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A PROPOSED MEASURE OF INTERNAL CONSISTENCY RELIABILITY: COEFFICIENT L-alpha

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Data sets in the social and behavioral sciences are often small or heavy-tailed. Previous studies have demonstrated that small samples or leptokurtic distributions adversely affect the performance of Cronbach's coefficient alpha. To address these concerns, we propose an alternative estimator of reliability based on L-comoments. The empirical results of this study demonstrate that when sample sizes are small and distributions are heavy-tailed that the proposed coefficient L-alpha has substantial advantages over the conventional Cronbach estimator of reliability in terms of relative bias and relative standard error.

1. Introduction

Coefficient alpha (Cronbach, 1951; Guttman, 1945) is a commonly used index for measuring internal consistency reliability. Consider alpha (α) in terms of a model that decomposes an observed score into the sum of two independent components: a true unobservable score t_i and a random error component e_{ij}. The model can be summarized as

$$X_{ij} = t_i + e_{ij} \tag{1}$$

where X_{ij} is the observed score associated with the i-th examinee on the j-th test item, and where $i = 1, \ldots, n$; $j = 1, \ldots, k$; and the error terms (e_{ij}) are independent with a mean of zero. Inspection of (1) indicates that this particular model restricts the true score t_i to be the same across all k test items. The reliability measure associated with the test items in (1) is a function of the true score variance and cannot be computed directly. Thus, estimates of reliability such as coefficient α have been derived and will be defined herein as (e.g., Christman and Van Aelst, 2006)

$$\alpha = \frac{k}{k-1}\left(1 - \frac{\sum_j \sigma_j^2}{\sum_j \sigma_j^2 + \sum\sum_{j \neq j'} \sigma_{jj'}}\right). \tag{2}$$

A conventional estimate of α can be obtained by substituting the usual OLS sample estimates associated with σ_j^2 and $\sigma_{jj'}$ into (2) as

$$\hat{\alpha}_C = \frac{k}{k-1}\left(1 - \frac{\sum_j s_j^2}{\sum_j s_j^2 + \sum\sum_{j \neq j'} s_{jj'}}\right) \tag{3}$$

where s_j^2 and $s_{jj'}$ are the diagonal and off-diagonal elements from the variance-covariance matrix, respectively.

Key Words and Phrases: Cronbach's alpha, L-moments, Reliability

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Although coefficient α is often used as an index for reliability, it is also well known that its use is limited when data are non-normal, in particular leptokurtic, or when sample sizes are small (e.g. Bay, 1973; Christman and Van Aelst, 2006; Sheng and Sheng, 2012; Wilcox, 1992). These limitations are of concern because data sets in the social and behavioral sciences can often possess heavy tails or consist of small sample sizes (e.g. Micceri, 1989; Yuan et al., 2004). Specifically, it has been demonstrated that $\hat{\alpha}_C$ can substantially underestimate α when heavy-tailed distributions are encountered. For example, Sheng and Sheng (2012, Table 1) sampled from a symmetric leptokurtic distribution and found the empirical estimate of α to be approximately $\hat{\alpha}_C = 0.70$ when the true population parameter was $\alpha = 0.80$. Further, it is not uncommon that data sets consist of small sample sizes e.g. $n = 10$ or 20 which are encountered in the contexts of rehabilitation (e.g. alcohol treatment programs, group therapy, etc.) and special education as student-teacher ratios are often small. Furthermore, Monte Carlo evidence has demonstrated that $\hat{\alpha}_C$ can underestimate α - even when small samples are drawn from a normal distribution (see Sheng and Sheng, 2012, Table 1).

L-moment estimators (e.g. Hosking, 1990; Hosking and Wallis, 1997) have demonstrated to be superior to the conventional product-moment estimators in terms of bias, efficiency, and their resistance to outliers (e.g. Headrick, 2011; Hodis et al., 2012; Hosking, 1992; Vogel and Fennessy, 1993). Further, L-comoment estimators (Serfling and Xiao, 2007) such as the L-correlation has demonstrated to be an attractive alternative to the conventional Pearson correlation in terms of relative bias when heavy-tailed distributions are of concern (Headrick and Pant, 2012a,b,c,d,e).

In view of the above, the present aim here is to propose a L-comoment based coefficient L-α, and its estimator denoted as $\hat{\alpha}_L$, as an alternative to conventional alpha $\hat{\alpha}_C$ in (3). Empirical results associated with the simulation study herein indicate that $\hat{\alpha}_L$ can be substantially superior to $\hat{\alpha}_C$ in terms of relative bias and relative standard error when distributions are heavy-tailed and sample sizes are small.

The rest of the paper is organized as follows. In Section 2, summaries of univariate L-moments and L-comoments are first provided. Coefficient L-α ($\hat{\alpha}_L$) is then introduced and numerical examples are provided to illustrate the computation and sampling distribution associated with $\hat{\alpha}_L$. In Section 3, a Monte Carlo study is carried out to evaluate the performance of $\hat{\alpha}_C$ and $\hat{\alpha}_L$. The results of the study are discussed in Section 4.

2. *L*-moments, *L*-comoments, and Coefficient *L*-α

The system of univariate L-moments (Hosking, 1990, 1992; Hosking and Wallis, 1997) can be considered in terms of the expectations of linear combinations of order statistics associated with a random variable Y. Specifically, the first four L-moments are expressed as

$$\lambda_1 = E[Y_{1:1}]$$

$$\begin{aligned}\lambda_2 &= \frac{1}{2}E[Y_{2:2} - Y_{1:2}]\\ \lambda_3 &= \frac{1}{3}E[Y_{3:3} - 2Y_{2:3} + Y_{1:3}]\\ \lambda_4 &= \frac{1}{4}E[Y_{4:4} - 3Y_{3:4} + 3Y_{2:4} - Y_{1:4}]\end{aligned}$$

where the order statistics $Y_{1:m} \leq Y_{2:m} \leq \ldots \leq Y_{m:m}$ are drawn from the random variable Y. The values of λ_1 and λ_2 are measures of location and scale and are the arithmetic mean and one-half the coefficient of mean difference (or Gini's index of spread), respectively. Higher order L-moments are transformed to dimensionless quantities referred to as L-moment ratios defined as $\tau_r = \lambda_r/\lambda_2$ for $r \geq 3$, and where τ_3 and τ_4 are the analogs to the conventional measures of skew and kurtosis. In general, L-moment ratios are bounded in the interval $-1 < \tau_r < 1$ as is the index of L-skew (τ_3) where a symmetric distribution implies that all L-moment ratios with odd subscripts are zero. Other smaller boundaries can be found for more specific cases. For example, the index of L-kurtosis (τ_4) has the boundary condition for continuous distributions of $(5\tau_3^2 - 1)/4 < \tau_4 < 1$.

L-comoments (Olkin and Yitzhuki, 1992; Serfling and Xiao, 2007) are introduced by considering two exchangeable random variables Y_j and Y_k with distribution functions $F(Y_j)$ and $F(Y_k)$. The second L-moment associated with Y_j can alternatively be expressed as

$$\lambda_2(Y_j) = 2\mathrm{Cov}(Y_j, F(Y_j)) \tag{4}$$

where $F(\cdot)$ denotes the cumulative distribution function (cdf). The second L-comoment associated with Y_j and Y_k is

$$\lambda_2(Y_j, Y_k) = 2\mathrm{Cov}(Y_j, F(Y_k)). \tag{5}$$

The ratio $\eta_{jk} = \lambda_2(Y_j, Y_k)/\lambda_2(Y_j)$ is defined as the L-correlation of Y_j with respect to Y_k, which measures the monotonic relationship (not just linear) between the two variables (Headrick and Pant, 2012e). The estimators of (4) and (5), based on the empirical cdf $\hat{F}(\cdot)$, are U-statistics (Serfling, 1980; Serfling and Xiao, 2007) and their sampling distributions converge to a normal distribution when the sample size is sufficiently large.

In terms of coefficient L-α, an approach that can be taken to equate the conventional and L-moment (comoment) definitions of α is to express (2) as

$$\alpha = \frac{1}{1 + (R-1)/k} = \frac{k}{k-1}\left(1 - \frac{\sum_j \sigma_j^2}{\sum_j \sigma_j^2 + \sum\sum_{j\neq j'} \sigma_{jj'}}\right) \tag{6}$$

where $R > 1$ is the common ratio between the main and off diagonal elements of the variance-covariance matrix i.e. $R = \sigma_j^2/\sigma_{jj'}$. As such, given a fixed value of R in (6) will allow us to define α in terms of the second L-moments and second L-comoments as

$$\alpha = \frac{1}{1+(R-1)/k} = \frac{k}{k-1}\left(1 - \frac{\sum_j \lambda_{2(j)}}{\sum_j \lambda_{2(j)} + \sum\sum_{j\neq j'} \lambda_{2(jj')}}\right) \tag{7}$$

where $R = \lambda_{2(j)}/\lambda_{2(jj')}$. Thus, the estimator of L-α is expressed as

$$\hat{\alpha}_L = \frac{k}{k-1}\left(1 - \frac{\sum_j \ell_{2(j)}}{\sum_j \ell_{2(j)} + \sum\sum_{j\neq j'} \ell_{2(jj')}}\right) \tag{8}$$

where $\ell_{2(j)}$ ($\ell_{2(jj')}$) denotes the sample estimate of the second L-moment (second L-comoment) in (4) and (5). An example demonstrating the computation of $\hat{\alpha}_L$ is provided below in equation (9). The computed estimator $\hat{\alpha}_L = 0.807$ in (9) is based on the data in Table 1 and the second L-moment-comoment matrix in Table 2. The corresponding conventional estimate for the data in Table 1 is $\hat{\alpha}_C = 0.798$.

Table 1: Data (Items) for computing the second L-moment-comoment matrix in Table 2. Note that $\hat{F}(\cdot)$ denotes the empirical cdf.

X_{i1}	X_{i2}	X_{i3}	$\hat{F}(X_{i1})$	$\hat{F}(X_{i2})$	$\hat{F}(X_{i3})$
2	4	3	0.15	0.45	0.15
5	7	7	0.75	0.95	1.00
3	5	5	0.35	0.65	0.40
6	6	6	0.90	0.80	0.75
7	7	6	1.00	0.95	0.75
5	2	6	0.75	0.10	0.75
2	3	3	0.15	0.25	0.15
4	3	6	0.55	0.25	0.75
3	5	5	0.35	0.65	0.40
4	4	5	0.55	0.45	0.40

The data are part of the “Satisfaction With Life Data” from McDonald (1999, p.47).

Table 2: Second L-moment-comoment matrix for coefficient $\hat{\alpha}_L$ in equation (9).

Item	1	2	3
1	$\ell_{2(1)} = 0.989$	$\ell_{2(12)} = 0.500$	$\ell_{2(13)} = 0.789$
2	$\ell_{2(21)} = 0.500$	$\ell_{2(2)} = 1.022$	$\ell_{2(23)} = 0.411$
3	$\ell_{2(31)} = 0.667$	$\ell_{2(32)} = 0.333$	$\ell_{2(3)} = 0.733$

$$\begin{aligned}\hat{\alpha}_L = 0.807 = (3/2)(1 - (\ell_{2(1)} + \ell_{2(2)} + \ell_{2(3)})/(\ell_{2(1)} + \ell_{2(2)} + \ell_{2(3)} +\\ \ell_{2(21)} + \ell_{2(31)} + \ell_{2(32)} + \ell_{2(12)} + \ell_{2(13)} + \ell_{2(23)})).\end{aligned} \tag{9}$$

The estimator $\hat{\alpha}_L$ in (8) and (9) is a ratio of the sums of U-statistics and thus a consistent estimator of α in (7) with a sampling distribution that converges, for large samples, to the normal distribution (e.g. Olkin and Yitzhuki, 1992; Schechtman and Yitzhaki, 1987; Serfling and Xiao, 2007). For convenience to the reader, provided in

Figure 1 is the sampling distribution of $\hat{\alpha}_L$ that is approximately normal and based on $\alpha = 0.50$, $n = 100,000$, and a symmetric heavy-tailed distribution (kurtosis of 25, see Figure 2) that would be associated with t_i in (1).

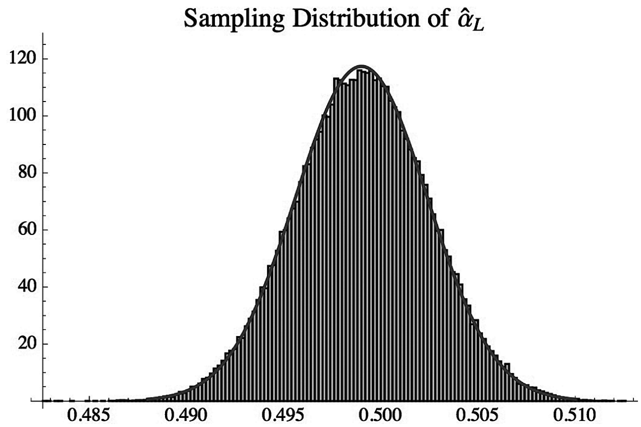

Figure 1: Approximate normal sampling distribution of $\hat{\alpha}_L$ with $\alpha = 0.50$. The distribution consists of 25,000 statistics based on samples of size $n = 100,000$ and the heavy-tailed distribution (kurtosis of 25) in Figure 2.

3. Monte Carlo Simulation

An algorithm was written in MATLAB (Mathworks, 2010) to generate 25,000 independent sample estimates of conventional and L-comoment α. The estimators $\hat{\alpha}_C$ and $\hat{\alpha}_L$ were based on the specified (a) distributions depicted in Figures 2–4 for the true score t_i in (1), (b) diagonal and off-diagonal values given in Tables 3 and 4 for σ_j^2, $\sigma_{jj'}$ in (6) and for $\lambda_{2(j)}$, $\lambda_{2(jj')}$ in (7), (c) variance (σ_e^2) values given in Tables 3 and 4 for the error term e_{ij} in (1), (d) number of test items $k = 4, 9, 10$, and (e) sample sizes of $n = 10, 20, 1000$ for all scenarios considered.

More specifically, the true score t_i in (1) followed each of the three distributions shown in Figures 2–4 and are referred to as (a) Distribution 1: symmetric and leptokurtic (skew = 0, kurtosis = 25; L-skew = 0, L-kurtosis = 0.4225), (b) Distribution 2: asymmetric and leptokurtic (skew = 3, kurtosis = 21; L-skew = 0.3130, L-kurtosis = 0.3335), and (c) Distribution 3: standard normal (skew = 0, kurtosis = 0; L-skew = 0, L-kurtosis = 0.1226). We would point out that Distributions 1 and 2 were considered for the purpose of comparing and contrasting heavy-tailed distributions that were symmetric with those that were skewed. Further, these two non-normal distributions have also been used in several studies in the social and behavioral sciences (e.g. Berkovits et al., 2000; Enders, 2001; Harwell and Serlin, 1988; Headrick and Sawilowsky, 1999, 2000; Olsson et al., 2003).

The three distributions described above were generated for the Monte Carlo simulation study using the L-moment based power method transformation derived by

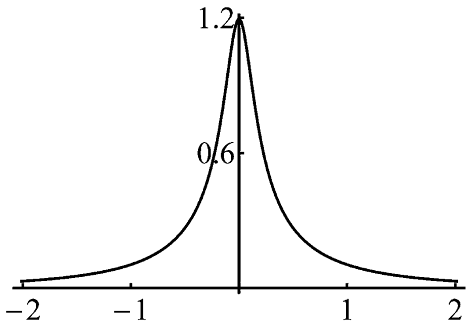

Figure 2: Distribution 1 with skew (L-skew) of 0 (0) and kurtosis (L-kurtosis) of 25 (0.4225).

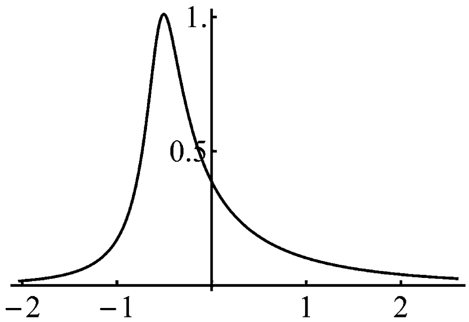

Figure 3: Distribution 2 with skew (L-skew) of 3 (0.3130) and kurtosis (L-kurtosis) of 21 (0.3335).

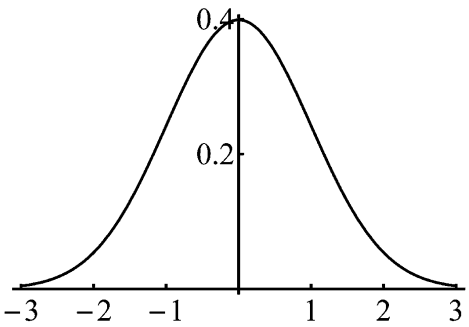

Figure 4: Distribution 3 is standard normal with skew (L-skew) of 0 (0) and kurtosis (L-kurtosis) of 0 (0.1226).

Headrick (2011). Specifically, the true scores t_i in (1) were generated using the following Fleishman (1978) type polynomial

$$t_i = c_1 + c_2 Z_i + c_3 Z_i^2 + c_4 Z_i^3 \tag{10}$$

where $Z_i \sim$ iid $N(0,1)$. The shape of the distribution associated with the true scores t_i in (10) is contingent on the values of the coefficients, which are computed based on Headrick's Equations (2.14)–(2.17) in Headrick (2011) as

Table 3: Conventional covariance (C) and L-comoment (L) matrices with the associated error variances (σ_e^2) for Distributions 1–3. The ratio of diagonal to off-diagonal is $R = 2$.

Distribution	Procedure	Diagonal	Off-Diagonal	σ_e^2
1	C	3.420	1.710	1.710
1	L	0.848	0.424	1.000
2	C	3.224	1.612	1.612
2	L	0.842	0.421	1.000
3	C	2.000	1.000	1.000
3	L	0.798	0.399	1.000

Reliability is $\alpha = 0.80$, 0.90; Number of Items are $k = 4$, 9.

Table 4: Conventional covariance (C) and L-comoment (L) matrices with the associated error variances (σ_e^2) for Distributions 1–3. The ratio of diagonal to off-diagonal is $R = 5$.

Distribution	Procedure	Diagonal	Off-Diagonal	σ_e^2
1	C	8.550	1.710	6.840
1	L	1.470	0.294	5.313
2	C	8.060	1.612	6.448
2	L	1.443	0.2886	5.135
3	C	5.000	1.000	4.000
3	L	1.262	0.2524	4.000

Reliability is $\alpha = 0.50$, 0.714; Number of Items are $k = 4$, 10.

$$\begin{aligned} c_1 &= -c_3 = -\tau_3\sqrt{\frac{\pi}{3}} \\ c_2 &= \frac{-16\delta_2 + \sqrt{2}(3+2\tau_4)\pi}{8(5\delta_1 - 2\delta_2)} \\ c_4 &= \frac{40\delta_1 - \sqrt{2}(3+2\tau_4)\pi}{20(5\delta_1 - 2\delta_2)}. \end{aligned} \tag{11}$$

The three sets of coefficients for the distributions in Figures 2–4 are (respectively): (1) $c_1 = 0.0$, $c_2 = 0.3338$, $c_3 = 0.0$, $c_4 = 0.2665$; (2) $c_1 = -0.3203$, $c_2 = 0.5315$, $c_3 = 0.3203$, $c_4 = 0.1874$; and (3) $c_1 = 0.0$, $c_2 = 1.0$, $c_3 = 0.0$, $c_4 = 0.0$. The values of the three sets of coefficients are based on the values of L-skew and L-kurtosis given in Figures 2–4 and where $\delta_1 = 0.36045147$ and $\delta_2 = 1.15112868$ in (11) (see Headrick, 2011, Eqs. A.1, A.2). The solutions to the coefficients in (11) ensure that $\lambda_1 = 0$ and $\lambda_2 = 1/\sqrt{\pi}$, which are associated with the unit normal distribution.

The values of α for both conventional and L-moment procedures were determined based on the three specified true score (t_i) distributions, main diagonal (σ_j^2, $\lambda_{2(j)}$) to off-diagonal ($\sigma_{jj'}$, $\lambda_{2(jj')}$) ratios (R) as in (6) and (7), and the number of items (k). As such, and given a specified true score (t_i) distribution, the error variances (σ_e^2) were subsequently determined so that the main diagonal and off-diagonal values in Table 3 and Table 4 yielded the appropriate ratios (R), i.e. $R = 2$ and $R = 5$, respectively. Thus, using (6) and (7) with $R = 2$ and $k = 4$ ($k = 9$) will yield $\alpha = 0.80$ ($\alpha = 0.90$) for all cases in Table 3. Analogously, the ratio $R = 5$ and $k = 4$ ($k = 10$) will yield $\alpha = 0.50$, ($\alpha = 5/7 = 0.714$) for all cases in Table 4. These four values of α represent commonly used references of various degrees of reliability i.e. 0.50 (poor); $5/7 = 0.714$

(acceptable); 0.80 (good); and 0.90 (excellent).

For all cases in the simulation, the error term e_{ij} in (1) was normally distributed with a mean of zero and the variance parameters (σ_e^2) listed in Table 3 and Table 4. We would note that it was required for the values of σ_e^2 to differ for the conventional moment and L-comoment procedures when the true score t_i followed the two non-normal distributions (i.e. Distributions 1 and 2) in order for the values of α to be the same for both procedures. This requirement was necessary because it has been demonstrated that the amount of bias associated with estimators of α depends on not only the distribution and sample size, but also the value of α being estimated (see Sheng and Sheng, 2012, Table 5).

The formulae used for computing the estimators $\hat{\alpha}_C$ and $\hat{\alpha}_L$ were computed using (3) and (8) and the empirical estimates of the cdfs in (4) and (5), $\hat{F}(\cdot)$, as in Tables 1 and 2. The estimators were both transformed to the form of an intraclass correlation (as the model in Eq. 1 assumes compound symmetry) as $\bar{\rho}_{C,L} = \hat{\alpha}_{C,L}/(1-(k-1)\hat{\alpha}_{C,L})$ (e.g. Headrick, 2010, p.104) and were subsequently Fisher z' transformed i.e. $z'_{\bar{\rho}_{C,L}}$. Bias-corrected accelerated bootstrapped average (mean) estimates, confidence intervals (C.I.s), and standard errors were subsequently obtained for $z'_{\bar{\rho}_{C,L}}$ using 10,000 resamples. The bootstrap results associated with the means and C.I.s were then transformed back to their original metrics (i.e. the estimators $\hat{\alpha}_C$ and $\hat{\alpha}_L$). Further, percentages of relative bias (RBias) and relative standard error (RSE) were computed for $\hat{\alpha}_{C,L}$ as: RBias $= ((\hat{\alpha}_{C,L} - \alpha)/\alpha) \times 100$ and RSE $=$ (standard error$/\hat{\alpha}_{C,L}) \times 100$. The results of the simulation are reported in Tables 5–7 and are discussed in the next section.

4. Discussion and Conclusion

One of the advantages that L-moment ratios have over conventional product-moment estimators is that they can be far less biased when sampling is from distributions with more severe departures from normality (Hosking and Wallis, 1997; Serfling and Xiao, 2007). And, inspection of the simulation results in Table 5 and Table 6 clearly indicates that this is the case. That is, the superiority that the L-comoment based estimator $\hat{\alpha}_L$ has over its corresponding conventional counterpart $\hat{\alpha}_C$ is obvious in the contexts of Distributions 1 and 2. For example, inspection of the first entry in Table 5 ($\alpha = 0.50$, $k = 4$, $n = 10$) indicates that the estimator $\hat{\alpha}_C$ associated with Distribution 1 was, on average, 88.32% of its associated population parameter whereas the estimator $\hat{\alpha}_L$ was 96.94% of its parameter. Further, it is also evident that $\hat{\alpha}_L$ is a more efficient estimator as its RSE is smaller than its corresponding conventional estimator. For example, in terms of Distribution 1, inspection of Table 5 ($\alpha = 0.50$, $k = 4$, $n = 10$) indicates RSE measures of: RSE($\hat{\alpha}_C$) $= 0.5661\%$ compared with RSE($\hat{\alpha}_L$) $= 0.4725\%$. This demonstrates that $\hat{\alpha}_L$ has more precision because it has less variance around its estimate. Moreover, one should note that $\hat{\alpha}_L$ or $\hat{\alpha}_C$ performs similarly in the two heavy-tailed distributions, namely, Distributions 1 and 2. This suggests that skewness does not affect $\hat{\alpha}_C$ or $\hat{\alpha}_L$, which agrees with

Table 5: Simulation results for estimating α using the Conventional (C) and L-moment (L) procedures (Proc) based on the number of items (k) and samples of size $n = 10$.

α	k	Dist-Proc	Estimate ($\hat{\alpha}$)	95% C.I.	RSE %	RBias %
0.50	4	1-C	0.4416	0.4367, 0.4465	0.5661	−11.68
0.50	4	1-L	0.4847	0.4801, 0.4891	0.4725	−3.06
0.50	4	2-C	0.4448	0.4400, 0.4495	0.3237	−11.04
0.50	4	2-L	0.4839	0.4796, 0.4883	0.2583	−3.22
0.50	4	3-C	0.4888	0.4852, 0.4922	0.3621	−2.24
0.50	4	3-L	0.5003	0.4968, 0.5040	0.3698	0.06
0.714	10	1-C	0.6617	0.6581, 0.6652	0.2720	−7.36
0.714	10	1-L	0.6960	0.6931, 0.6989	0.2155	−2.56
0.714	10	2-C	0.6662	0.6628, 0.6697	0.2612	−6.73
0.714	10	2-L	0.6975	0.6946, 0.7003	0.2079	−2.35
0.714	10	3-C	0.7069	0.7051, 0.7086	0.1273	−1.03
0.714	10	3-L	0.7131	0.7113, 0.7149	0.1290	−0.17
0.80	4	1-C	0.7306	0.7275, 0.7336	0.2053	−8.67
0.80	4	1-L	0.7887	0.7866, 0.7908	0.1357	−1.41
0.80	4	2-C	0.7398	0.7371, 0.7426	0.1906	−7.52
0.80	4	2-L	0.7924	0.7904, 0.7944	0.1287	−0.95
0.80	4	3-C	0.7908	0.7893, 0.7922	0.0923	−1.15
0.80	4	3-L	0.8030	0.8016, 0.8044	0.0909	0.37
0.90	9	1-C	0.8591	0.8575, 0.8609	0.0989	−4.54
0.90	9	1-L	0.8924	0.8914, 0.8936	0.0628	−0.84
0.90	9	2-C	0.8636	0.8620, 0.8651	0.0926	−4.04
0.90	9	2-L	0.8933	0.8922, 0.8944	0.0605	−0.74
0.90	9	3-C	0.8934	0.8927, 0.8941	0.0381	−0.73
0.90	9	3-L	0.8991	0.8985, 0.8998	0.0378	−0.10

See Tables 3 and 4 for the parameters and Figures 2–4 for the distributions (Dist).

Table 6: Simulation results for estimating α using the Conventional (C) and L-moment (L) procedures (Proc) based on the number of items (k) and samples of size $n = 20$.

α	k	Dist-Proc	Estimate ($\hat{\alpha}$)	95% C.I.	RSE %	RBias %
0.50	4	1-C	0.4643	0.4606, 0.4679	0.3977	−7.15
0.50	4	1-L	0.4903	0.4870, 0.4933	0.3263	−1.94
0.50	4	2-C	0.4697	0.4663, 0.4732	0.3732	−6.05
0.50	4	2-L	0.4938	0.4909, 0.4967	0.306	−1.24
0.50	4	3-C	0.4945	0.4921, 0.4968	0.2389	−1.11
0.50	4	3-L	0.4995	0.4971, 0.5019	0.2456	−0.11
0.714	10	1-C	0.6852	0.6826, 0.6878	0.1926	−4.07
0.714	10	1-L	0.7056	0.7036, 0.7077	0.1485	−1.22
0.714	10	2-C	0.6858	0.6834, 0.6882	0.1831	−3.98
0.714	10	2-L	0.7047	0.7028, 0.7066	0.1414	−1.34
0.714	10	3-C	0.7098	0.7086, 0.7111	0.0881	−0.62
0.714	10	3-L	0.7130	0.7117, 0.7142	0.0882	−0.19
0.80	4	1-C	0.7569	0.7549, 0.7591	0.1404	−5.39
0.80	4	1-L	0.7937	0.7923, 0.7952	0.0917	−0.78
0.80	4	2-C	0.7612	0.7592, 0.7631	0.1330	−4.85
0.80	4	2-L	0.7940	0.7926, 0.7954	0.0893	−0.75
0.80	4	3-C	0.7944	0.7935, 0.7954	0.0627	−0.7
0.80	4	3-L	0.8000	0.7990, 0.8010	0.0613	−0.002
0.90	9	1-C	0.8750	0.8737, 0.8761	0.0690	−2.79
0.90	9	1-L	0.8958	0.8950, 0.8966	0.0431	−0.47
0.90	9	2-C	0.8784	0.8773, 0.8795	0.0644	−2.4
0.90	9	2-L	0.8965	0.8958, 0.8972	0.0411	−0.39
0.90	9	3-C	0.8969	0.8965, 0.8974	0.0247	−0.34
0.90	9	3-L	0.8998	0.8994, 0.9002	0.0250	−0.02

See Tables 3 and 4 for the parameters and Figures 2–4 for the distributions (Dist).

Table 7: Simulation results for estimating α using the Conventional (C) and L-moment (L) procedures (Proc) based on the number of items (k) and samples of size $n = 1000$.

α	k	Dist-Proc	Estimate ($\hat{\alpha}$)	95% C.I.	RSE %	RBias %
0.50	4	1-C	0.4988	0.4982, 0.4994	0.05814	−0.24
0.50	4	1-L	0.4988	0.4984, 0.4992	0.04210	−0.24
0.50	4	2-C	0.4993	0.4987, 0.4998	0.05613	−0.14
0.50	4	2-L	0.5001	0.4997, 0.5005	0.04200	0.02
0.50	4	3-C	0.5000	0.4997, 0.5003	0.03200	0.00
0.50	4	3-L	0.5000	0.4997, 0.5004	0.03400	0.00
0.714	10	1-C	0.7134	0.7129, 0.7138	0.03084	−0.12
0.714	10	1-L	0.7132	0.7129, 0.7135	0.02103	−0.15
0.714	10	2-C	0.7133	0.7129, 0.7137	0.02804	−0.14
0.714	10	2-L	0.7140	0.7137, 0.7143	0.01961	−0.04
0.714	10	3-C	0.7141	0.7140, 0.7143	0.01120	−0.03
0.714	10	3-L	0.7142	0.7140, 0.7144	0.01260	−0.01
0.80	4	1-C	0.7991	0.7987, 0.7994	0.02127	−0.11
0.80	4	1-L	0.8017	0.8015, 0.8019	0.01247	0.21
0.80	4	2-C	0.7990	0.7987, 0.7993	0.02003	−0.12
0.80	4	2-L	0.8011	0.8009, 0.8013	0.01248	0.14
0.80	4	3-C	0.7999	0.7998, 0.8000	0.00875	−0.01
0.80	4	3-L	0.8000	0.7998, 0.8001	0.00875	0.00
0.90	9	1-C	0.8992	0.8990, 0.8994	0.01001	−0.09
0.90	9	1-L	0.9008	0.9007, 0.9009	0.00555	0.09
0.90	9	2-C	0.8994	0.8992, 0.8995	0.01000	−0.07
0.90	9	2-L	0.9005	0.9004, 0.9006	0.00556	0.06
0.90	9	3-C	0.8999	0.8999, 0.9000	0.00333	−0.01
0.90	9	3-L	0.9000	0.8999, 0.9000	0.00333	0.00

See Tables 3 and 4 for the parameters and Figures 2–4 for the distributions (Dist).

results from Headrick and Pant (2012e, see Remark 2).

In summary, the L-comoment based $\hat{\alpha}_L$ is an attractive alternative to the traditional Cronbach alpha $\hat{\alpha}_C$ when distributions with heavy tails and small samples sizes are encountered. It is also worthy to point out that $\hat{\alpha}_L$ had a slight advantage over $\hat{\alpha}_C$ when sampling was from normal populations (see Table 5; $\alpha = 0.50$, $k = 4$, $n = 10$, 3-C, 3-L). When sample sizes were large the performance of the two estimators $\hat{\alpha}_{C,L}$ were similar (see Table 7; $n = 1000$). It is noted that the data in this study were generated assuming (essential) tau-equivalence (Lord and Novick, 1968) and uncorrelated error terms (Guttman, 1945; Novick and Lewis, 1967). Thus, it would also be interesting to see the performance of $\hat{\alpha}_L$ in comparison with $\hat{\alpha}_C$ in situations where one or both of these assumptions are violated.

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