\mathcal{U}}(\omega)$*, there is a unique continuous-time "history" process also denoted by* $y(\cdot,\omega) : (-\infty, 0] \to M_2$ *such that* $y(0,\omega) = (v,\eta)$*,* $X(t, y(s,\omega), \theta(s,\omega)) = y(t+s,\omega)$ *for all* $s \leq 0, 0 \leq t \leq -s$*, and*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|y(-t,\omega) - Y(\theta(-t,\omega))\| \leq -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}(\omega)$ *of the linearized semiflow* D_2X *is tangent at* $Y(\omega)$ *to* $\tilde{\mathcal{U}}(\omega)$*, viz.* $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$*. In particular,* $\dim\tilde{\mathcal{U}}(\omega)$ *is finite and non-random.*

(e) Let $y(\cdot,(v_i,\eta_i),\omega), i = 1,2,$ *be the history processes associated with* $(v_i,\eta_i) = y(0,(v_i,\eta_i),\omega) \in \tilde{\mathcal{U}}(\omega)$*,* $i = 1,2$*. Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log\left[\sup\left\{ \frac{\|y(-t,(v_1,\eta_1),\omega) - y(-t,(v_2,\eta_2),\omega)\|}{\|(v_1,\eta_1) - (v_2,\eta_2)\|} : (v_1,\eta_1) \neq (v_2,\eta_2),\ (v_i,\eta_i) \in \tilde{\mathcal{U}}(\omega), i = 1,2 \right\}\right] \leq -\lambda_{i_0-1}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ *such that*

$$\tilde{\mathcal{U}}(\omega)) \subseteq X(t,\cdot,\theta(-t,\omega))(\tilde{\mathcal{U}}(\theta(-t,\omega)))$$

for all $t \geq \tau_2(\omega)$*. Also*

$$D_2X(t,\cdot,\theta(-t,\omega))(\mathcal{U}(\theta(-t,\omega))) = \mathcal{U}(\omega), \quad t \geq 0;$$

and the restriction

$$D_2X(t,\cdot,\theta(-t,\omega))|\mathcal{U}(\theta(-t,\omega)) : \mathcal{U}(\theta(-t,\omega)) \to \mathcal{U}(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto.

(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ *and* $\tilde{\mathcal{S}}(\omega)$ *are transversal, viz.*

$$M_2 = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

(h) The mappings

$$\begin{array}{ll} \Omega \to \mathcal{C}(M_2), & \Omega \to \mathcal{C}(M_2), \\ \omega \mapsto \tilde{\mathcal{S}}(\omega) & \omega \mapsto \tilde{\mathcal{U}}(\omega) \end{array}$$

are $(\mathcal{F}, \mathcal{B}(\mathcal{C}(M_2)))$*-measurable.*

Assume, in addition, that the smoothness hypotheses hold for every $k \geq 1$ *and* $\delta \in (0,1]$*. Then the local stable and unstable manifolds* $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ *are* C^∞*.*

Sketch of Proof-SODE Case

Broad Outline:

- Linearize along stationary solution and use substitution formula:

$$\left.\begin{aligned} d\phi(t,Y) &= h(\phi(t,Y))\,dt + \sum_{i=1}^{m} g_i(\phi(t,Y)) \circ dW_i(t), \qquad t>0 \\ \phi(0,Y) &= Y. \end{aligned}\right\} \quad (II)$$

([N-P]).

$$\left.\begin{aligned} dD_2\phi(t,Y) &= Dh(\phi(t,Y))D_2\phi(t,Y)\,dt \\ &\quad + \sum_{i=1}^{m} Dg_i(\phi(t,Y))D_2\phi(t,Y) \circ dW_i(t), \qquad t>0 \\ D_2\phi(0,Y) &= I. \end{aligned}\right\} \quad (III)$$

D_2, D denotes spatial (Fréchet) derivatives.

$$\left.\begin{aligned} d\phi(t,Y) &= -h(\phi(t,Y))\,dt - \sum_{i=1}^{m} g_i(\phi(t,Y)) \circ \hat{d}W_i(t), \qquad t<0 \\ \phi(0,Y) &= Y. \end{aligned}\right\} \quad (II^-)$$

$$\left.\begin{aligned} dD_2\phi(t,Y) &= -Dh(\phi(t,Y))D_2\phi(t,Y)\,dt \\ &\quad - \sum_{i=1}^{m} Dg_i(\phi(t,Y))D_2\phi(t,Y) \circ \hat{d}W_i(t), \qquad t<0 \\ D_2\phi(0,Y) &= I. \end{aligned}\right\} \quad (III^-)$$

Above SODE's (II)-(III)$^-$ give dynamic characterizations of stable/unstable manifolds.

- "Perfection" of ergodic theorem and Kingman's subadditive ergodic theorem under suitable integrability hypotheses.
- Apply the Oseledec theorem to the linearized system. Get a fixed Lyapunov spectrum. Hyperbolicity is well-defined.
- Continuous-time integrability estimates on the non-linear cocycle in a neighborhood of the stationary trajectory. Uses sharp global spatial estimates on the stochastic flow via Kolmogorov's theorem; viz. the following quantities have q-th moments for all $q \geq 1$:

$$\sup_{\substack{0\leq s,t\leq T,\\ x\in\mathbf{R}^d}} \frac{|\phi_{s,t}(x,\omega)|}{[1+|x|(\log^+|x|)^\gamma]}, \qquad \sup_{\substack{0\leq s,t\leq T,\\ x\in\mathbf{R}^d}} \frac{|D_x^\alpha \phi_{s,t}(x,\omega)|}{(1+|x|^\gamma)},$$

$$\sup_{x\in\mathbf{R}^d} \sup_{\substack{0\leq s,t\leq T,\\ 0<|x'-x|\leq\rho}} \frac{|D_x^\alpha\phi_{s,t}(x,\omega)-D_x^\alpha\phi_{s,t}(x',\omega)|}{|x-x'|^\epsilon(1+|x|)^\gamma},$$

$$\epsilon\in(0,\delta), \gamma, \rho, T>0, 1\leq|\alpha|\leq k$$

- Use Ruelle's discrete non-linear ergodic theorem on the auxiliary perfect cocycle

$$Z(t,x,\omega) := \phi(t,x+Y(\omega),\omega)-Y(\theta(t,\omega)), \quad t\in\mathbf{R}, x\in\mathbf{R}^d, \omega\in\Omega$$

to construct the stable/unstable manifolds. Based on difficult computations using implicit function theorem, perfection arguments and local perturbation of discrete cocycle under the norm

$$\|D\phi\|_\omega := \sup_{n \geq 0} \|D_2\phi(1, Y(\theta(n-1,\omega)), \theta(n-1,\omega))\| e^{n\eta}$$

for small $\eta > 0$.

- Use the continuous-time integrability estimates and the perfect subadditive ergodic theorem to interpolate between discrete time units (or delay periods in SFDE case). This gives asymptotic invariance of the stable/unstable manifolds.

Linearization and Substitution

Assume regularity conditions on the coefficients h, g_i. By the *Substitution Rule*, $\phi(t, Y(\omega), \omega)$ is a stationary *solution* of the *anticipating* Stratonovich SODE

$$\left.\begin{aligned} d\phi(t,Y) &= h(\phi(t,Y))\,dt + \sum_{i=1}^{m} g_i(\phi(t,Y)) \circ dW_i(t), \qquad t>0 \\ \phi(0,Y) &= Y. \end{aligned}\right\} \qquad (II)$$

([N-P]).

Linearize the SODE (I) along the stationary trajectory. By substitution, match the solution of the linearized equation with the linearized cocycle $D_2\phi(t, Y(\omega), \omega)$. Hence $D_2\phi(t, Y(\omega), \omega)$, $t \geq 0$, solves the SODE:

$$\left.\begin{aligned} dD_2\phi(t,Y) &= Dh(\phi(t,Y))D_2\phi(t,Y)\,dt \\ &\quad + \sum_{i=1}^{m} Dg_i(\phi(t,Y))D_2\phi(t,Y) \circ dW_i(t), \qquad t>0 \\ D_2\phi(0,Y) &= I. \end{aligned}\right\} \qquad (III)$$

D_2, D denotes spatial (Fréchet) derivatives.

Similarly, the backward trajectories

$$\phi(t,Y),\ D_2\phi(t,Y),\ \ t<0,$$

solve the corresponding backward Stratonovich SODE's:

$$\left.\begin{aligned} d\phi(t,Y) &= -h(\phi(t,Y))\,dt - \sum_{i=1}^{m} g_i(\phi(t,Y)) \circ \hat{d}W_i(t), \qquad t<0 \\ \phi(0,Y) &= Y. \end{aligned}\right\} \tag{II^-}$$

$$\left.\begin{aligned} dD_2\phi(t,Y) &= -Dh(\phi(t,Y))D_2\phi(t,Y)\,dt \\ &\qquad - \sum_{i=1}^{m} Dg_i(\phi(t,Y))D_2\phi(t,Y) \circ \hat{d}W_i(t), \qquad t<0 \\ D_2\phi(0,Y) &= I. \end{aligned}\right\} \tag{III^-}$$

Above SODE's (II)-(III)$^-$ give dynamic characterizations of the stable/unstable manifolds.

The following lemma is used to construct the shift-invariant sure event appearing in the statement of the local stable manifold theorem. Gives "perfect versions" of the ergodic theorem and Kingman's subadditive ergodic theorem.

Lemma 1

(i) Let $h : \Omega \to \mathbf{R}^+$ be $\mathcal{F}$-measurable and

$$\int_\Omega \sup_{0 \le u \le 1} h(\theta(u,\omega))\,dP(\omega) < \infty.$$

Then there is a sure event $\Omega_1 \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega_1) = \Omega_1$ for all $t \in \mathbf{R}$, and

$$\lim_{t\to\infty} \frac{1}{t} h(\theta(t,\omega)) = 0$$

for **all** $\omega \in \Omega_1$.

(ii) *Suppose $f : \mathbf{R}^+ \times \Omega \to \mathbf{R} \cup \{-\infty\}$ is a measurable process on $(\Omega, \mathcal{F}, P)$ satisfying the following conditions*

(a) $E \sup_{0\le u\le 1} f^+(u) < \infty, \quad E \sup_{0\le u\le 1} f^+(1-u,\theta(u)) < \infty$

(b) $f(t_1+t_2,\omega) \le f(t_1,\omega) + f(t_2,\theta(t_1,\omega))$ *for all* $t_1, t_2 \ge 0$ *and* **all** $\omega \in \Omega$.

Then there is sure event $\Omega_2 \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega_2) = \Omega_2$ for all $t \in \mathbf{R}$, and a fixed number $f^* \in \mathbf{R} \cup \{-\infty\}$ such that

$$\lim_{t\to\infty} \frac{1}{t} f(t,\omega) = f^*$$

for **all** $\omega \in \Omega_2$.

Proof

[Mo.1], Lemma 7. $\square$

Theorem 2 ([O], 1968)

Let $(\Omega, \mathcal{F}, P)$ *be a probability space and* $\theta : \mathbf{R}^+ \times \Omega \to \Omega$ *a measurable family of ergodic* P*-preserving transformations. Let* $T : \mathbf{R}^+ \times \Omega \to L(\mathbf{R}^d)$ *be measurable, such that* (T, θ) *is an* $L(\mathbf{R}^d)$*-valued cocycle. Suppose that*

$$E \sup_{0 \le t \le 1} \log^+ \|T(t, \cdot)\| < \infty, \quad E \sup_{0 \le t \le 1} \log^+ \|T(1-t, \theta(t, \cdot))\| < \infty.$$

Then there is a set $\Omega_0 \in \mathcal{F}$ *of full* P*-measure such that* $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ *for all* $t \in \mathbf{R}^+$*, and for each* $\omega \in \Omega_0$*, the limit*

$$\lim_{n \to \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. Each $\Lambda(\omega)$ *has a non-random spectrum*

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_p}$$

where the λ_i*'s are distinct. Each* e^{λ_i} *has a fixed non-random multiplicity* m_i *and eigen-space* $F_i(\omega)$*, with* $m_i := \dim F_i(\omega)$*. Define*

$$E_1(\omega) := \mathbf{R}^d, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^\perp, \ 1 < i \le p.$$

Then

$$E_p(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = \mathbf{R}^d,$$

$$\lim_{t \to \infty} \frac{1}{t} \log \|T(t, \omega)x\| = \lambda_i(\omega) \quad \textit{if} \quad x \in E_i(\omega) \backslash E_{i+1}(\omega),$$

and

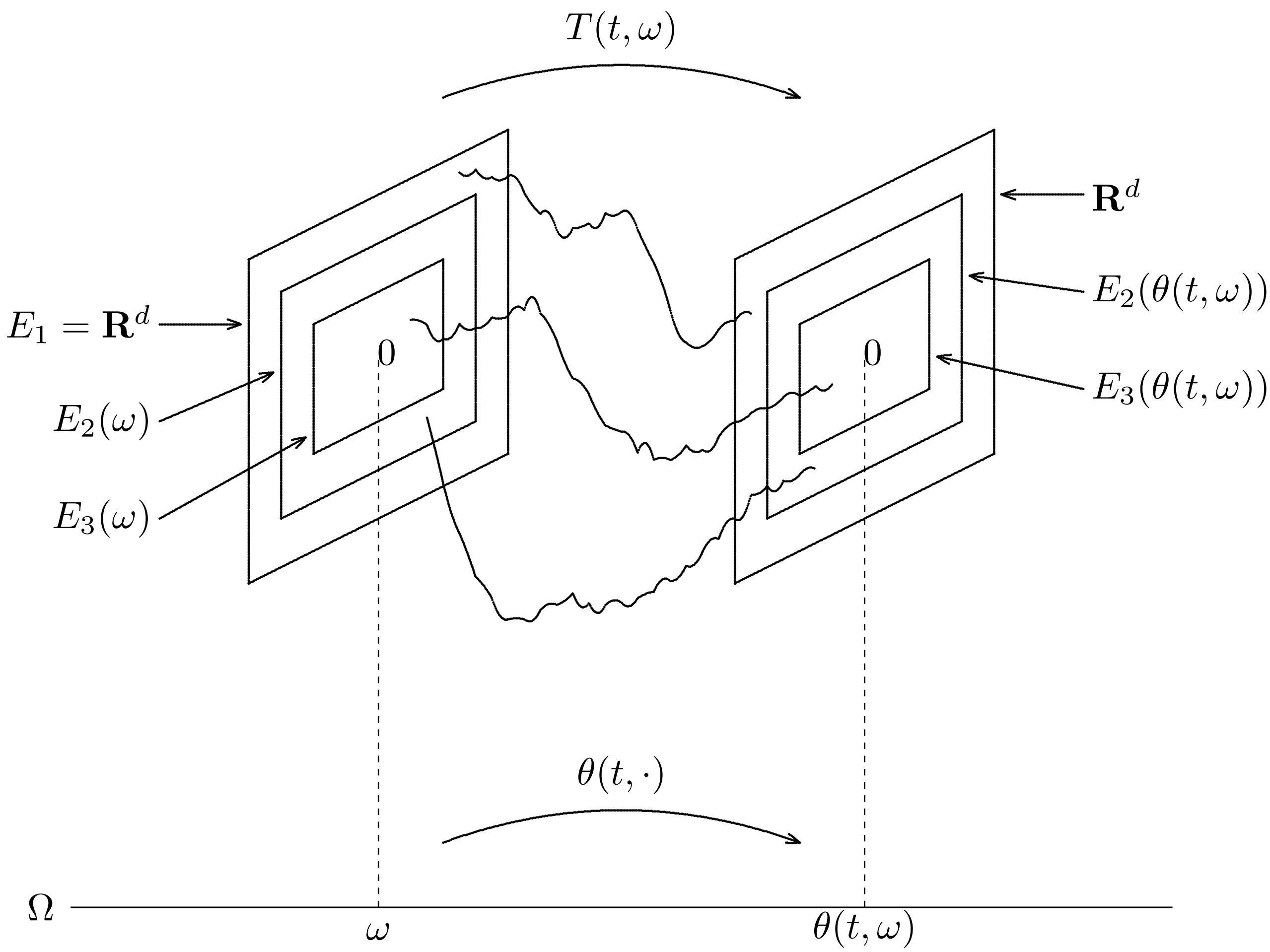
$$T(t,\omega)(E_i(\omega)) \subseteq E_i(\theta(t,\omega))$$

for all $t \geq 0,\ 1 \leq i \leq p.$

Proof.

Based on the discrete version of Oseledec's multiplicative ergodic theorem and Lemma 1. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]), "perfect" infinite-dimensional version and application to regular linear SFDE's. □

Spectral Theorem

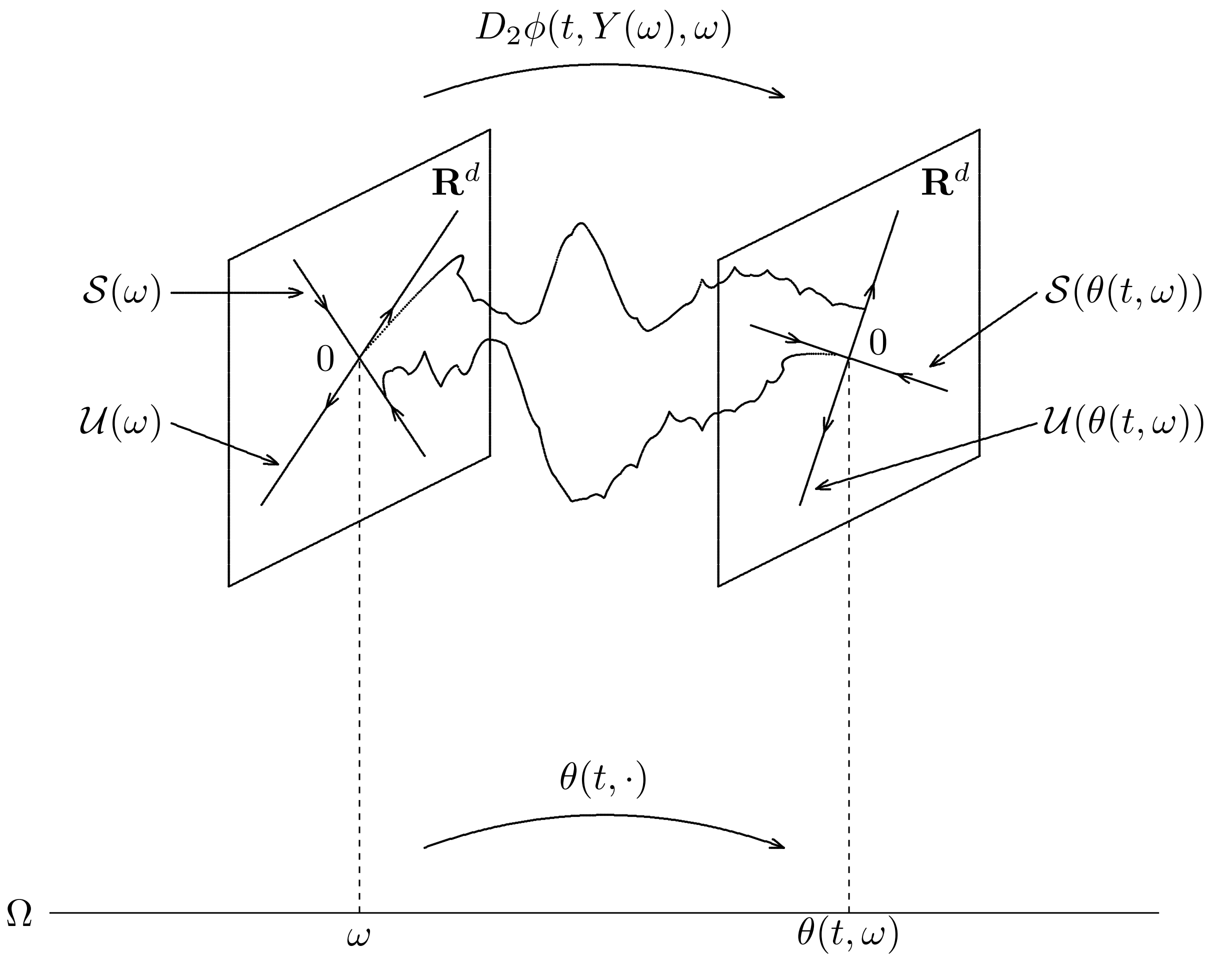


Apply Theorem 2 with $T(t,\omega) := D_2\phi(t, Y(\omega), \omega)$. Then linearized cocycle has random invariant stable and unstable subspaces $\{S(\omega), U(\omega) : \omega \in \Omega\}$:

$$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t,\omega)),$$

$$D_2\phi(-t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(-t,\omega)), \quad t \geq 0.$$

[Mo.1].



Estimates on the non-linear cocycle

Theorem 3 ([M-S.2])

There exists a jointly measurable modification of the trajectory random field of (I), denoted by $\{\phi_{s,t}(x) : -\infty < s,t < \infty,\ x \in \mathbf{R}^d\}$*, with the following properties:*

Define $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ *by*

$$\phi(t, x, \omega) := \phi_{0,t}(x, \omega), \quad x \in \mathbf{R}^d, \omega \in \Omega, t \in \mathbf{R}.$$

Then the following is true for all $\omega \in \Omega$, $\epsilon \in (0, \delta)$*:*

(i) *For each* $x \in \mathbf{R}^d$*, and* $s, t \in \mathbf{R}$, $\phi_{s,t}(x, \omega) = \phi(t - s, x, \theta(s, \omega))$.

(ii) (ϕ, θ) *is a perfect cocycle:*

$$\phi(t + s, \cdot, \omega) = \phi(t, \cdot, \theta(s, \omega)) \circ \phi(s, \cdot, \omega),$$

for all $s, t \in \mathbf{R}$.

(iii) *For each* $t \in \mathbf{R}$, $\phi(t, \cdot, \omega) : \mathbf{R}^d \to \mathbf{R}^d$ *is a* $C^{k,\epsilon}$ *diffeomorphism.*

(iv) *The mapping* $\mathbf{R}^2 \ni (s, t) \mapsto \phi_{s,t}(\cdot, \omega) \in \mathit{Diff}^k(\mathbf{R}^d)$ *is continuous, where* $\mathit{Diff}^k(\mathbf{R}^d)$ *denotes the group of all* C^k *diffeomorphisms of* $\mathbf{R}^d$*, given the* C^k*-topology.*

(v) The quantities

$$\sup_{\substack{0\le s,t\le T,\\ x\in\mathbf{R}^d}} \frac{|\phi_{s,t}(x,\omega)|}{[1+|x|(\log^+|x|)^\gamma]}, \qquad \sup_{\substack{0\le s,t\le T,\\ x\in\mathbf{R}^d}} \frac{|D_x^\alpha \phi_{s,t}(x,\omega)|}{(1+|x|^\gamma)},$$

$$\sup_{x\in\mathbf{R}^d} \sup_{\substack{0\le s,t\le T,\\ 0<|x'-x|\le\rho}} \frac{|D_x^\alpha \phi_{s,t}(x,\omega) - D_x^\alpha \phi_{s,t}(x',\omega)|}{|x-x'|^\epsilon (1+|x|)^\gamma},$$

$$\gamma,\rho,T>0,\ 1\le|\alpha|\le k$$

are finite. The random variables defined by the above expressions have q-th moments for all $q \ge 1$.

$$\| \cdot \|_{k,\epsilon} := C^{k,\epsilon}\text{-norm on } C^{k,\epsilon} \text{ mappings } \bar{B}(0,\rho) \to \mathbf{R}^d.$$

Lemma 2

Assume that $\log^+ |Y(\cdot)|$ *is integrable. Then*

$$\int_\Omega \log^+ \sup_{-T \le t_1, t_2 \le T} \|\phi(t_2, Y(\theta(t_1,\omega)) + (\cdot), \theta(t_1,\omega))\|_{k,\epsilon} \, dP(\omega) < \infty \tag{10}$$

for any fixed $0 < T, \rho < \infty$ *and any* $\epsilon \in (0, \delta)$. *The linearized flow* $(D_2\phi(t, Y(\omega), \omega), \theta(t,\omega))$, $t \ge 0$, *is an* $L(\mathbf{R}^d)$*-valued perfect cocycle and*

$$\int_\Omega \log^+ \sup_{-T \le t_1, t_2 \le T} \|D_2\phi(t_2, Y(\theta(t_1,\omega)), \theta(t_1,\omega))\|_{L(\mathbf{R}^d)} \, dP(\omega) < \infty \tag{11}$$

for any fixed $0 < T < \infty$. *The forward cocycle* $(D_2\phi(t, Y(\omega), \omega), \theta(t,\omega), t > 0)$ *has a non-random finite Lyapunov spectrum* $\{\lambda_p < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. *Each Lyapunov exponent* λ_i *has a non-random multiplicity* q_i, $1 \le i \le p$, *and* $\sum_{i=1}^{p} q_i = d$. *The backward linearized cocycle* $(D_2\phi(t, Y(\omega), \omega), \theta(t,\omega), t < 0)$, *admits a "backward" non-random finite Lyapunov spectrum:*

$$\lim_{t \to -\infty} \frac{1}{t} \log |D_2\phi(t, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbf{R}^d),$$

taking values $\{-\lambda_i\}_{i=1}^p$ *with non-random multiplicities* q_i, $1 \le i \le p$, *and* $\sum_{i=1}^{p} q_i = d$.

The Auxiliary Cocycle

To apply Ruelle's discrete non-linear ergodic theorem ([Ru.1], Theorem 5.1, p. 292), introduce the following auxiliary cocycle $Z : \mathbf{R} \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$. This a "centering" of the flow ϕ about the stationary solution:

$$Z(t,x,\omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t,\omega))$$

for $t \in \mathbf{R}, x \in \mathbf{R}^d, \omega \in \Omega$.

Lemma 3

(Z,θ) is a perfect cocycle on $\mathbf{R}^d$ and $Z(t,0,\omega) = 0$ for all $t \in \mathbf{R}$, and all $\omega \in \Omega$.

Proof.

$$\begin{aligned} Z(t_2,&Z(t_1,x,\omega),\theta(t_1,\omega)) \\ &= \phi(t_2, Z(t_1,x,\omega) + Y(\theta(t_1,\omega)), \theta(t_1,\omega)) - Y(\theta(t_2,\theta(t_1,\omega))) \\ &= \phi(t_2, \phi(t_1, x + Y(\omega),\omega), \theta(t_1,\omega)) - Y(\theta(t_2 + t_1,\omega)) \\ &= Z(t_1 + t_2, x, \omega), \qquad t_1, t_2 \in \mathbf{R}, \omega \in \Omega, x \in \mathbf{R}^d. \end{aligned}$$

$Z(t,0,\omega) \equiv 0$ by definition of Z and stationary solution. $\square$

The proof of the local stable-manifold theorem (Theorem 1) uses a discretization argument that requires the following lemma.

Lemma 4

Suppose that $\log^+ |Y(\cdot)|$ *is integrable. Then there is a sure event* $\Omega_3 \in \mathcal{F}$ *with the following properties:*

(i) $\theta(t, \cdot)(\Omega_3) = \Omega_3$ *for all* $t \in \mathbf{R}$,

(ii) For every $\omega \in \Omega_3$ *and any* $x \in \mathbf{R}^d$, *the statement*

$$\limsup_{n\to\infty} \frac{1}{n} \log |Z(n, x, \omega)| < 0 \tag{17}$$

implies

$$\limsup_{t\to\infty} \frac{1}{t} \log |Z(t, x, \omega)| = \limsup_{n\to\infty} \frac{1}{n} \log |Z(n, x, \omega)|. \tag{18}$$

Ruelle's Non-linear Ergodic Theorem

Theorem 4 ([Ru.1], 1979)

Let $\Omega \ni \omega \mapsto F_\omega \in C^{k,\delta}(\mathbf{R}^d, 0; \mathbf{R}^d, 0)$ *be measurable such that* $E \log^+ \|F_\cdot|\bar{B}(0,1)\| < \infty$. *Set* $F^n(\omega) := F_{\theta(n-1,\omega)} \circ \cdots \circ F_{\theta(1,\omega)} \circ F_\omega$. *Suppose* $\lambda < 0$ *is not in the spectrum of the cocycle* $(DF^n_\omega(0), \theta(n,\omega))$. *Then there is a sure event* $\Omega_0 \in \mathcal{F}$ *such that* $\theta(1,\cdot)(\Omega_0) \subseteq \Omega_0$, *and measurable functions* $0 < \alpha(\omega) < \beta(\omega) < 1, \gamma(\omega) > 1$ *with the following properties:*

(a) If $\omega \in \Omega_0$, *the set*

$$V^\lambda_\omega := \{x \in \bar{B}(0,\alpha(\omega)) : \|F^n_\omega(x)\| \leq \beta(\omega)e^{n\lambda} \text{for all } n \geq 0\}$$

is a $C^{k,\delta}$ *submanifold of* $\bar{B}(0,\alpha(\omega))$.

(b) If $x_1, x_2 \in V^\lambda_\omega$, *then*

$$\|F^n_\omega(x_1) - F^n_\omega(x_2)\| \leq \gamma(\omega)\|x_1 - x_2\|e^{n\lambda}$$

for all integers $n \geq 0$. *If* $\lambda' < \lambda$ *and* $[\lambda', \lambda]$ *is disjoint from the spectrum of* $(DF^n_\omega(0), \theta(n,\omega))$, *then there exists a measurable* $\gamma'(\omega) > 1$ *such that*

$$\|F^n_\omega(x_1) - F^n_\omega(x_2)\| \leq \gamma'(\omega)\|x_1 - x_2\|e^{n\lambda'}$$

for all $x_1, x_2 \in V^\lambda_\omega$ *and all integers* $n \geq 0$.

Proof

[Ru.1], Theorem 5.1, p. 292.

Construction of the Stable/Unstable Manifolds

Assume the hypotheses of Theorem 1.

Consider the auxiliary cocycle (Z, θ). Define the family of maps $F_\omega : \mathbf{R}^d \to \mathbf{R}^d$ by $F_\omega(x) := Z(1, x, \omega)$ for all $\omega \in \Omega$ and $x \in \mathbf{R}^d$. Let $\tau := \theta(1, \cdot) : \Omega \to \Omega$. Define $F_\omega^n := F_{\tau^{n-1}(\omega)} \circ \cdots \circ F_{\tau(\omega)} \circ F_\omega$. Then cocycle property for Z gives $F_\omega^n = Z(n, \cdot, \omega)$ for each $n \geq 1$. F_ω is $C^{k,\epsilon}$ $(\epsilon \in (0, \delta))$ and $(DF_\omega)(0) = D_2\phi(1, Y(\omega), \omega)$. By measurability of the flow ϕ, the map $\omega \mapsto (DF_\omega)(0)$ is $\mathcal{F}$-measurable. By (11) of Lemma 2, the map $\omega \mapsto \log^+ \|D_2\phi(1, Y(\omega), \omega)\|_{L(\mathbf{R}^d)}$ is integrable. The discrete cocycle $((DF_\omega^n)(0), \theta(n, \omega), n \geq 0)$ has a non-random Lyapunov spectrum which coincides with that of the linearized continuous cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$, viz. $\{\lambda_p < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, where each λ_i has fixed multiplicity q_i, $1 \leq i \leq p$ (Lemma 2). If $\lambda_i > 0$ for all $1 \leq i \leq m$, then take $\tilde{\mathcal{S}}(\omega) := \{Y(\omega)\}$ for all $\omega \in \Omega$. Theorem is trivial in this case. Suppose that at least one $\lambda_i < 0$.

Use discrete non-linear ergodic theorem of Ruelle (Theorem 4) and its proof to obtain a sure event $\Omega_1^* \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$, $\mathcal{F}$-measurable positive random variables $\rho_1, \beta_1 : \Omega_1^* \to (0, 1)$, $\rho_1 < \beta_1$, and a random family of $C^{k,\epsilon}$ $\big(\epsilon \in (0, \delta)\big)$ submanifolds of $\bar{B}(0, \rho_1(\omega))$ denoted by $\tilde{\mathcal{S}}_d(\omega)$, $\omega \in \Omega_1^*$, and satisfying the following properties for each $\omega \in \Omega_1^*$:

$$\tilde{\mathcal{S}}_d(\omega) = \{x \in \bar{B}(0, \rho_1(\omega)) : |Z(n, x, \omega)| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1) n} \text{ for all } n \in \mathbf{Z}^+\}. \tag{21}$$

$\tilde{\mathcal{S}}_d(\omega)$ is tangent at 0 to the stable subspace $\mathcal{S}(\omega)$ of the linearized flow $D_2\phi$, viz. $T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$. Therefore dim $\tilde{\mathcal{S}}_d(\omega)$ is non-random by ergodicity of θ. Also

$$\limsup_{n\to\infty} \frac{1}{n} \log \left[\sup_{\substack{x_1 \neq x_2, \\ x_1, x_2 \in \tilde{\mathcal{S}}_d(\omega)}} \frac{|Z(n, x_1, \omega) - Z(n, x_2, \omega)|}{|x_1 - x_2|} \right] \leq \lambda_{i_0}. \tag{22}$$

The $\theta(t, \cdot)$-invariant sure event $\Omega_1^* \in \mathcal{F}$ is constructed using the ideas in Ruelle's proof (of Theorem 5.1 in [Ru.1], p. 293), combined with the estimate (10) of Lemma 2 and the subadditive ergodic theorem (Lemma 1 (ii)).

For each $\omega \in \Omega_1^*$, let $\tilde{\mathcal{S}}(\omega)$ be the set defined in part (a) of the theorem. Then by definition of $\tilde{\mathcal{S}}_d(\omega)$ and Z:

$$\tilde{\mathcal{S}}(\omega) = \tilde{\mathcal{S}}_d(\omega) + Y(\omega). \tag{23}$$

Since $\tilde{\mathcal{S}}_d(\omega)$ is a $C^{k,\epsilon}$ $\big(\epsilon \in (0, \delta)\big)$ submanifold of $\bar{B}(0, \rho_1(\omega))$, then $\tilde{\mathcal{S}}(\omega)$ is a $C^{k,\epsilon}$ $\big(\epsilon \in (0, \delta)\big)$ submanifold of $\bar{B}(Y(\omega), \rho_1(\omega))$. Furthermore, $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$. Hence dim $\tilde{\mathcal{S}}(\omega) =$ dim $\mathcal{S}(\omega) = \displaystyle\sum_{i=i_0}^{p} q_i$, and is non-random.

Now (22) implies that

$$\limsup_{n\to\infty} \frac{1}{n} \log |Z(n,x,\omega)| \leq \lambda_{i_0} \tag{24}$$

for all ω in Ω_1^* and all $x \in \tilde{\mathcal{S}}_d(\omega)$. Therefore by Lemma 4, there is a sure event $\Omega_2^* \subseteq \Omega_1^*$ such that $\theta(t,\cdot)(\Omega_2^*) = \Omega_2^*$ for all $t \in \mathbf{R}$, and

$$\limsup_{t\to\infty} \frac{1}{t} \log |Z(t,x,\omega)| \leq \lambda_{i_0} \tag{25}$$

for all $\omega \in \Omega_2^*$ and all $x \in \tilde{\mathcal{S}}_d(\omega)$. Therefore (2) holds.

To prove (b), let $\omega \in \Omega_1^*$. By (22), there is a positive integer $N_0 := N_0(\omega)$ (independent of $x \in \tilde{\mathcal{S}}_d(\omega)$) such that $Z(n,x,\omega) \in \bar{B}(0,1)$ for all $n \geq N_0$. Let $\Omega_4^* := \Omega_2^* \cap \Omega_3$, where Ω_3 is the shift-invariant sure event defined in the proof of Lemma 4. Then Ω_4^* is a sure event and $\theta(t,\cdot)(\Omega_4^*) = \Omega_4^*$ for all $t \in \mathbf{R}$. By cocycle property, Mean-Value theorem and the ergodic theorem (Lemma 1(i)), we get (b).

To prove the invariance property (4), apply the Oseledec theorem to the cocycle $(D_2\phi(t,Y(\omega),\omega),\theta(t,\omega))$ ([Mo.1], Theorem 4, Corollary 2). This gives a sure $\theta(t,\cdot)$-invariant event, also denoted by Ω_1^*, such that
$D_2\phi(t,Y(\omega),\omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$. Equality holds because $D_2\phi(t,Y(\omega),\omega)$ is injective and $\dim \mathcal{S}(\omega) = \dim \mathcal{S}(\theta(t,\omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$.

To prove the asymptotic invariance property (3), use the ideas in the proofs of Theorems 5.1 and 4.1 in [Ru.1], pp. 285-297, to pick random variables ρ_1, β_1 and a sure event (also denoted by) Ω_1^* such that $\theta(t,\cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$, and

$$\rho_1(\theta(t,\omega)) \geq \rho_1(\omega)e^{(\lambda_{i_0}+\epsilon_1)t}, \quad \beta_1(\theta(t,\omega)) \geq \beta_1(\omega)e^{(\lambda_{i_0}+\epsilon_1)t} \tag{26}$$

for all $t \geq 0, \omega \in \Omega_1^*$. Use (b) to obtain a sure event $\Omega_5^* \subseteq \Omega_4^*$ such that $\theta(t,\cdot)(\Omega_5^*) = \Omega_5^*$ for all $t \in \mathbf{R}$, and for any $0 < \epsilon < \epsilon_1$ and $\omega \in \Omega_4^*$, there exists $\beta^\epsilon(\omega) > 0$ (independent of x) with

$$|\phi(t,x,\omega) - Y(\theta(t,\omega))| \leq \beta^\epsilon(\omega)e^{(\lambda_{i_0}+\epsilon)t} \tag{27}$$

for all $x \in \tilde{\mathcal{S}}(\omega)$, $t \geq 0$. Fix $t \geq 0$, $\omega \in \Omega_5^*$ and $x \in \tilde{\mathcal{S}}(\omega)$. Let n be a non-negative integer. Then the cocycle property and (27) imply that

$$\begin{aligned} |\phi(n,\phi(t,x,\omega),\theta(t,\omega)) - & Y(\theta(n,\theta(t,\omega)))| \\ &= |\phi(n+t,x,\omega) - Y(\theta(n+t,\omega))| \\ &\leq \beta^\epsilon(\omega)e^{(\lambda_{i_0}+\epsilon)(n+t)} \\ &\leq \beta^\epsilon(\omega)e^{(\lambda_{i_0}+\epsilon)t}e^{(\lambda_{i_0}+\epsilon_1)n}. \end{aligned} \tag{28}$$

If $\omega \in \Omega_5^*$, then it follows from (26),(27), (28) and the definition of $\tilde{\mathcal{S}}(\theta(t,\omega))$ that there exists $\tau_1(\omega) > 0$ such that

$\phi(t,x,\omega) \in \tilde{\mathcal{S}}(\theta(t,\omega))$ for all $t \geq \tau_1(\omega)$. This proves asymptotic invariance.

Prove (d) by running both the flow ϕ and the shift θ backward in time:

$$\tilde{\phi}(t,x,\omega) := \phi(-t,x,\omega),\ \tilde{Z}(t,x,\omega) := Z(-t,x,\omega),\ \tilde{\theta}(t,\omega) := \theta(-t,\omega)$$

for all $t \geq 0$ and all $\omega \in \Omega$. $(\tilde{Z}(t,\cdot,\omega), \tilde{\theta}(t,\omega), t \geq 0)$ is a smooth cocycle, with $\tilde{Z}(t,0,\omega) = 0$ for all $t \geq 0$. The linearized flow $(D_2\tilde{\phi}(t,Y(\omega),\omega), \tilde{\theta}(t,\omega), t \geq 0)$ is an $L(\mathbf{R}^d)$-valued perfect cocycle with a non-random finite Lyapunov spectrum $\{-\lambda_1 < -\lambda_2 < \cdots < -\lambda_i < -\lambda_{i+1} < \cdots < -\lambda_p\}$ where $\{\lambda_p < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ is the Lyapunov spectrum of the forward linearized flow $(D_2\phi(t,Y(\omega),\omega), \theta(t,\omega),\ t \geq 0)$. Apply first part of the proof to get *stable manifolds* for the backward flow $\tilde{\phi}$ satisfying assertions (a), (b), (c). This translates into the existence of *unstable manifolds* for the original flow ϕ, and (d), (e), (f) automatically hold. Hence there is a sure event $\Omega_6^* \in \mathcal{F}$ such that $\theta(-t,\cdot)(\Omega_6^*) = \Omega_6^*$ for all $t \in \mathbf{R}$, and (d), (e) and (f) hold for all $\omega \in \Omega_6^*$.

Define the sure event $\Omega^* := \Omega_6^* \cap \Omega_5^*$. Then $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$. Assertions (a)-(f) hold for all $\omega \in \Omega^*$.

Measurability of the stable manifolds follows from

$$\tilde{\mathcal{S}}(\omega) = Y(\omega) + \tilde{\mathcal{S}}_d(\omega) \tag{29}$$

$$\tilde{\mathcal{S}}_d(\omega) = \lim_{m\to\infty} \bar{B}(0, \rho_1(\omega)) \cap \bigcap_{i=1}^{m} f_i(\cdot, \omega)^{-1}(\bar{B}(0,1)) \qquad (30)$$

$$f_n(x,\omega) := \beta_1(\omega)^{-1} e^{-(\lambda_{i_0}+\epsilon_1)n} Z(n,x,\omega), \quad x \in \mathbf{R}^d, \, \omega \in \Omega_1^*,$$

for all integers $n \geq 0$. (Above limit is taken in the metric d^* on $\mathcal{C}(\mathbf{R}^d)$.) Use joint continuity of translation and measurability of Y, f_i, ρ_1, finite intersections and the continuity of the maps

$$\mathbf{R}^+ \ni r \mapsto \bar{B}(0,r) \in \mathcal{C}(\mathbf{R}^d).$$

$$\mathrm{Hom}(\mathbf{R}^d) \ni f \mapsto f^{-1}(\bar{B}(0,1)) \in \mathcal{C}(\mathbf{R}^d).$$

When h, g_i are in C_b^∞, adapt above argument to give a sure event in $\mathcal{F}$, also denoted by Ω^* such that $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞ for all $\omega \in \Omega^*$. $\square$

Remarks

(i) Replace the stationary random variable Y by its invariant distribution ρ, with $\int_{\mathbf{R}^d} |x|\, d\rho(x) < \infty$. Formulate result with respect to the product measure $P \otimes \rho$ and the underlying skew-product flow. This would give stable and unstable manifolds that are defined a.e.$(P \otimes \rho)$; cf. [C] for the globally asymptotically stable case on a compact manifold.

(ii) In Stratonovich SODE (I), replace global boundedness on $g_i's$ by requiring

$$\mathbf{R}^d \ni x \mapsto \sum_{l=1}^{m} \frac{\partial g_l^i(x)}{\partial x_j} g_l^j(x) \in \mathbf{R},\ 1 \le i, j \le d$$

to be in $C_b^{k,\delta}$.

(iii) Conjecture: The global boundedness condition is not needed. This conjecture is not hard to check if the vector fields g_i, $1 \le i \le m$, are C_b^∞ and generate a finite-dimensional solvable Lie algebra. See [Ku], Theorem 4.9.10, p. 212.

(iv) Theorem holds for the Itô SODE

$$dx(t) = h(x(t))\, dt + \sum_{i=1}^{m} g_i(x(t))\, dW_i(t), \qquad (II)$$

with $h, g_i : \mathbf{R}^d \to \mathbf{R}^d, 1 \le i \le m$, in $C_b^{k,\delta}$.

(v) Replace (I) with Kunita-type SODE

$$\left.\begin{aligned} d\phi(t) &= \overset{\circ}{F}(\circ dt, \phi(t)), \quad t > s \\ \phi(s) &= x \end{aligned}\right\}$$

where $\overset{\circ}{F}$ is a spatial semimartingale helix (i.e. with stationary ergodic increments) and with local characteristics of class $(B_{ub}^{k+1,\delta}, B_{ub}^{k,\delta})$ and the function

$$[0, \infty) \times \mathbf{R}^d \ni (t, x) \mapsto \sum_{j=1}^{d} \frac{\partial a^{\cdot,j}(t, x, y)}{\partial x_j}\bigg|_{y=x} \in \mathbf{R}^d$$

belongs to $B_{ub}^{k,\delta}$. In the Itô case, last condition is not needed.

$$\begin{aligned} \overset{\circ}{F}(t, x) &= V(t, x) + M(t, x) \\ a^{i,j}(t, x, y) &:= \frac{d}{dt} < M^i(\cdot, x), M^j(\cdot, y) > (t) \\ b^i(t, x) &:= \frac{d}{dt} V^i(t, x), \quad x, y \in \mathbf{R}^d, 1 \le i, j \le d \end{aligned}$$

Proof of Theorem 3

Cocycle property (ii): approximate the flow using helix mollifiers of Brownian motion:

$$W^k(t) := k \int_{t-1/k}^{t} W(s)\,ds - k \int_{-1/k}^{0} W(s)\,ds.$$

$$W^k(t_2, \theta(t_1,\omega)) = W^k(t_1+t_2,\omega) - W^k(t_1,\omega), \quad k \geq 1$$

([I-W], cf. [Mo.1], [Mo.2] for linear infinite-dimensional case).

(iii) and (iv) are well-known to hold for a.a. $\omega \in \Omega$ ([Ku], Theorem 4.6.5).

A perfect version of $\phi_{s,t}$ satisfying (i)-(iv) for *all* $\omega \in \Omega$, is obtained in [A-S] by perfection techniques and the diffeomorphism theorem for flows ([Ku], Theorem 4.6.5; cf. also [M-S.1]).

By [M-S.2], the random variables

$$X_1 := \sup_{\substack{0\leq s\leq t\leq T,\\ x\in \mathbf{R}^d}} \frac{|\phi_{s,t}(x,\cdot)|}{[1+|x|(\log^+|x|)^\gamma]},$$

$$X_2 := \sup_{\substack{0\leq s\leq t\leq T,\\ x\in \mathbf{R}^d}} \frac{|x|}{[1+|\phi_{s,t}(x,\cdot)|(\log^+|x|)^\gamma]}$$

have q-th moments for all $q \geq 1$. It is sufficient to show that the random variable

$$\hat{X}_1 := \sup_{\substack{0 \leq s \leq t \leq T, \\ x \in \mathbf{R}^d}} \frac{|\phi_{t,s}(x, \cdot)|}{[1 + |x|(\log^+ |x|)^\gamma]}$$

has q-th moments for all $q \geq 1$. Assume (without loss of generality) that $\gamma \in (0,1)$. From the definition of X_2,

$$|y| \leq X_2[1 + |\phi_{s,t}(y, \cdot)|(\log^+ |y|)^\gamma]$$

for all $0 \leq s \leq t \leq T, y \in \mathbf{R}^d$. Use the substitution

$$y = \phi_{t,s}(x, \omega) = \phi_{s,t}^{-1}(x, \omega),\ \phi_{s,t}(y, \omega) = x, 0 \leq s \leq t \leq T, \omega \in \Omega, x \in \mathbf{R}^d,$$

to rewrite above inequality as

$$|y| \leq X_2[1 + |x|(\log^+ |y|)^\gamma].$$

Solve above inequality (by taking $\log^+$) for $\log^+ |y|$. Therefore, there exists a non-random constant $K_1 := K_1(\gamma) > 0$ such that

$$|y| \leq K_1 X_2[1 + |x|\{1 + (\log^+ |X_2|)^\gamma + (\log^+ |x|)^\gamma\}].$$

Since X_2 has moments of all orders, the above inequality implies that $\hat{X}_1$ also has q-th moments for all $q \geq 1$.

Complete proof by [Ku], [M-S.2] and GRR. $\square$

Proof of Lemma 2

We first prove (11). Start with the perfect cocycle property for (ϕ, θ):

$$\phi(t_1 + t_2, \cdot, \omega) = \phi(t_2, \cdot, \theta(t_1, \omega)) \circ \phi(t_1, \cdot, \omega) \tag{12}$$

for all $t_1, t_2 \in \mathbf{R}$ and all $\omega \in \Omega$. Cocycle property for $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$ follows directly by taking Fréchet derivatives at $Y(\omega)$ on both sides of (12); viz.

$$\begin{aligned} D_2\phi(t_1 + t_2,&Y(\omega), \omega) \\ &= D_2\phi(t_2, \phi(t_1, Y(\omega), \omega), \theta(t_1, \omega)) \circ D_2\phi(t_1, Y(\omega), \omega) \\ &= D_2\phi(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega)) \circ D_2\phi(t_1, Y(\omega), \omega) \end{aligned} \tag{13}$$

for all $\omega \in \Omega, t_1, t_2 \in \mathbf{R}$. Existence of a fixed discrete spectrum for $D_2\phi(t, Y)$ follows from [Mo.1] and [M-S.1], using the integrability property (11) and the ergodicity of θ. ((11) follows from (13) and Theorem 3 (v)). But (10) implies (11)! Therefore it is sufficient to prove (10).

In view of (1) and the identity

$$\phi_{t_1, t_1+t_2}(x, \omega) = \phi(t_2, x, \theta(t_1, \omega)), \quad x \in \mathbf{R}^d, t_1, t_2 \in \mathbf{R},$$

(Theorem 3(i)), (10) (for $\epsilon = 0$) will follow from

$$\int_\Omega \log^+ \sup_{\substack{0\le s,t\le T,\\ |x'|\le\rho}} |D_x^\alpha \phi_{s,t}(\phi_{0,s}(Y(\omega),\omega)+x',\omega)|\, dP(\omega) < \infty, \quad 0 \le |\alpha| \le k. \tag{14}$$

Denote random "constants" by $K_i, i = 1,2,3,4$. Each $K_i := K_i(\rho, T), i = 1,2,3,4$, has q-th moments for all $q \ge 1$. The following inequalities follow easily from Theorem 3 (v).

$$\begin{aligned}
\log^+ &\sup_{\substack{s,t\in[0,T],\\ |x'|\le\rho}} |D_x^\alpha \phi_{s,t}(\phi_{0,s}(Y(\omega),\omega)+x',\omega)| \\
&\le \log^+ \sup_{s\in[0,T]} \left\{K_1(\omega)[1+(\rho+|\phi_{0,s}(Y(\omega),\omega)|)^2]\right\} \\
&\le \log^+ K_2(\omega) + \log^+[1+2\rho^2+K_3(\omega)(1+|Y(\omega)|^4)] \\
&\le \log^+ K_4(\omega) + \log[1+2\rho^2] + 4\log^+|Y(\omega)| \qquad (15)
\end{aligned}$$

for all $\omega \in \Omega$. (15)+ integrability hypothesis on Y imply (14). $\square$

Proof of Lemma 4

The integrability condition (10) of Lemma 2 implies that

$$\int_\Omega \log^+ \sup_{\substack{0\le t_1,t_2\le 1,\\ x^*\in\bar{B}(0,1)}} \|D_2 Z(t_1, x^*, \theta(t_2,\omega))\|_{L(\mathbf{R}^d)}\, dP(\omega) < \infty. \tag{19}$$

Therefore by (the perfect version of) the ergodic theorem (Lemma 1(i)), there is a sure event $\Omega_3 \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega_3) = \Omega_3$ for all $t \in \mathbf{R}$, and

$$\lim_{t\to\infty} \frac{1}{t} \log^+ \sup_{\substack{0\le u\le 1,\\ x^*\in \bar{B}(0,1)}} \|D_2 Z(u, x^*, \theta(t,\omega))\|_{L(\mathbf{R}^d)} = 0 \tag{20}$$

for all $\omega \in \Omega_3$.

Let $\omega \in \Omega_3$ and suppose $x \in \mathbf{R}^d$ satisfies (17). Then (17) implies that there exists a positive integer $N_0(x,\omega)$ such that $Z(n,x,\omega) \in \bar{B}(0,1)$ for all $n \ge N_0$. Let $n \le t < n+1$, $n \ge N_0$. Then by the cocycle property for (Z,θ) and the Mean Value Theorem:

$$\begin{aligned}&\sup_{n\le t\le n+1} \frac{1}{t} \log |Z(t,x,\omega)| \\ &\le \frac{1}{n} \log^+ \sup_{\substack{0\le u\le 1,\\ x^*\in \bar{B}(0,1)}} \|D_2 Z(u, x^*, \theta(n,\omega))\|_{L(\mathbf{R}^d)} + \frac{n}{(n+1)} \frac{1}{n} \log |Z(n,x,\omega)|.\end{aligned}$$

Take $\limsup\limits_{n\to\infty}$ in the above relation and use (20) to get

$$\limsup_{t\to\infty} \frac{1}{t} \log |Z(t,x,\omega)| \le \limsup_{n\to\infty} \frac{1}{n} \log |Z(n,x,\omega)|.$$

The inequality

$$\limsup_{n\to\infty} \frac{1}{n} \log |Z(n,x,\omega)| \le \limsup_{t\to\infty} \frac{1}{t} \log |Z(t,x,\omega)|,$$

is obvious. Hence (18) holds. $\square$

References

[A-S] Arnold, L., and Scheutzow, M. K. R., Perfect cocycles through stochastic differential equations, *Probab. Th. Rel. Fields,* 101, (1995), 65-88.

[A-I] Arnold, L., and Imkeller, P., Stratonovich calculus with spatial parameters and anticipative problems in multiplicative ergodic theory, *Stoch. Proc. Appl.* 62 (1996), 19-54.

[C] Carverhill, A., Flows of stochastic dynamical systems: Ergodic theory, *Stochastics*, 14 (1985), 273-317.

[I-W] Ikeda, N., and Watanabe, S., *Stochastic Differential Equations and Diffusion Processes,* Second Edition, North-Holland-Kodansha (1989).

[Ku] Kunita, H., *Stochastic Flows and Stochastic Differential Equations*, Cambridge University Press, Cambridge, New York, Melbourne, Sydney (1990).

[M-N-S] Millet, A., Nualart, D., and Sanz, M., Large deviations for a class of anticipating stochastic differential equations, *The Annals of Probability*, 20 (1992), 1902-1931.

[Mo.1] Mohammed, S.-E. A., The Lyapunov spectrum and stable manifolds for stochastic linear delay equations, *Stochastics and Stochastic Reports*, Vol. 29 (1990), 89-131.

[Mo.2] Mohammed, S.-E. A., Lyapunov exponents and stochastic flows of linear and affine hereditary Systems, *Diffusion Processes and Related Problems in Analysis*, Vol. II, edited by Mark Pinsky and Volker Wihstutz, Birkhauser (1992), 141-169.

[M-S.1] Mohammed, S.-E. A., and Scheutzow, M. K. R., Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales, Part I: The multiplicative ergodic theory, *Ann. Inst. Henri Poincaré, Probabilités et Statistiques,* Vol. 32, 1, (1996), 69-105. pp. 43.

[M-S.2] Mohammed, S.-E. A., and Scheutzow, M. K. R., Spatial estimates for stochastic flows in Euclidean space, *The Annals of Probability*, 26, 1, (1998), 56-77.

[Nu] Nualart, D., Analysis on Wiener space and anticipating stochastic calculus, *St. Flour Notes.*

[N-P] Nualart, D., and Pardoux, E., Stochastic calculus with anticipating integrands, Analysis on Wiener space and anticipating stochastic calculus, *Probab. Th. Rel. Fields*, 78 (1988), 535-581.

[O] Oseledec, V. I., A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems, *Trudy Moskov. Mat. Obšč.* 19 (1968), 179-210. English transl. *Trans. Moscow Math. Soc.* 19 (1968), 197-221.

[O-P] Ocone, D., and Pardoux, E., A generalized Itô-Ventzell formula. Application to a class of anticipating stochastic differential equations, *Ann. Inst. Henri Poincaré, Probabilités et Statistiques,* Vol. 25, no. 1 (1989), 39-71.

[Ru.1] Ruelle, D., Ergodic theory of differentiable dynamical systems, *Publ. Math. Inst. Hautes Etud. Sci.* (1979), 275-306.

[Ru.2] Ruelle, D., Characteristic exponents and invariant manifolds in Hilbert space, *Annals of Mathematics 115* (1982), 243–290.

[Sc] Scheutzow, M. K. R., On the perfection of crude cocycles, *Random and Computational Dynamics, 4,* (1996), 235-255.

[Wa] Wanner, T., Linearization of random dynamical systems, in *Dynamics Reported,* vol. 4, edited by U. Kirchgraber and H.O. Walther, Springer (1995), 203-269.