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THE STABLE MANIFOLD

THEOREM FOR STOCHASTIC

DIFFERENTIAL EQUATIONS

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SDE's: Stable Manifolds

- Formulate a *Local Stable Manifold Theorem* for SDE's driven by Brownian motion (or general noise with stationary ergodic increments): Stratonovich or Itô type.

- Start with the existence of a stochastic flow for SDE.

- Concept of a hyperbolic stationary trajectory. The stationary trajectory is a solution of the forward/backward anticipating SDE for all time (Stratonovich case).

- Existence of a stationary random family of asymptotically invariant stable and unstable manifolds within a stationary neighborhood of the hyperbolic stationary solution.

- Stable and unstable manifolds dynamically characterized using forward and backward solutions of anticipating versions of the (Stratonovich) SDE.

- Proof based on Ruelle-Oseledec (nonlinear) multiplicative ergodic theory and anticipating stochastic calculus.

Formulation of the Theorem

Stratonovich SDE on $\mathbf{R}^d$

$$dx(t) = h(x(t))\,dt + \sum_{i=1}^{m} g_i(x(t)) \circ dW_i(t), \qquad (I)$$

driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$.

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}}, P) :=$ canonical filtered Wiener space.

$\Omega :=$ space of all continuous paths $\omega : \mathbf{R} \to \mathbf{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbf{R}^m$, with compact open topology;

$\mathcal{F} :=$ Borel σ-field of Ω;

$\mathcal{F}_t :=$ sub-σ-field of $\mathcal{F}$ generated by the evaluations $\omega \to \omega(u)$, $u \le t$, $\quad t \in \mathbf{R}$.

$P :=$ Wiener measure on Ω.

$h : \mathbf{R}^d \to \mathbf{R}^d, 1 \leq i \leq m,\ C_b^{k,\delta}$ vector fields on $\mathbf{R}^d$; viz. h has all derivatives $D^j h, 1 \leq j \leq k$, continuous and globally bounded, $D^k h$ Hölder continuous with exponent $\delta \in (0, 1)$.

$g_i,\ 1 \leq i \leq m$, globally bounded and $C_b^{k+1,\delta}$.

$\theta : \mathbf{R} \times \Omega \to \Omega$ is the (ergodic) Brownian shift

$$\theta(t,\omega)(s) := \omega(t + s) - \omega(t), \qquad t, s \in \mathbf{R},\ \omega \in \Omega.$$

Let $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ be the stochastic flow generated by (I) $\left(\phi(t, \cdot, \omega) = [\phi(-t, \cdot, \theta(t,\omega))]^{-1},\right.$
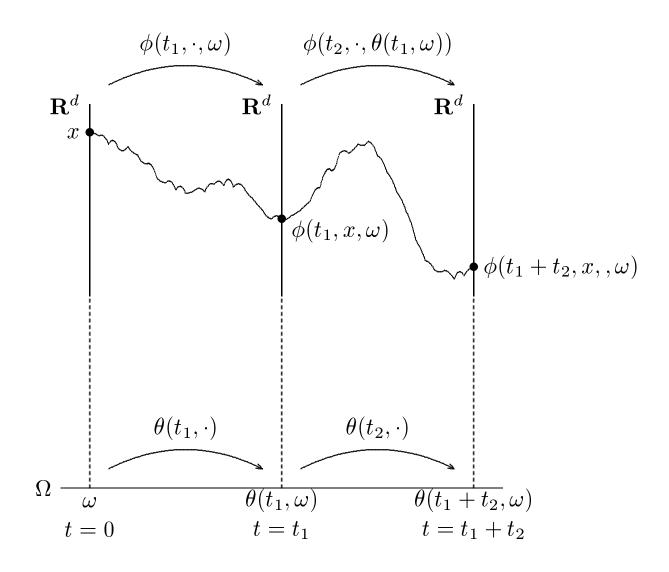
$t < 0$). Then ϕ is a perfect *cocycle:*

$$\phi(t_1 + t_2, \cdot, \omega) = \phi(t_2, \cdot, \theta(t_1, \omega)) \circ \phi(t_1, \cdot, \omega),$$

for all $t_1, t_2 \in \mathbf{R}$ and all $\omega \in \Omega$ ([I-W], [A-S], [A]).

Figure illustrates the cocycle property. Vertical solid lines represent random fibers consisting of copies of $\mathbf{R}^d$. (ϕ, θ) is a "random vector-bundle morphism" over the "base" probability space Ω.

The Cocycle

Definition

The SDE (I) has a *stationary trajectory* if there exists an $\mathcal{F}$-measurable random variable $Y : \Omega \to \mathbf{R}^d$ such that

$$\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{1}$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $\phi(t, Y) = Y(\theta(t))$.

Examples of Stationary Solutions

1. Fixed points:

$$d\phi(t) = h(\phi(t))\, dt + \sum_{i=1}^{m} g_i(\phi(t)) \circ dW_i(t)$$

$$h(x_0) = g_i(x_0) = 0, \qquad 1 \leq i \leq m$$

Take $Y(\omega) = x_0$ for all $\omega \in \Omega$.

2. Linear affine case $d = 1$:

$$d\phi(t) = \lambda\phi(t)\, dt + dW(t)$$

$\lambda > 0$ fixed, $W(t) \in \mathbf{R}$. Take

$$\phi(t, x, \omega) = e^{\lambda t}\left[x + \int_0^t e^{-\lambda u}\, dW(u) \right],$$

$$Y(\omega) := -\int_0^\infty e^{-\lambda u}\, dW(u),$$

$$\theta(t, \omega)(s) = \omega(t + s) - \omega(t).$$

Check that $\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega))$, using integration by parts and variation of parameters.

3. Affine linear SDE in $d = 2$:

$$d\phi(t) = A\phi(t)\,dt + G dW(t)$$

with A a fixed hyperbolic 2×2-diagonal matrix; G a constant 2×2-matrix, and W 2-dimensional Brownian motion.

4. Non-linear transforms of (3) under a global diffeomorphism.

5. Invariant measure for SDE: Enlarge probability space ([M-S.3]).

Let $\phi(t, Y)$ be a stationary solution of (I). Cocycle property of ϕ implies that the linearization

$$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$$

along the stationary solution is also a $d \times d$-matrix-valued cocycle. Using Kolmogorov's theorem, the random variables

$$\sup_{x \in \mathbf{R}^d} \frac{\|D_2\phi(t, x)\|}{(1 + |x|^\gamma)}, \ \gamma > 0,$$

have moments of all orders. If $E \log^+ |Y| < \infty$, then $E \log^+ \|D_2\phi(1, Y)\| < \infty$. Apply Oseledec's Theorem to get a *non-random* finite Lyapunov spectrum:

$$\lim_{n \to \infty} \frac{1}{n} \log |D_2\phi(n, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbf{R}^d).$$

Spectrum takes finitely many *fixed* values $\{\lambda_i\}_{i=1}^p$ with non-random multiplicities q_i, $1 \leq i \leq p$, and $\sum_{i=1}^p q_i = d$ ([Ru.1], Theorem I.6).

Definition

Stationary trajectory $\phi(t, Y)$ of (I) is *hyperbolic* if $E \log^+ |Y(\cdot)| < \infty$, and if the linearized cocycle $(D_2\phi(n, Y(\omega), \omega), \theta(n, \omega))$ has a non-vanishing Lyapunov spectrum

$$\{\lambda_p < \cdots < \lambda_{i_0+1} < \lambda_{i_0} < 0 < \lambda_{i_0-1} < \cdots < \lambda_2 < \lambda_1\}$$

i.e. $\lambda_i \neq 0$ for all $1 \leq i \leq p$.

Define $\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all $\lambda_i > 0$, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest

positive Lyapunov exponent of the linearized flow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Let $\rho \in \mathbf{R}^+$, $x \in \mathbf{R}^d$.

$B(x, \rho) :=$ open ball in $\mathbf{R}^d$, center x and radius ρ;

$\bar{B}(x, \rho) :=$ corresponding closed ball;

$\mathcal{K}(\mathbf{R}^d) :=$ the class of all non-empty compact subsets of $\mathbf{R}^d$ with Hausdorff metric d^*:

$d^*(A_1, A_2) := \sup\{d(x, A_1) : x \in A_2\} \vee \sup\{d(y, A_2) : y \in A_1\}$ where $A_1, A_2 \in \mathcal{K}(\mathbf{R}^d)$;

$d(x, A_i) := \inf\{|x - y| : y \in A_i\}$, $x \in \mathbf{R}^d$, $i = 1, 2$;

$\mathcal{B}(\mathcal{K}(\mathbf{R}^d)) :=$ Borel σ-algebra on $\mathcal{K}(\mathbf{R}^d)$ with respect to the metric d^*.

Theorem 1 (The Stable Manifold Theorem) (M.+ Scheutzow, AOP '99)

Assume that the coefficients of SDE (I) satisfy the given hypotheses. Suppose $\phi(t, Y)$ is a hyperbolic stationary trajectory of (I) with $E \log^+ |Y| < \infty$.

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

(i) *a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,*

(ii) *$\mathcal{F}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:*

There are $C^{k,\epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) $\tilde{\mathcal{S}}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$|\phi(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) \, e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \to \infty} \frac{1}{t} \log |\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0} \qquad (2)$$

for all $x \in \tilde{\mathcal{S}}(\omega)$. Each stable subspace $\mathcal{S}(\omega)$ of the linearized flow $D_2\phi$ is tangent at $Y(\omega)$ to the submanifold $\tilde{\mathcal{S}}(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. In particular, $\dim \tilde{\mathcal{S}}(\omega) = \dim \mathcal{S}(\omega)$ and is non-random.

(b) $\displaystyle \limsup_{t \to \infty} \frac{1}{t} \log \left[\sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{\mathcal{S}}(\omega)}} \left\{ \frac{|\phi(t, x_1, \omega) - \phi(t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq$

λ_{i_0}.

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ such that

$$\phi(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega)), \quad t \geq \tau_1(\omega). \qquad (3)$$

Also

$$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t, \omega)), \quad t \geq 0. \qquad (4)$$

(d) $\tilde{\mathcal{U}}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_2(\omega))$ with the property that

$$|\phi(-n, x, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega)\, e^{(-\lambda_{i_0 - 1} + \epsilon_2)n}$$

$$(5)$$

for all integers $n \geq 0$. Also

$$\limsup_{t \to \infty} \frac{1}{t} \log |\phi(-t, x, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0 - 1}.$$

$$(6)$$

for all $x \in \tilde{\mathcal{U}}(\omega)$. Furthermore, the unstable subspace $\mathcal{U}(\omega)$ of $D_2\phi$ is the tangent space to $\tilde{\mathcal{U}}(\omega)$ at $Y(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. In particular, $\dim \tilde{\mathcal{U}}(\omega) = \dim \mathcal{U}(\omega)$ and is non-random.

(e) $\displaystyle \limsup_{t\to\infty} \frac{1}{t}\log\left[\sup_{\substack{x_1\neq x_2 \\ x_1,x_2\in\tilde{\mathcal{U}}(\omega)}}\left\{\frac{|\phi(-t,x_1,\omega)-\phi(-t,x_2,\omega)|}{|x_1-x_2|}\right\}\right]\leq$

$-\lambda_{i_0-1}.$

(f) *(Cocycle-invariance of the unstable manifolds):*

There exists $\tau_2(\omega)\geq 0$ such that

$$\phi(-t,\cdot,\omega)(\tilde{\mathcal{U}}(\omega))\subseteq\tilde{\mathcal{U}}(\theta(-t,\omega)),\quad t\geq\tau_2(\omega).\qquad(7)$$

Also

$$D_2\phi(-t,Y(\omega),\omega)(\mathcal{U}(\omega))=\mathcal{U}(\theta(-t,\omega)),\quad t\geq 0.\qquad(8)$$

(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ and $\tilde{\mathcal{S}}(\omega)$ are transversal, viz.

$$\mathbf{R}^d=T_{Y(\omega)}\tilde{\mathcal{U}}(\omega)\oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).\qquad(9)$$
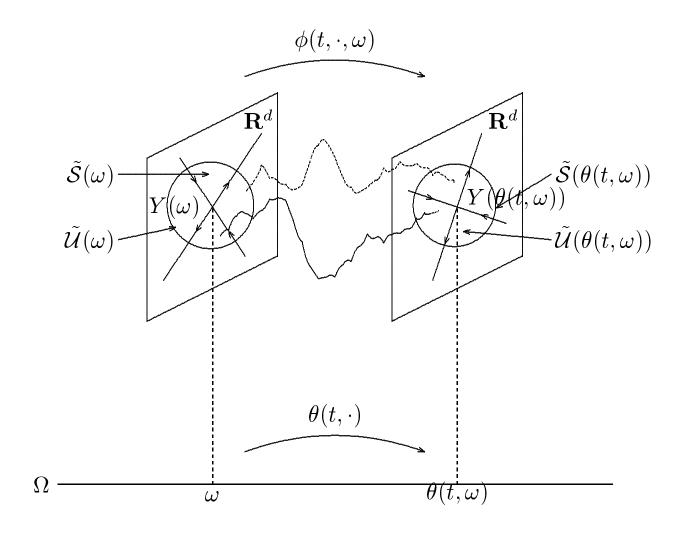
(h) The mappings

$$\Omega\to\mathcal{K}(\mathbf{R}^d),\qquad \Omega\to\mathcal{K}(\mathbf{R}^d),$$

$$\omega\mapsto\tilde{\mathcal{S}}(\omega)\qquad\qquad\omega\mapsto\tilde{\mathcal{U}}(\omega)$$

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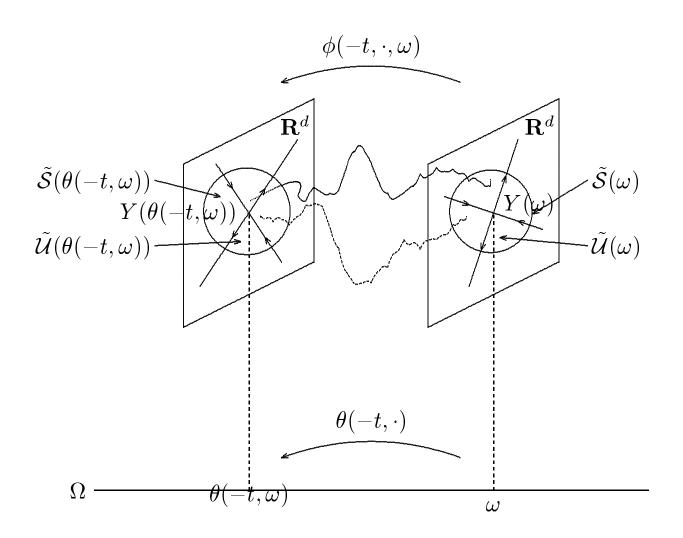
are $(\mathcal{F}, \mathcal{B}(\mathcal{K}(\mathbf{R}^d)))$-measurable.

Assume, further, that h, g_i, $1 \leq i \leq m$, are C_b^∞. Then the local stable and unstable manifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞.

$$\phi(t,\cdot,\omega)$$

$\mathbf{R}^d$

$\mathbf{R}^d$

$\tilde{\mathcal{S}}(\omega)$

$Y(\omega)$

$\tilde{\mathcal{U}}(\omega)$

$\tilde{\mathcal{S}}(\theta(t,\omega))$

$Y(\theta(t,\omega))$

$\tilde{\mathcal{U}}(\theta(t,\omega))$

$$\theta(t,\cdot)$$

Ω

ω

$\theta(t,\omega)$

$$t > \tau_1(\omega)$$

A picture is worth a 1000 words!

$$\phi(-t, \cdot, \omega)$$

$\mathbf{R}^d$ $\mathbf{R}^d$

$\tilde{\mathcal{S}}(\theta(-t, \omega))$ $\tilde{\mathcal{S}}(\omega)$

$Y(\theta(-t, \omega))$ $Y(\omega)$

$\tilde{\mathcal{U}}(\theta(-t, \omega))$ $\tilde{\mathcal{U}}(\omega)$

$$\theta(-t, \cdot)$$

Ω $\theta(-t, \omega)$ ω

$$t > \tau_2(\omega)$$

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Sketch of Proof

Linearization and Substitution

Assume regularity conditions on the coefficients h, g_i. By the *Substitution Rule*, $\phi(t, Y(\omega), \omega)$ is a stationary *solution* of the *anticipating* Stratonovich SDE

$$\left.\begin{aligned}
d\phi(t, Y) &= h(\phi(t, Y))\, dt + \sum_{i=1}^{m} g_i(\phi(t, Y)) \circ dW_i(t), \quad t > 0 \\
\phi(0, Y) &= Y.
\end{aligned}\right\}$$

$$(II)$$

([N-P]).

Linearize the SDE (I) along the stationary trajectory. By substitution, match the solution of the linearized equation

with the linearized cocycle $D_2\phi(t, Y(\omega), \omega)$.
Hence $D_2\phi(t, Y(\omega), \omega)$, $t \geq 0$, solves the SDE:

$$
\left.
\begin{aligned}
dD_2\phi(t, Y) &= Dh(\phi(t, Y))D_2\phi(t, Y)\, dt \\
&\quad + \sum_{i=1}^{m} Dg_i(\phi(t, Y))D_2\phi(t, Y) \circ dW_i(t), \quad t > 0 \\
D_2\phi(0, Y) &= I.
\end{aligned}
\right\}
$$

$$(III)$$

D_2, D denotes spatial (Fréchet) derivatives.

Similarly, the backward trajectories

$$\phi(t, Y), \ D_2\phi(t, Y), \ \ t < 0,$$

solve the corresponding backward Stratonovich SDE's:

$$
\left.
\begin{aligned}
d\phi(t, Y) &= -h(\phi(t, Y))\, dt - \sum_{i=1}^{m} g_i(\phi(t, Y)) \circ \hat{d}W_i(t), \quad t < 0 \\
\phi(0, Y) &= Y.
\end{aligned}
\right\}
$$

$$(II^-)$$

$$dD_2\phi(t,Y) = -Dh(\phi(t,Y))D_2\phi(t,Y)\,dt$$
$$\left. \begin{array}{l} \\ \qquad\qquad - \sum_{i=1}^{m} Dg_i(\phi(t,Y))D_2\phi(t,Y) \circ \hat{d}W_i(t), \quad t < 0 \\ \\ D_2\phi(0,Y) = I. \end{array} \right\}$$

$$(III^-)$$

Above SDE's (II)-(III)$^-$ give dynamic characterizations of the stable and unstable manifolds.

The following lemma is used to construct the shift-invariant sure event appearing in the statement of the local stable manifold theorem. Gives "perfect versions" of the ergodic theorem and Kingman's subadditive ergodic theorem.

Lemma 1

(i) Let $h : \Omega \to \mathbf{R}^+$ be $\mathcal{F}$-measurable and such that

$$\int_\Omega \sup_{0 \leq u \leq 1} h(\theta(u, \omega)) \, dP(\omega) < \infty.$$

Then there is a sure event $\Omega_1 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_1) = \Omega_1$ for all $t \in \mathbf{R}$, and

$$\lim_{t \to \infty} \frac{1}{t} h(\theta(t, \omega)) = 0$$

for **all** $\omega \in \Omega_1$.

(ii) Suppose $f : \mathbf{R}^+ \times \Omega \to \mathbf{R} \cup \{-\infty\}$ is a measurable process on $(\Omega, \mathcal{F}, P)$ satisfying the following conditions

(a) $E \sup_{0 \leq u \leq 1} f^+(u) < \infty, \quad E \sup_{0 \leq u \leq 1} f^+(1 - u, \theta(u)) < \infty$

(b) $f(t_1 + t_2, \omega) \leq f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ for all $t_1, t_2 \geq 0$ and **all** $\omega \in \Omega$.

Then there is sure event $\Omega_2 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_2) = \Omega_2$ for all $t \in \mathbf{R}$, and a fixed number $f^* \in \mathbf{R} \cup \{-\infty\}$ such that

$$\lim_{t \to \infty} \frac{1}{t} f(t, \omega) = f^*$$

*for **all** $\omega \in \Omega_2$.*

Proof

[Mo.1], Lemma 7. $\quad\square$

Theorem 2 ([O], 1968)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\theta : \mathbf{R}^+ \times \Omega \to \Omega$ a measurable family of ergodic P-preserving transformations. Let $T : \mathbf{R}^+ \times \Omega \to L(\mathbf{R}^d)$ be measurable, such that (T, θ) is an $L(\mathbf{R}^d)$-valued cocycle. Suppose that

$$E \sup_{0 \le t \le 1} \log^+ \|T(t, \cdot)\| < \infty, \quad E \sup_{0 \le t \le 1} \log^+ \|T(1-t, \theta(t, \cdot))\| < \infty.$$

Then there is a set $\Omega_0 \in \mathcal{F}$ of full P-measure such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \in \mathbf{R}^+$, and for each $\omega \in \Omega_0$, the limit

$$\lim_{t \to \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. Each $\Lambda(\omega)$ has a discrete non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_p}$$

where the λ_i's are distinct. Each e^{λ_i} has an eigen-space $F_i(\omega)$ and a fixed non-random multiplicity $m_i := \dim F_i(\omega)$. Define

$$E_1(\omega) := \mathbf{R}^d, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^\perp, \ 1 < i \le p.$$

Then

$$E_p(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = \mathbf{R}^d$$

$$\lim_{t \to \infty} \frac{1}{t} \log \|T(t,\omega)x\| = \lambda_i(\omega), \quad \text{if} \quad x \in E_i(\omega) \backslash E_{i+1}(\omega),$$
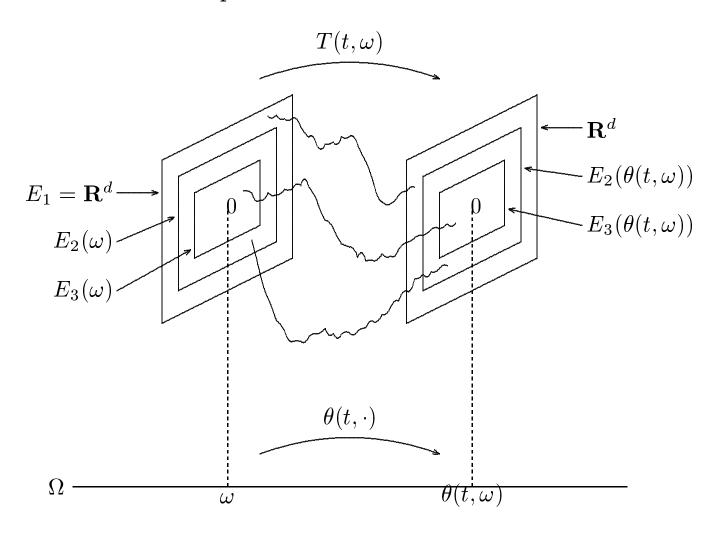
and

$$T(t,\omega)(E_i(\omega)) \subseteq E_i(\theta(t,\omega))$$

for all $t \ge 0$, $1 \le i \le p$.

Proof.

Based on the discrete version of Os-eledec's multiplicative ergodic theorem

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and Lemma 1. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]), "perfect" infinite-dimensional version and application to SFDE's. □

Spectral Theorem

$$T(t, \omega)$$

$$E_1 = \mathbf{R}^d$$

$$E_2(\omega)$$

$$E_3(\omega)$$

$$\mathbf{R}^d$$

$$E_2(\theta(t, \omega))$$

$$E_3(\theta(t, \omega))$$

$$0 \qquad\qquad 0$$

$$\theta(t, \cdot)$$

$$\Omega \qquad\qquad \omega \qquad\qquad \theta(t, \omega)$$
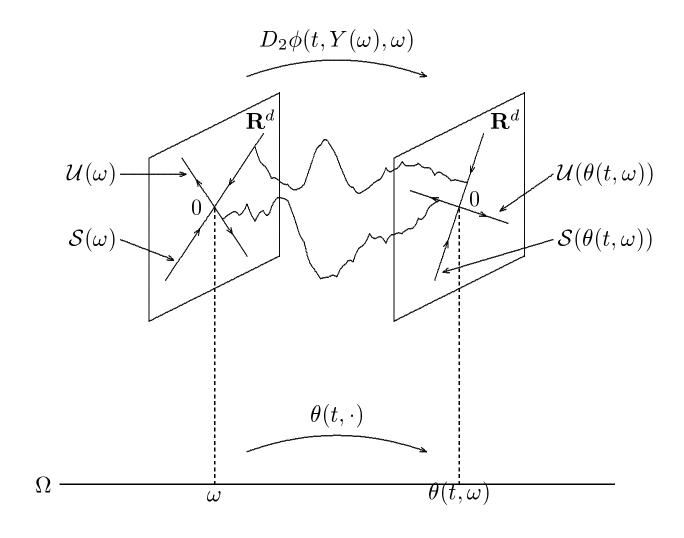
Apply Theorem 2 with

$$T(t, \omega) := D_2 \phi(t, Y(\omega), \omega)$$

Then linearized cocycle has random invariant stable and unstable subspaces $\{S(\omega), U(\omega) : \omega \in \Omega\}$:

$$D_2 \phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) = \mathcal{S}(\theta(t, \omega)),$$

$$D_2 \phi(-t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(-t, \omega)), \quad t \geq 0.$$

[Mo.1].

Estimates on the non-linear cocycle

Theorem 3 (M. + Scheutzow [M-S.2])

There exists a jointly measurable modification of the trajectory random field of (I) (with initial condition x at $t = s$), denoted by $\{\phi_{s,t}(x) : -\infty < s, t < \infty, \ x \in \mathbf{R}^d\}$, having the following properties:

The cocycle $\phi : \mathbf{R} \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ is given by

$$\phi(t, x, \omega) := \phi_{0,t}(x, \omega), \quad x \in \mathbf{R}^d, \omega \in \Omega, t \in \mathbf{R}.$$

Then for all $\omega \in \Omega$, $\epsilon \in (0, \delta), \gamma, \rho, T > 0, 1 \leq |\alpha| \leq k$, $\phi(t, \cdot, \omega)$ is $C^{k,\epsilon}, 0 < \epsilon < \delta$, and the quantities

$$\sup_{\substack{0 \leq s,t \leq T, \\ x \in \mathbf{R}^d}} \frac{|\phi_{s,t}(x, \omega)|}{[1 + |x|(\log^+ |x|)^\gamma]}, \qquad \sup_{\substack{0 \leq s,t \leq T, \\ x \in \mathbf{R}^d}} \frac{\|D_x^\alpha \phi_{s,t}(x, \omega)\|}{(1 + |x|^\gamma)},$$

$$\sup_{x \in \mathbf{R}^d} \sup_{\substack{0 \leq s,t \leq T, \\ 0 < |x' - x| \leq \rho}} \frac{\|D_x^\alpha \phi_{s,t}(x, \omega) - D_x^\alpha \phi_{s,t}(x', \omega)\|}{|x - x'|^\epsilon (1 + |x|)^\gamma},$$

are finite. The random variables defined by the above expressions have p-th moments for all $p \geq 1$.

Ruelle's Non-linear Ergodic Theorem

Theorem 4 ([Ru.1], 1979)

Let $\Omega \ni \mapsto F_\omega \in C^{k,\epsilon}(\mathbf{R}^d, 0; \mathbf{R}^d, 0)$ be measurable such that $E \log^+ \|F. |\bar{B}(0,1)\|_{k,\epsilon} < \infty$. Set $F^n(\omega) := F_{\theta(n-1,\omega)} \circ \cdots \circ F_{\theta(1,\omega)} \circ F_\omega$. Suppose $\lambda < 0$ is not in the spectrum of the cocycle $(DF_\omega^n(0), \theta(n,\omega))$. Then there is a sure event $\Omega_0 \in \mathcal{F}$ such that $\theta(1,\cdot)(\Omega_0) \subseteq \Omega_0$, and measurable functions $0 < \alpha(\omega) < \beta(\omega) < 1, \gamma(\omega) > 1$ with the following properties:

(a) If $\omega \in \Omega_0$, the set

$$V_\omega^\lambda := \{x \in \bar{B}(0, \alpha(\omega)) : |F_\omega^n(x)| \le \beta(\omega)e^{n\lambda} \text{ for all } n \ge 0\}$$

is a $C^{k,\epsilon}$ submanifold of $\bar{B}(0, \alpha(\omega))$.

(b) If $x_1, x_2 \in V_\omega^\lambda$, then

$$|F_\omega^n(x_1) - F_\omega^n(x_2)| \le \gamma(\omega)|x_1 - x_2|e^{n\lambda}$$

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for all integers $n \geq 0$. If $\lambda' < \lambda$ and $[\lambda', \lambda]$ is disjoint

from the spectrum of $(DF_\omega^n(0), \theta(n, \omega))$, then there

exists a measurable $\gamma'(\omega) > 1$ such that

$$|F_\omega^n(x_1) - F_\omega^n(x_2)| \leq \gamma'(\omega)|x_1 - x_2|e^{n\lambda'}$$

for all $x_1, x_2 \in V_\omega^\lambda$ and all integers $n \geq 0$.

Proof

[Ru.1], Theorem 5.1, p. 292.

Construction of the Stable/Unstable Manifolds

- Use auxiliary cocycle (Z, θ):

$$Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega)) \quad (16)$$

for $t \in \mathbf{R}, x \in \mathbf{R}^d, \omega \in \Omega$. Set $\tau := \theta(1, \cdot) : \Omega \to \Omega$. Define maps $F_\omega, F_\omega^n : \mathbf{R}^d \to \mathbf{R}^d$:

$$F_\omega(x) := Z(1, x, \omega) \quad x \in \mathbf{R}^d$$

$$F_\omega^n := F_{\tau^{n-1}(\omega)} \circ \cdots \circ F_{\tau(\omega)} \circ F_\omega$$

for all $\omega \in \Omega$. Then cocycle property for Z gives $F_\omega^n = Z(n, \cdot, \omega)$ for each $n \geq 1$. F_ω is $C^{k,\epsilon}$ ($\epsilon \in (0, \delta)$) and $(DF_\omega)(0) = D_2\phi(1, Y(\omega), \omega)$.

- Integrability of the map

$$\omega \mapsto \log^+ \|D_2\phi(1, Y(\omega), \omega)\|_{L(\mathbf{R}^d)}$$

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(Lemma 2) implies discrete cocycle $((DF_\omega^n)(0), \theta(n,\omega), n \geq 0)$ has same non-random Lyapunov spectrum as that of linearized continuous cocycle

$$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0),$$

viz. $\{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, where each λ_i has fixed multiplicity q_i, $1 \leq i \leq m$ (Lemma 2).

- If $\lambda_i > 0$ for all $1 \leq i \leq m$, then take $\tilde{S}(\omega) := \{Y(\omega)\}$ for all $\omega \in \Omega$. Theorem is trivial in this case. Hence assume there is at least one $\lambda_i < 0$.

- Use discrete non-linear ergodic theorem of Ruelle (Theorem 4) and its proof to obtain a sure event $\Omega_1^* \in \mathcal{F}$

such that $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$, $\mathcal{F}$-measurable positive random variables $\rho_1, \beta_1 : \Omega_1^* \to (0, \infty)$, $\rho_1 < \beta_1$, and a random family of $C^{k,\epsilon}$ $\left(\epsilon \in (0, \delta)\right)$ submanifolds of $\bar{B}(0, \rho_1(\omega))$ denoted by $\tilde{\mathcal{S}}_d(\omega)$, $\omega \in \Omega_1^*$, and satisfying the following properties for each $\omega \in \Omega_1^*$: $\tilde{\mathcal{S}}_d(\omega)$ is the set of all $x \in \bar{B}(0, \rho_1(\omega))$ such that

$$|Z(n, x, \omega)| \le \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}, \qquad n \in \mathbf{Z}^+ \qquad (21)$$

$\tilde{\mathcal{S}}_d(\omega)$ is tangent at 0 to the stable subspace $\mathcal{S}(\omega)$ of the linearized flow $D_2\phi$, viz. $T_0 \tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$. Therefore dim $\tilde{\mathcal{S}}_d(\omega)$ is non-random by ergodicity of θ. Also

$$\limsup_{n \to \infty} \frac{1}{n} \log \left[\sup_{\substack{x_1 \ne x_2, \\ x_1, x_2 \in \tilde{\mathcal{S}}_d(\omega)}} \frac{|Z(n, x_1, \omega) - Z(n, x_2, \omega)|}{|x_1 - x_2|} \right] \le \lambda_{i_0}.$$

$$(22)$$

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The $\theta(t, \cdot)$-invariant sure event $\Omega_1^* \in \mathcal{F}$ is constructed using the ideas in Ruelle's proof (of Theorem 5.1 in [Ru.1], p. 293), combined with the estimate (10) of Lemma 2 and the subadditive ergodic theorem (Lemma 1 (ii)).

- For each $\omega \in \Omega_1^*$, let $\tilde{\mathcal{S}}(\omega)$ be as defined in part (a) of the theorem. Then by definition of $\tilde{\mathcal{S}}_d(\omega)$ and z:

$$\tilde{\mathcal{S}}(\omega) = \tilde{\mathcal{S}}_d(\omega) + Y(\omega). \tag{23}$$

Since $\tilde{\mathcal{S}}_d(\omega)$ is a $C^{k,\epsilon} \left(\epsilon \in (0, \delta)\right)$ submanifold of $\bar{B}(0, \rho_1(\omega))$, then $\tilde{\mathcal{S}}(\omega)$ is a $C^{k,\epsilon}$ $\left(\epsilon \in (0, \delta)\right)$ submanifold of $\bar{B}(Y(\omega), \rho_1(\omega))$. Furthermore, $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = T_0\tilde{\mathcal{S}}_d(\omega) = \mathcal{S}(\omega)$.

Hence $\dim \tilde{\mathcal{S}}(\omega) = \dim \mathcal{S}(\omega) = \sum\limits_{i=i_0}^{m} q_i$, and is non-random.

- (22) implies that

$$\limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)| \leq \lambda_{i_0} \qquad (24)$$

for all ω in Ω_1^* and all $x \in \tilde{\mathcal{S}}_d(\omega)$. Lemma 4 implies there is a sure event $\Omega_2^* \subseteq \Omega_1^*$ such that $\theta(t, \cdot)(\Omega_2^*) = \Omega_2^*$ for all $t \in \mathbf{R}$, and

$$\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| \leq \lambda_{i_0} \qquad (25)$$

for all $\omega \in \Omega_2^*$ and all $x \in \tilde{\mathcal{S}}_d(\omega)$. Therefore (2) holds.

- To prove (b), let $\omega \in \Omega_1^*$. By (22), there is a positive integer $N_0 := N_0(\omega)$ (independent of $x \in \tilde{\mathcal{S}}_d(\omega)$) such that

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$Z(n, x, \omega) \in \bar{B}(0, 1)$ for all $n \geq N_0$. Let $\Omega_4^* :=$ $\Omega_2^* \cap \Omega_3$, where Ω_3 is the shift-invariant sure event defined in the proof of Lemma 4. Then Ω_4^* is a sure event and $\theta(t, \cdot)(\Omega_4^*) = \Omega_4^*$ for all $t \in \mathbf{R}$. By cocycle property, Mean-Value theorem and the ergodic theorem (Lemma 1(i)), we get (b).

- To prove the invariance property (4), apply the Oseledec theorem to $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$. Get a sure $\theta(t, \cdot)$-invariant event, also denoted by Ω_1^*, such that

$D_2\phi(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$. Equality holds because $D_2\phi(t, Y(\omega), \omega)$ is injective and $\dim \mathcal{S}(\omega) = \dim \mathcal{S}(\theta(t, \omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$.

- To prove the asymptotic invariance property (3), use ideas from Ruelle's Theorems 5.1 and 4.1 in [Ru.1], to pick random variables ρ_1, β_1 and a sure event (also denoted by) Ω_1^* such that $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbf{R}$, and for any $\epsilon \in (0, \epsilon_1)$ and every $\omega \in \Omega_1^*$, there exists a positive $K_1^\epsilon(\omega)$ for which the inequalities

$$
\begin{aligned}
\rho_1(\theta(t, \omega)) &\geq K_1^\epsilon(\omega) \rho_1(\omega) e^{(\lambda_{i_0} + \epsilon)t}, \\
\beta_1(\theta(t, \omega)) &\geq K_1^\epsilon(\omega) \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon)t}
\end{aligned}
\tag{26}
$$

hold for all $t \geq 0$. Use (b) to obtain a sure event $\Omega_5^* \subseteq \Omega_4^*$ such that $\theta(t, \cdot)(\Omega_5^*) = \Omega_5^*$ for all $t \in \mathbf{R}$, and for any $0 < \epsilon < \epsilon_1$

and $\omega \in \Omega_4^*$, there exists $\beta^\epsilon(\omega) > 0$ (inde-pendent of x) with

$$|\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t} \qquad (27)$$

for all $x \in \tilde{\mathcal{S}}(\omega)$, $t \geq 0$. Fix $t \geq 0$, $\omega \in \Omega_5^*$ and $x \in \tilde{\mathcal{S}}(\omega)$. Let n be a non-negative integer. Then the cocycle property and (27) imply that

$$|\phi(n, \phi(t, x, \omega), \theta(t, \omega)) - Y(\theta(n, \theta(t, \omega)))|$$
$$= |\phi(n + t, x, \omega) - Y(\theta(n + t, \omega))|$$
$$\leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)(n+t)}$$
$$\leq \beta^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t} e^{(\lambda_{i_0} + \epsilon_1)n}.$$
$$(28)$$

If $\omega \in \Omega_5^*$, then it follows from (26),(27), (28) and the definition of $\tilde{\mathcal{S}}(\theta(t, \omega))$ that

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there exists $\tau_1(\omega) > 0$ such that $\phi(t, x, \omega) \in \tilde{\mathcal{S}}(\theta(t, \omega))$ for all $t \geq \tau_1(\omega)$. This proves asymptotic invariance.

- Prove (d), the existence of the local unstable manifolds $\tilde{\mathcal{U}}(\omega)$, by running both the flow ϕ and the shift θ backward in time getting the cocycle $(\tilde{Z}(t, \cdot, \omega), \tilde{\theta}(t, \omega), t \geq 0)$:

$$\tilde{\phi}(t, x, \omega) := \phi(-t, x, \omega), \ \tilde{Z}(t, x, \omega) := Z(-t, x, \omega),$$

$$\tilde{\theta}(t, \omega) := \theta(-t, \omega)$$

for all $t \geq 0, \omega \in \Omega$. The linearized flow $(D_2\tilde{\phi}(t, Y(\omega), \omega), \tilde{\theta}(t, \omega), t \geq 0)$ is an $L(\mathbf{R}^d)$-valued perfect cocycle with a non-random finite Lyapunov spectrum $\{-\lambda_1 < -\lambda_2 < \cdots < -\lambda_i < -\lambda_{i+1} < \cdots < -\lambda_m\}$ where $\{\lambda_m <$

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$\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ is the Lyapunov spectrum of the forward linearized flow $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$. Apply first part of the proof to get *stable manifolds* for the backward flow $\tilde{\phi}$ satisfying assertions (a), (b), (c). This gives *unstable manifolds* for the original flow ϕ, and (d), (e), (f) automatically hold.

- Measurability of the stable manifolds follows from the representations:

$$\tilde{\mathcal{S}}(\omega) = Y(\omega) + \tilde{\mathcal{S}}_d(\omega) \tag{29}$$

$$\tilde{\mathcal{S}}_d(\omega) = \lim_{n \to \infty} \bar{B}(0, \rho_1(\omega)) \cap \bigcap_{i=1}^{n} f_i(\cdot, \omega)^{-1}(\bar{B}(0, 1)) \tag{30}$$

$$f_i(x, \omega) := \beta_1(\omega)^{-1} e^{-(\lambda_{i_0} + \epsilon_1)i} Z(i, x, \omega), \ x \in \mathbf{R}^d, \ \omega \in \Omega_1^*,$$

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for all integers $i \geq 0$. (Above limit is taken in the metric d^* on $\mathcal{K}(\mathbf{R}^d)$.) Use joint continuity of translation and measurability of Y, f_i, ρ_1, finite intersections and the continuity of the maps

$$\mathbf{R}^+ \ni r \mapsto \bar{B}(0, r) \in \mathcal{K}(\mathbf{R}^d).$$

$$\mathrm{Hom}(\mathbf{R}^d) \ni f \mapsto f^{-1}(\bar{B}(0, 1)) \in \mathcal{K}(\mathbf{R}^d).$$

- For h, g_i in C_b^∞, can adapt above argument to give a sure event in $\mathcal{F}$, also denoted by Ω^* such that $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞ for all $\omega \in \Omega^*$. $\quad \square$

Some Technical Lemmas

$\|\cdot\|_{k,\epsilon} := C^{k,\epsilon}$-norm on $C^{k,\epsilon}$ mappings $\bar{B}(0,\rho) \to$ $\mathbf{R}^d$.

Lemma 2

Assume that $\log^+ |Y(\cdot)|$ is integrable. Then the cocycle ϕ satisfies

$$\int_\Omega \log^+ \sup_{-T \le t_1, t_2 \le T} \|\phi(t_2, Y(\theta(t_1,\omega))+(\cdot), \theta(t_1,\omega))\|_{k,\epsilon} \, dP(\omega) < \infty$$

$$(10)$$

for any fixed $0 < T, \rho < \infty$ and any $\epsilon \in (0,\delta)$. Furthermore, the linearized flow $(D_2\phi(t, Y(\omega), \omega), \theta(t,\omega))$, $t \ge 0$, is an $L(\mathbf{R}^d)$-valued perfect cocycle and

$$\int_\Omega \log^+ \sup_{-T \le t_1, t_2 \le T} \|D_2\phi(t_2, Y(\theta(t_1,\omega)), \theta(t_1,\omega))\|_{L(\mathbf{R}^d)} \, dP(\omega) < \infty$$

$$(11)$$

for any fixed $0 < T < \infty$. The forward cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t > 0)$ has a non-random finite Lyapunov spectrum $\{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. Each Lyapunov exponent λ_i has a non-random multiplicity q_i, $1 \le i \le m$, and $\sum_{i=1}^{m} q_i = d$. The backward linearized cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t < 0)$, admits a "backward" non-random finite Lyapunov spectrum:

$$\lim_{t \to -\infty} \frac{1}{t} \log |D_2\phi(t, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbf{R}^d),$$

taking values in $\{-\lambda_i\}_{i=1}^{m}$ with non-random multiplicities q_i, $1 \le i \le m$, and $\sum_{i=1}^{m} q_i = d$.

The Auxiliary Cocycle

To apply Ruelle's discrete non-linear ergodic theorem ([Ru.1], Theorem 5.1, p. 292), introduce the following auxiliary cocycle $Z : \mathbf{R} \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$. This a "centering" of the flow ϕ about the stationary solution:

$$Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega)) \qquad (16)$$

for $t \in \mathbf{R}, x \in \mathbf{R}^d, \omega \in \Omega$.

Lemma 3

(Z, θ) is a perfect cocycle on $\mathbf{R}^d$ and $Z(t, 0, \omega) = 0$ for all $t \in \mathbf{R}$, and all $\omega \in \Omega$.

The proof of the local stable-manifold theorem (Theorem 1) uses a discretization argument that requires the following lemma.

Lemma 4

Suppose that $\log^+ |Y(\cdot)|$ is integrable. Then there is a sure event $\Omega_3 \in \mathcal{F}$ with the following properties:

(i) $\theta(t, \cdot)(\Omega_3) = \Omega_3$ for all $t \in \mathbf{R}$,

(ii) For every $\omega \in \Omega_3$ and any $x \in \mathbf{R}^d$, the statement

$$\limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)| < 0 \qquad (17)$$

implies

$$\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| = \limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)|.$$
$$(18)$$

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DYNAMICS

OF

STOCHASTIC SYSTEMS

WITH MEMORY

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Deterministic ODE's: Stable Manifolds

ODE on $\mathbf{R}^d$:

$$dx(t) = h(x(t))\, dt \qquad\qquad (ODE)$$

driven by a vector field $h : \mathbf{R}^d \to \mathbf{R}^d$, C_b^k; viz. all derivatives $D^j h, 1 \le j \le k$, continuous and globally bounded.

Assume *hyperbolic equilibrium* at 0: $h(0) = 0$; $Dh(0) \in L(\mathbf{R}^d)$ has all eigenvalues off imaginary axis.

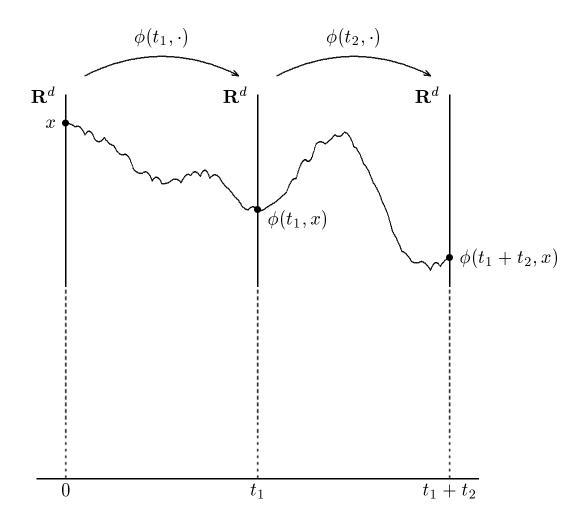Then (ODE) has a C_b^k flow $\phi : \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$ s.t.

(i) $\phi(\cdot, x) =$ unique solution of (ODE) through $x \in \mathbf{R}^d$.

(ii) $\phi(t, 0) = 0, t \in \mathbf{R}$.

(iii) Group property:

$$\phi(t_1 + t_2, \cdot) = \phi(t_2, \cdot) \circ \phi(t_1, \cdot), \qquad t_1, t_2 \in \mathbf{R}$$
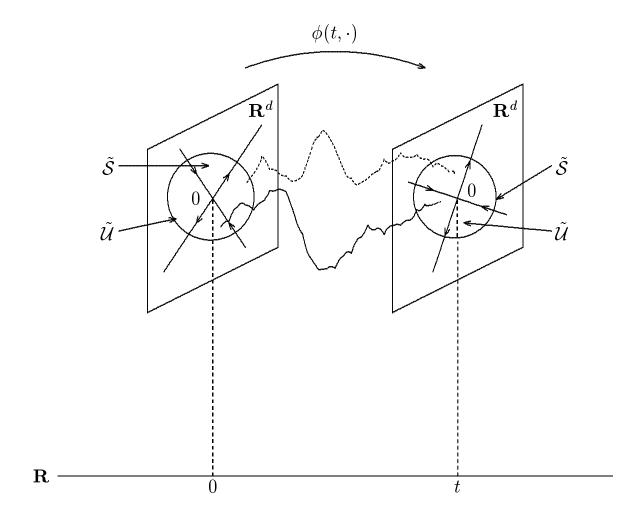
(iv) Local flow-invariant stable/unstable C^k manifolds in a neighborhood of 0.
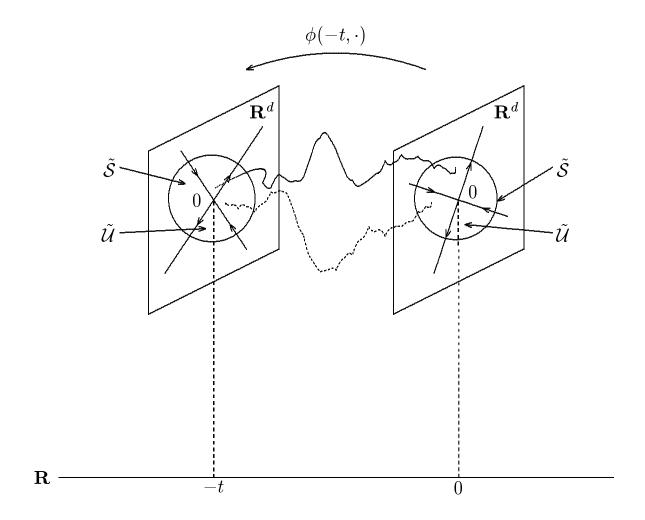
Properties (i)-(iv) are "generic" among all vector fields.

The Flow

$$\phi(t_1, \cdot) \qquad\qquad \phi(t_2, \cdot)$$

$\mathbf{R}^d \qquad\qquad \mathbf{R}^d \qquad\qquad \mathbf{R}^d$

x

$\phi(t_1, x)$

$\phi(t_1 + t_2, x)$

$0 \qquad\qquad t_1 \qquad\qquad t_1 + t_2$

Local Stable/Unstable Manifolds

$$\phi(t, \cdot)$$

$\mathbf{R}^d$

$\tilde{\mathcal{S}}$

$\tilde{\mathcal{U}}$

0

$\mathbf{R}^d$

$\tilde{\mathcal{S}}$

$\tilde{\mathcal{U}}$

0

$\mathbf{R}$

0

t

What happens

if vector field

is noisy??

Stable Manifolds

Outline

- Smooth cocycles in Hilbert space. Stationary trajectories.

- Linearization of a cocycle along a stationary trajectory.

- Ergodic theory of cocycles in Hilbert space.

- Hyperbolicity of stationary trajectories. Lyapunov exponents.

- Cocycles generated by stochastic systems with memory. Via random diffeomorphism groups.

- *Local Stable Manifold Theorem* for stochastic differential equations with memory (SFDE's): Existence of smooth stable and unstable manifolds in a neighborhood of a hyperbolic stationary trajectory.

- Proof: Ruelle-Oseledec multiplicative ergodic theory+ perfection techniques.

The Cocycle

$(\Omega, \mathcal{F}, P) :=$ complete probability space.

$\theta : \mathbf{R}^+ \times \Omega \to \Omega$ a P-preserving (ergodic) semigroup on $(\Omega, \mathcal{F}, P)$.
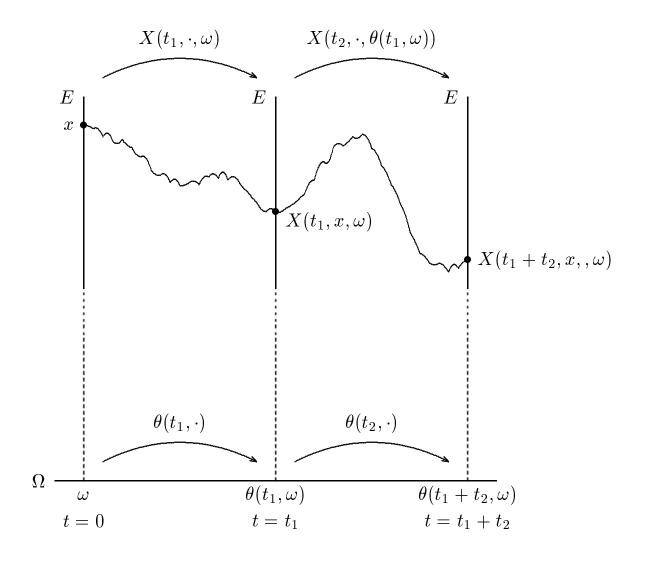
$E :=$ real (separable) Hilbert space, norm $\|\cdot\|$, Borel σ-algebra.

Definition.

$k =$ non-negative integer, $\epsilon \in (0, 1]$. A $C^{k,\epsilon}$ *perfect cocycle* (X, θ) on E is a measurable random field $X : \mathbf{R}^+ \times E \times \Omega \to E$ such that:

(i) For each $\omega \in \Omega$, the map $\mathbf{R}^+ \times E \ni (t, x) \mapsto X(t, x, \omega) \in E$ is continuous; for fixed $(t, \omega) \in \mathbf{R}^+ \times \Omega$, the map $E \ni x \mapsto X(t, x, \omega) \in E$ is $C^{k,\epsilon}$ $\left(D^k X(t, x, \omega) \text{ is } C^\epsilon \text{ in } x \right)$.

(ii) $X(t_1 + t_2, \cdot, \omega) = X(t_2, \cdot, \theta(t_1, \omega)) \circ X(t_1, \cdot, \omega)$ for all $t_1, t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.

(iii) $X(0, x, \omega) = x$ for all $x \in E, \omega \in \Omega$.

Cocycle Property

Vertical solid lines represent random fibers: copies of E. (X, θ) is a "vector-bundle morphism".

Definition

A random variable $Y : \Omega \to E$ is a *stationary point* for the cocycle (X, θ) if

$$X(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \qquad (1)$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $X(t, Y) = Y(\theta(t))$.

Linearization. Hyperbolicity.

Linearize a $C^{k,\epsilon}$ cocycle (X, θ) along a stationary random point Y: Get an $L(E)$-valued cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega))$. (Follows from cocycle property of X and chain rule.)

Theorem. *(Oseledec-Ruelle)*

Let $T : \mathbf{R}^+ \times \Omega \to L(E)$ be strongly measurable, such that (T, θ) is an $L(E)$-valued cocycle, with each $T(t, \omega)$ compact. Suppose that

$$E \sup_{0 \le t \le 1} \log^+ \|T(t, \cdot)\|_{L(E)} < \infty, \quad E \sup_{0 \le t \le 1} \log^+ \|T(1-t, \theta(t, \cdot))\|_{L(E)} < \infty.$$

Then there is a sure event $\Omega_0 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \in \mathbf{R}^+$, and for each $\omega \in \Omega_0$,

$$\lim_{t \to \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. $\Lambda(\omega)$ is self-adjoint with a non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots$$

where the λ_i's are distinct. Each e^{λ_i} has a fixed finite non-random multiplicity m_i and eigen-space $F_i(\omega)$, with $m_i := \dim F_i(\omega)$. Define

$$E_1(\omega) := E, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega) \right]^\perp, \ i > 1, \ E_\infty := \ker \Lambda(\omega).$$

Then

$$E_\infty \subset \cdots \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$\lim_{t \to \infty} \frac{1}{t} \log \|T(t, \omega)x\| = \begin{cases} \lambda_i & if \quad x \in E_i(\omega) \backslash E_{i+1}(\omega), \\ -\infty & if \quad x \in E_\infty(\omega), \end{cases}$$

and

$$T(t, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

for all $t \geq 0, \ i \geq 1.$
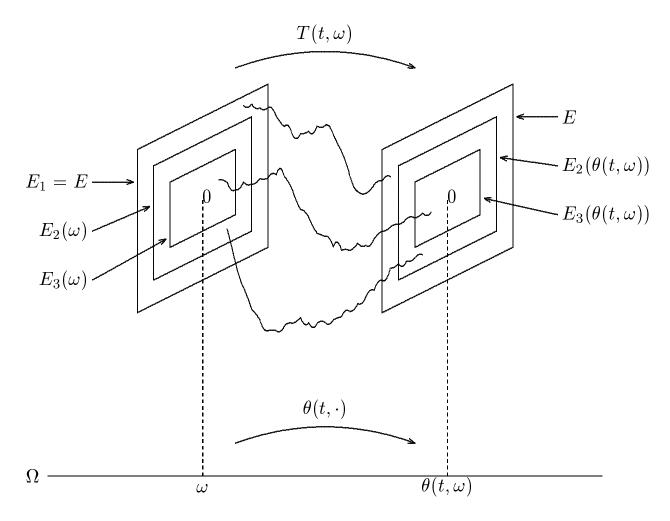
Proof.

Based on discrete version of Oseledec's multiplicative ergodic theorem and the perfect ergodic theorem. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]). □

Lyapunov Spectrum:

$\{\lambda_1, \lambda_2, \lambda_3, \cdots\} :=$ *Lyapunov spectrum* of (T, θ).

Spectral Theorem

Definition

A stationary point $Y(\omega)$ of (X, θ) is *hyperbolic* if the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega))$ has a non-vanishing Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, viz. $\lambda_i \neq 0$ for all $i \geq 1$.

Let $i_0 > 1$ be s.t. $\lambda_{i_0} < 0 < \lambda_{i_0 - 1}$.

Assume $X(t, \cdot, \omega)$ *locally compact* and

$$E \log^+ \sup_{0 \leq t_1, t_2 \leq r} \|D_2 X(t_2, Y(\theta(t_1)), \theta(t_1))\|_{L(E)} < \infty.$$

By Oseledec-Ruelle Theorem, there is a sequence of closed finite-codimensional (Oseledec) spaces

$$\cdots E_{i-1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$E_i(\omega) = \{x \in E : \lim_{t \to \infty} \frac{1}{t} \log \|DX(t, Y(\omega), \omega)(x)\| \leq \lambda_i\}, \quad i \geq 1,$$

for all $\omega \in \Omega^*$, a sure event in $\mathcal{F}$ satisfying $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$.

Let $\{U(\omega), S(\omega) : \omega \in \Omega^*\}$ be the unstable and stable subspaces associated with the linearized cocycle (DX, θ) ([Mo.1], Theorem 4, Corollary 2; [M-S.1], Theorem 5.3). Then get a measurable invariant splitting

$$E = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \Omega^*,$$
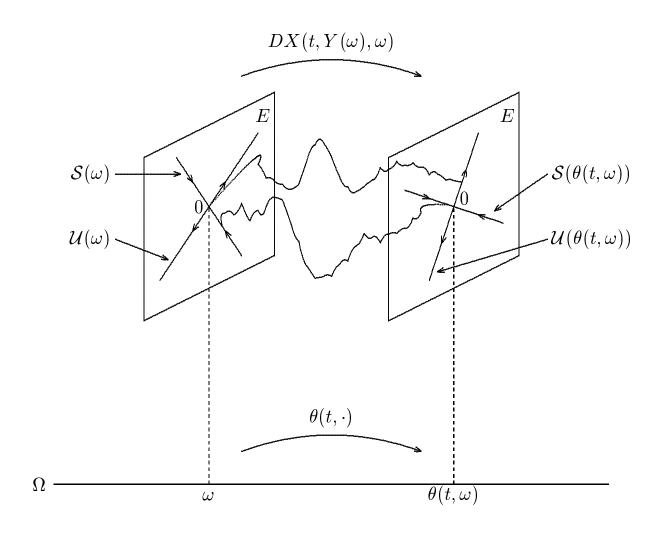
$$DX(t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)), \ DX(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)),$$

for all $t \geq 0$, with *exponential dichotomies*

$$\|DX(t, Y(\omega), \omega)(x)\| \geq \|x\| e^{\delta_1 t} \quad \text{for all} \quad t \geq \tau_1^*, x \in \mathcal{U}(\omega),$$

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$$\|DX(t, Y(\omega), \omega)(x)\| \leq \|x\| e^{-\delta_2 t} \quad \text{for all} \quad t \geq \tau_2^*, x \in \mathcal{S}(\omega),$$

with $\tau_i^* = \tau_i^*(x, \omega) > 0, i = 1, 2$, random times and $\delta_i > 0, i = 1, 2$, fixed.

Nonlinear Stochastic Systems with Memory

"Regular" Itô SFDE with finite memory:

$$\left.\begin{array}{l} dx(t) = H(x(t), x_t)\, dt + \sum_{i=1}^{m} G_i(x(t))\, dW_i(t), \\[2mm] (x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{array}\right\} \quad (I)$$

Solution segment $x_t(s) := x(t+s)$, $t \geq 0, s \in [-r, 0]$.

m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$, $W(0) = 0$.

Ergodic Brownian shift θ on Wiener space $(\Omega, \mathcal{F}, P)$. $\bar{\mathcal{F}} := P-$completion of $\mathcal{F}$.

State space M_2, Hilbert with usual norm $\|\cdot\|$.

Can allow for "smooth memory" in diffusion coefficient.

$H : M_2 \to \mathbf{R}^d$, $C^{k,\delta}$, globally bounded.

$G : \mathbf{R}^d \to L(\mathbf{R}^m, \mathbf{R}^d)$, $C_b^{k+1,\delta}$; $G := (G_1, \cdots, G_m)$.

$B((v, \eta), \rho)$ open ball, radius ρ, center $(v, \eta) \in M_2$;

$\bar{B}((v, \eta), \rho)$ closed ball.

Then (I) has a stochastic semiflow $X : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ with $X(t, (v, \eta), \cdot) = (x(t), x_t)$. X is $C^{k,\epsilon}$ for any $\epsilon \in (0, \delta)$, takes

bounded sets into relatively compact sets in M_2. (X, θ) is a perfect cocycle on M_2 ([M-S.4]).

Theorem. *([M-S], 1999) (Local Stable and Unstable Manifolds)*

Assume smoothness hypotheses on H and G. Let $Y : \Omega \to M_2$ be a hyperbolic stationary point of the SFDE (I) such that $E(\|Y(\cdot)\|^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$

Suppose the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ of (I) has a Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. Define $\lambda_{i_0} := max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all finite λ_i are positive, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized semiflow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,

(ii) $\bar{\mathcal{F}}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k,\epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) $\tilde{\mathcal{S}}(\omega)$ is the set of all $(v, \eta) \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$\|X(nr, (v, \eta), \omega) - Y(\theta(nr, \omega))\| \leq \beta_1(\omega) e^{(\lambda_{i_0}+\epsilon_1)nr}$$

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for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega) - Y(\theta(t, \omega))\| \leq \lambda_{i_0}$$

for all $(v, \eta) \in \tilde{\mathcal{S}}(\omega)$. Each stable subspace $\mathcal{S}(\omega)$ of the linearized semiflow DX is tangent at $Y(\omega)$ to the submanifold $\tilde{\mathcal{S}}(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. In particular, codim $\tilde{\mathcal{S}}(\omega) =$ codim $\mathcal{S}(\omega)$, is fixed and finite.

(b) $\displaystyle \limsup_{t \to \infty} \frac{1}{t} \log \left[\sup \left\{ \frac{\|X(t, (v_1, \eta_1), \omega) - X(t, (v_2, \eta_2), \omega)\|}{\|(v_1, \eta_1) - (v_2, \eta_2)\|} : (v_1, \eta_1) \neq \right. \right.$

$\displaystyle \left. \left. (v_2, \eta_2), \; (v_1, \eta_1), \; (v_2, \eta_2) \in \tilde{\mathcal{S}}(\omega) \right\} \right] \leq \lambda_{i_0}.$

(c) *(Cocycle-invariance of the stable manifolds):*

There exists $\tau_1(\omega) \geq 0$ such that

$$X(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega))$$

for all $t \geq \tau_1(\omega)$. Also

$$DX(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)), \quad t \geq 0.$$

(d) $\tilde{\mathcal{U}}(\omega)$ *is the set of all $(v, \eta) \in \bar{B}(Y(\omega), \rho_2(\omega))$ with the property that there is a unique "history" process $y(\cdot, \omega) : \{-nr : n \geq$*

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$0\} \to M_2$ *such that* $y(0, \omega) = (v, \eta)$ *and for each integer* $n \geq 1,$ *one has* $X(r, y(-nr, \omega), \theta(-nr, \omega)) = y(-(n-1)r, \omega)$ *and*

$$\|y(-nr, \omega) - Y(\theta(-nr, \omega))\|_{M_2} \leq \beta_2(\omega)e^{-(\lambda_{i_0-1}-\epsilon_2)nr}.$$

Furthermore, for each $(v, \eta) \in \tilde{\mathcal{U}}(\omega),$ *there is a unique continuous-time "history" process also denoted by* $y(\cdot, \omega) : (-\infty, 0] \to M_2$ *such that* $y(0, \omega) = (v, \eta),$ $X(t, y(s, \omega), \theta(s, \omega)) = y(t+s, \omega)$ *for all* $s \leq 0, 0 \leq t \leq -s,$ *and*

$$\limsup_{t \to \infty} \frac{1}{t} \log \|y(-t, \omega) - Y(\theta(-t, \omega))\| \leq -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}(\omega)$ *of the linearized semiflow* DX *is tangent at* $Y(\omega)$ *to* $\tilde{\mathcal{U}}(\omega),$ *viz.* $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega).$ *In particular,* $\dim \tilde{\mathcal{U}}(\omega)$ *is finite and non-random.*

(e) *Let* $y(\cdot, (v_i, \eta_i), \omega), i = 1, 2,$ *be the history processes associated with* $(v_i, \eta_i) = y(0, (v_i, \eta_i), \omega) \in \tilde{\mathcal{U}}(\omega),$ $i = 1, 2.$ *Then*

$$\limsup_{t \to \infty} \frac{1}{t} \log \left[\sup\left\{ \frac{\|y(-t, (v_1, \eta_1), \omega) - y(-t, (v_2, \eta_2), \omega)\|}{\|(v_1, \eta_1) - (v_2, \eta_2)\|} : \right.\right.$$
$$\left.\left. (v_1, \eta_1) \neq (v_2, \eta_2), (v_i, \eta_i) \in \tilde{\mathcal{U}}(\omega), i = 1, 2 \right\} \right] \leq -\lambda_{i_0-1}.$$

(f) *(Cocycle-invariance of the unstable manifolds):*

There exists $\tau_2(\omega) \geq 0$ *such that*

$$\tilde{\mathcal{U}}(\omega) \subseteq X(t, \cdot, \theta(-t, \omega))(\tilde{\mathcal{U}}(\theta(-t, \omega)))$$

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for all $t \geq \tau_2(\omega)$. Also

$$DX(t, \cdot, \theta(-t, \omega))(\mathcal{U}(\theta(-t, \omega))) = \mathcal{U}(\omega), \quad t \geq 0;$$

and the restriction

$$DX(t, \cdot, \theta(-t, \omega))|\mathcal{U}(\theta(-t, \omega)) : \mathcal{U}(\theta(-t, \omega)) \to \mathcal{U}(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto.
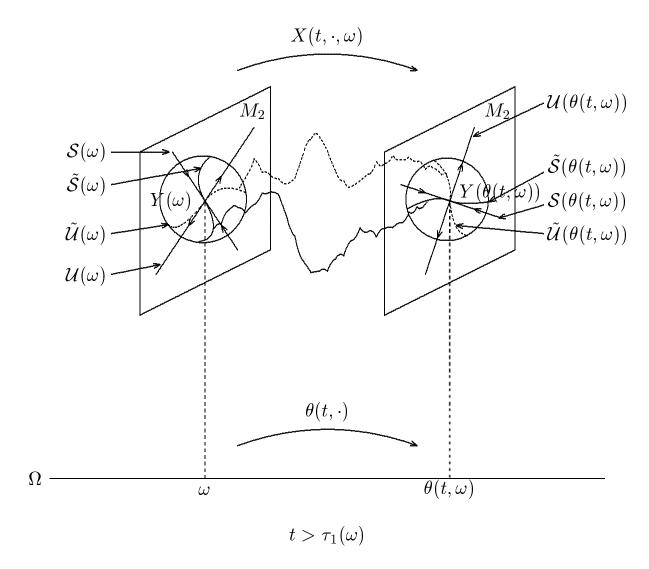
(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ and $\tilde{\mathcal{S}}(\omega)$ are transversal, viz.

$$M_2 = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

Assume, in addition, that H, G are C_b^∞. Then the local stable and unstable manifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞.

Figure summarizes essential features of Stable Manifold Theorem:

Stable Manifold Theorem

$$X(t, \cdot, \omega)$$

M_2 $\mathcal{U}(\theta(t, \omega))$

$\mathcal{S}(\omega)$

$\tilde{\mathcal{S}}(\omega)$

$Y(\omega)$ M_2 $\tilde{\mathcal{S}}(\theta(t, \omega))$

$Y(\theta(t, \omega))$ $\mathcal{S}(\theta(t, \omega))$

$\tilde{\mathcal{U}}(\omega)$ $\tilde{\mathcal{U}}(\theta(t, \omega))$

$\mathcal{U}(\omega)$

$$\theta(t, \cdot)$$

Ω ω $\theta(t, \omega)$

$$t > \tau_1(\omega)$$

A picture is worth a 1000 words!

Example

Affine linear sfde:

$$dx(t) = H(x(t), x_t)\, dt + G\, dW(t), \quad t > 0 \\ x(0) = v \in \mathbf{R}^d, \quad x_0 = \eta \in L^2([-r, 0], \mathbf{R}^d) \Bigg\} \quad (I'')$$

$H : M_2 \to \mathbf{R}^d$ continuous linear map, G a fixed $(d \times p)$-matrix, and W p-dimensional Brownian motion. Assume that the $(d \times d)$-matrix-valued FDE

$$dy(t) = H \circ (y(t), y_t)\, dt$$

has a semiflow

$$T_t : L(\mathbf{R}^d) \times L^2([-r, 0], L(\mathbf{R}^d)) \to L(\mathbf{R}^d) \times L^2([-r, 0], L(\mathbf{R}^d)), t \geq 0,$$

which is uniformly asymptotically stable. Set

$$Y := \int_{-\infty}^{0} T_{-u}(I, 0) G\, dW(u) \tag{2}$$

where I is the identity $(d \times d)$-matrix. Integration by parts and

$$W(t, \theta(t_1, \omega)) = W(t + t_1, \omega) - W(t_1, \omega), \quad t, t_1 \in \mathbf{R}, \tag{3}$$

imply that Y has a measurable version satisfying (1). Y is Gaussian and thus has finite moments of all orders. See

([Mo], Pitman Books, 1984, Theorem 4.2, Corollary 4.2.1, pp. 208-217.) More generally, when H is hyperbolic, one can show that a stationary point of (I'') exists ([Mo]).

For general white-noise case with invariant measure, get stationary point in M_2 by enlarging probability space. Conversely, let $Y : \Omega \to M_2$ be a stationary point independent of the Brownian motion $W(t)$, $t \geq 0$. Then $\rho := P \circ Y^{-1}$ (distribution of Y) is an invariant measure for the one-point motion (by independence of Y and W).

Outline of Proof

- By definition, a *stationary* random point $Y(\omega) \in M_2$ is invariant under the semiflow X; viz $X(t, Y) = Y(\theta(t, \cdot))$ for all times t.

- Linearize the semiflow X along the stationary point $Y(\omega)$ in M_2. By stationarity of Y and the cocycle property of X, this gives a linear perfect cocycle $(DX(t, Y), \theta(t, \cdot))$ in $L(M_2)$, where $D = $ spatial (Fréchet) derivatives.

- Ergodicity of θ allows for the notion of *hyperbolicity* of a stationary solution of (I) via Oseledec-Ruelle theorem: Use local compactness of the semiflow for times greater than the delay r ([M-S.4]), and apply multiplicative ergodic theorem to get a discrete non-random Lyapunov spectrum $\{\lambda_i : i \geq 1\}$ for the linearized cocycle. Y is *hyperbolic* if $\lambda_i \neq 0$ for every i.

- Assume that $\|Y\|^{\epsilon_0}$ is integrable (for small ϵ_0). Variational method of construction of the semiflow shows that the linearized cocycle satisfies hypotheses of "perfect versions" of ergodic theorem and Kingman's subadditive ergodic theorem. These refined versions give

invariance of the Oseledec spaces under the continuous-time linearized cocycle. Thus the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear semiflow X.

- Establish continuous-time integrability estimates on the spatial derivatives of the non-linear cocycle X in a neighborhood of the stationary point Y. Estimates follow from the variational construction of the stochastic semiflow coupled with known global spatial estimates for finite-dimensional stochastic flows.

- Introduce the auxiliary perfect cocycle

$$Z(t, \cdot, \omega) := X(t, (\cdot) + Y(\omega), \omega) - Y(\theta(t, \omega)), \quad t \in \mathbf{R}^+, \omega \in \Omega.$$

Refine arguments in ([Ru.2], Theorems 5.1 and 6.1) to construct local stable/unstable manifolds for the discrete cocycle $(Z(nr, \cdot, \omega), \theta(nr, \omega))$ near 0 and hence (by translation) for $X(nr, \cdot, \omega)$ near $Y(\omega)$ for all ω sampled from a $\theta(t, \cdot)$-invariant sure event in Ω. This is possible because of the continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between delay periods of length r and further refining

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the arguments in [Ru.2], show that the above mani-folds also serve as local stable/unstable manifolds for the *continuous-time* semiflow X near Y.

- Final key step: Establish the asymptotic invariance of the local stable manifolds under the stochastic semi-flow X. Use arguments underlying the proofs of Theorems 4.1 and 5.1 in [Ru.2] and some difficult esti-mates using the continuous-time integrability prop-erties, and the perfect subadditive ergodic theorem. Asymptotic invariance of the local unstable manifolds follows by employing the concept of a *stochastic his-tory process* for X coupled with similar arguments to the above. Existence of history process compensates for the lack of invertibility of the semiflow.

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STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH CONSTRAINTS

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Outline

- Theory of stochastic functional differential equations (SFDE's) in flat space: Itô and Nisio ([IN], Kushner ([Ku]), Mohammed ([Mo$_2$], [Mo$_3$]) and Mohammed-Scheutzow ([MoS$_1$], [MoS$_2$]).

- **Objective:** to constrain the solution to live on a smooth submanifold of Euclidean space.

- **Main difficulty:** Tangent space along a solution path is random (cf. unlike flat case).

- Difficulty resolved by pulling back the calculus on the tangent space at the starting point of the initial semi-martingale using stochastic parallel transport. Get SFDE on a linear space of semimartingales with values in the tangent space at a given point on the manifold.

- Solve SFDE on flat space by Picard's iteration method. (cf. Driver [Dr]). But two levels of randomness:
  (1) stochastic parallel transport over initial semimartingale path;
  (2) driving Brownian motion.

Law of solution at a given time may not be absolutely continuous with respect to law of initial semimartingale.

- Example of SDDE on the manifold with a type of Markov property in space of semimartingales.

- Regularity of solution of SDDE in initial semimartingale: stochastic Chen-Souriau calculus (Léandre [Le$_2$], [Le$_3$]). Requires Fréchet topology on semimartingales.

The Existence Theorem

Notation:

M smooth compact Riemannian manifold, dimension d.

Delay $\delta > 0$, $T > 0$.

$(\Omega, \mathcal{F}_t, t \geq -\delta, P)$ filtered probability space-usual conditions.

$W : [-\delta, \infty) \times \Omega \to \mathbf{R}^p$ Brownian motion on $(\Omega, \mathcal{F}_t, t \geq -\delta, P)$, $W(-\delta) = 0$.
($p = 1$ for simplicity.)

N any smooth finite-dimensional Riemannian manifold; $x \in N$.

$\mathcal{S}([-\delta, T], N; -\delta, x) :=$ space of all N-valued $(\mathcal{F}_t)_{t \geq -\delta}$-adapted continuous semimartingales

$$\gamma : [-\delta, T] \times \Omega \to N$$

with $\gamma(-\delta) = x$.

The Itô Map:

Fix $x \in M$.

$T(M) :=$ tangent bundle over M.

Define the *Itô map* by

$$\mathcal{S}([-\delta, T], M; -\delta, x) \ni \gamma \to \tilde{\gamma} \in \mathcal{S}([-\delta, T], T_x(M); -\delta, 0)$$

$$\left. \begin{array}{l} d\tilde{\gamma}(t) = \tau_{t,-\delta}^{-1}(\gamma) \circ d\gamma(t) \\[2mm] \tilde{\gamma}(-\delta) = 0 \end{array} \right\} \quad (1)$$

(Stratonovich).

$\tau_{t,-\delta}(\gamma) :=$ (stochastic) parallel transport from $x = \gamma(-\delta)$ to $\gamma(t)$ along semimartingale γ.([E.E], [Em])

Itô map is a bijection.

$\tilde{\mathcal{S}}_2^T :=$ Hilbert space of all semimartin-gales $\tilde{\gamma} \in \mathcal{S}([-\delta, T], T_x(M); -\delta, 0)$ such that

$$\tilde{\gamma}(t) = \int_{-\delta}^{t} A(s)\, dW(s) + \int_{-\delta}^{t} B(s)\, ds, \qquad -\delta \leq t \leq T \tag{2}$$

and

$$\|\tilde{\gamma}\|_2^2 := E[\int_{-\delta}^{T} |A(s)|^2\, ds] + E[\int_{-\delta}^{T} |B(s)|^2\, ds] < \infty \tag{3}$$

$A(s), \quad B(s) \in T_x(M)$ adapted previsible processes-*characteristics* of $\tilde{\gamma}$ (or γ). $\|\cdot\|_2$ gives slightly different topology than traditional semi-martingale topologies ([D.M]).

$\mathcal{S}_2^T :=$ inverse image of $\tilde{\mathcal{S}}_2^T$ under the Itô map with induced topology.

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Let $\gamma \in \mathcal{S}_2^T$, $t \in [-\delta, T]$. Set

$$\gamma^t(s) := \gamma(s \wedge t), \qquad s \in [-\delta, T].$$

Then $\widetilde{(\gamma^t)} = (\tilde{\gamma})^t$.

Evaluation map

$$e : [0, T] \times \mathcal{S}_2^T \to L^0(\Omega, M)$$

$$e(t, \gamma) := \gamma(t)$$

Vector bundle $L^0(\Omega, T(M))$ over $L^0(\Omega, M)$ with fiber over $Z \in L^0(\Omega, M)$ given by

$$L^0(\Omega, T(M))_Z := \{Y : Y(\omega) \in T_{Z(\omega)} M \text{ a.a. } \omega \in \Omega\}$$

$e^* L^0(\Omega, T(M)) :=$ pull-back bundle of $L^0(\Omega, T(M))$ over $[0, T] \times \mathcal{S}_2^T$ by e.

A SFDE on M is a map

$$F : [0, T] \times \mathcal{S}_2^T \to L^0(\Omega, T(M))$$

such that $F(t, \gamma^t) \in T_{\gamma(t)}(M)$ a.s. for all $\gamma \in \mathcal{S}_2^T$, $0 \leq t \leq T$. I.e. F is a section of $e^* L^0(\Omega, T(M))$.

Consider SFDE

$$\left.\begin{aligned} dx(t) &= F(t, x^t) \circ dW(t), \quad t \geq 0 \\ x^0 &= \gamma^0 \end{aligned}\right\} \tag{4}$$

- Pullback SFDE (4) over $T_x(M)$. Then:

$$\left.\begin{aligned} d\tilde{x}(t) &= \tau_{t,-\delta}^{-1}(x^t) F(t, x^t) \circ dW(t) \\ &= \tilde{F}(t, \tilde{x}^t) \circ dW(t), \quad t \geq 0 \\ \tilde{x}^0 &= \tilde{\gamma}^0 \end{aligned}\right\} \tag{5}$$

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$(t, \tilde{\gamma}) \mapsto \tilde{F}(t, \tilde{\gamma}) := \tau_{t, -\delta}^{-1}(\gamma) F(t, \gamma)$ can be viewed as a functional

$$[0, T] \times \tilde{\mathcal{S}}_2^T \to L^0(\Omega, T_x(M))$$

on the flat space $\tilde{\mathcal{S}}_2^T$,

- Use Stratonovich correction $\Delta \tilde{F}(t, \tilde{\gamma}^t)$ and impose "boundedness" and "Lipschitz condition" on F in terms of $\tilde{F}$ to get existence and uniqueness:

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Hypothesis (H):

(i) **"Boundedeness"**. There exists a deterministic constant C_1 such that

$$|\tilde{F}(t, \tilde{\gamma}^t)| + |\Delta \tilde{F}(t, \tilde{\gamma}^t)| < C_1 < \infty, \quad \text{a.s.} \quad (6)$$

for all $(t, \tilde{\gamma}) \in [0, T] \times \tilde{\mathcal{S}}_2^T$.

(ii) **"Local Lipschitz property"**. Suppose $\tilde{\gamma}, \tilde{\gamma}' \in \mathcal{S}_2^T$ have characteristics $(A(.), B(.))$ and $(A'(.), B'(.))$ respectively which are a.s. bounded by a deterministic constant R. Then

$$E[|\tilde{F}(t, \tilde{\gamma}^t) - \tilde{F}(t, (\tilde{\gamma}')^t)|^2 + |\Delta \tilde{F}(t, \tilde{\gamma}^t) - \Delta \tilde{F}(t, (\tilde{\gamma}')^t)|^2]$$

$$\leq K(R) \|\tilde{\gamma}^t - (\tilde{\gamma}')^t\|_2^2 \quad (7)$$

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Examples:

1. $X :=$ a smooth section of k-frame bundle $L(\underline{\mathbf{R}}^k, T(M)) \to M$.

SDDE:

$$dx(t) = \tau_{t, t-\delta}(x) X(x(t - \delta)), \quad t > 0 \qquad (8)$$

with

$$F(t, \gamma) := \tau_{t, t-\delta}(\gamma) X(\gamma(t - \delta));$$

and

$$\tilde{F}(t, \tilde{\gamma}) = \tau^{-1}_{t-\delta, -\delta}(\gamma) X(\gamma(t - \delta)). \qquad (8')$$

$\tilde{F}$ satisfies (H)(i) because parallel transport is a rotation and M is compact.

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2. $X_1, X_2 :=$ smooth sections of k-frame bundle $L(\underline{\mathbf{R}^k}, T(M)) \to M$.

SFDE:

$$dx_{c,t} = \left\{ \int_{t-\delta}^{t} \tau_{t,s}(x_{c,.}) X_1(x_{c,s}) ds + X_2(x_{c,t}) \right\} \circ dw_t,$$

$$(9)$$

for $0 < t < T$.

For (H)(ii) embed M (isometrically) into $R^{d'}$ and extend the Riemannian structure over $R^{d'}$: the Riemannian metric has bounded derivatives of all orders and is uniformly non-degenerate. Extend the Levi-Civita connection over M to a connection which preserves the metric over $R^{d'}$ on the trivial tangent bundle of $R^{d'}$ with

Christoffel symbols having bounded derivatives of all order. The pair $(\gamma(t), \tau_{t,-\delta})$ corresponds to a process $\hat{x}(t) \in R^{d'} \times R^{d' \times d'}$ which solves the Stratonovitch SDE:

$$
\left.
\begin{aligned}
d\hat{x}(t) &= \hat{Z}(\hat{x}(t)) \circ A(t)\, dW(t) + \hat{Z}(\hat{x}(t))B(t)\, dt \\
\hat{x}(-\delta) &= (x, Id_{T_x(M)})
\end{aligned}
\right\}
$$
(10)

on $R^{d'} \times R^{d' \times d'}$

$\hat{Z}$ is smooth (and hence has derivatives of all orders bounded over the range of existence of $\hat{x}$).

(10) in Itô form:

$$
\left.
\begin{aligned}
d\hat{x}(t) &= \hat{Z}(\hat{x}(t))A(t)\, dW(t) + \hat{Y}(\hat{x}(t))A(t)^2\, dt \\
&\quad + \hat{Z}(\hat{x}(t))B(t)\, dt
\end{aligned}
\right\}
$$
(11)

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In (11), $A(t) \in T_x(M)$, but we consider the one-dimensional case $d = 1$ for simplicity.

$\hat{Y}$ satisfies same hypotheses as the vector field $\hat{Z}$.

$\hat{x}(A, B)$ denotes dependence of $\hat{x}$ on A and B.

Lemma 1.

Suppose

$$|A(t)| + |B(t)| + |A'(t)| + |B'(t)| \leq R,$$

a.s. for all $t \in [-\delta, T]$ and some deterministic $R > 0$.

Then there exists a constant $K(R) > 0$ such that:

$$E[\sup_{-\delta \leq s \leq t} |\hat{x}(A, B)(s) - \hat{x}(A', B')(s)|^2]$$

$$\leq K(R)E[\int_{-\delta}^{t} (|A(s) - A'(s)|^2 + |B(s) - B'(s)|^2) \, ds] \tag{12}$$

Proof.

Follows from (11) by Burkholder's inequality and Gronwall's lemma. □

Put $t = 0$ in Lemma to show that SFDE's (8) and (9) satisfy (H)(ii).

Theorem 1.

Assume hypotheses (H).

Suppose that $\gamma^0 \in \mathcal{S}_2^0$ has characteristics $(A(t), B(t))$, $t \in [-\delta, 0]$, a.s. bounded by a deterministic constant $C > 0$.

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Then the SFDE (4) has a unique global solution x such that $x|[-\delta, T] \in \mathcal{S}_2^T$ for every $T > 0$.

Proof.

Sufficient to prove theorem for the SFDE (5) in flat space.

Define $\tilde{x}^n$ inductively:

$$\left.\begin{array}{ll} d\tilde{x}^{n+1}(t) = \tilde{F}(t, \tilde{x}^{n,t})\, dW(t) + \Delta\tilde{F}(t, \tilde{x}^{n,t})\, dt, & t \geq 0 \\[2mm] \tilde{x}^{n+1,0} = \tilde{\gamma}^0 & \end{array}\right\} \tag{13}$$

By (H)(i),(ii),

$$\|\tilde{x}^{n+1,t} - \tilde{x}^{n,t}\|_2^2 \leq C \int_0^t \|\tilde{x}^{n,s} - \tilde{x}^{n-1,s}\|_2^2 ds \tag{14}$$

By induction:

$$\|\tilde{x}^{n+1,t} - \tilde{x}^{n,t}\|_2^2 \leq \frac{C^n t^n}{n!} \tag{15}$$

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This gives existence.

For uniqueness, take two solutions $\tilde{x}^1, \tilde{x}^2$ of (5). By (H)(i), their characteristics are a.s. bounded. Then

$$
\left.\begin{aligned}
d\tilde{x}^1(t) &= \tilde{F}(t, \tilde{x}^{1,t})\, dW(t) + \Delta\tilde{F}(t, \tilde{x}^{1,t})\, dt \\
d\tilde{x}^2(t) &= \tilde{F}(t, \tilde{x}^{2,t})\, dW(t) + \Delta\tilde{F}(t, \tilde{x}^{2,t})\, dt \\
\tilde{x}^{1,0} &= \tilde{x}^{2,0} = \tilde{\gamma}^0
\end{aligned}\right\} \qquad (16)
$$

imply

$$
\|\tilde{x}^{1,t} - \tilde{x}^{2,t}\|_2^2 \leq C \int_0^t \|\tilde{x}^{1,s} - \tilde{x}^{2,s}\|_2^2 ds \qquad (17)
$$

Hence $\|\tilde{x}^{1,t} - \tilde{x}^{2,t}\|_2^2 = 0$. $\qquad \square$

Under the

Delay Condition:

$$\tilde{F}(t, \tilde{\gamma}^t) = \tilde{F}(t, \tilde{\gamma}^{t-\delta})$$

the *Stratonovich* equation (5) now becomes also the *Itô* equation:

$$\left.\begin{aligned} d\tilde{x}(t) =& \tilde{F}(t, \tilde{x}^{(t-\delta)}) \, dW(t) \\ \tilde{x}^0 =& \tilde{\gamma}^0 \end{aligned}\right\}$$

Existence and uniqueness hold by forward steps of length δ.

Continuous dependence on initial process:

Theorem 2.

Assume hypotheses (H). Let $\mathcal{B}^T \subset \mathcal{S}_2^T$ be the family of all $\gamma \in \mathcal{S}_2^T$ with characteristics (A, B) a.s. uniformly bounded on $[-\delta, 0]$ by a deterministic constant. Denote by $x(\gamma^0)$ the unique solution of SFDE (4) with initial semimartingale $\gamma^0 \in \mathcal{B}^0$. Then the mapping

$$\mathcal{B}^0 \ni \gamma^0 \mapsto x(\gamma^0) \in \mathcal{B}^T$$

is continuous.

Proof.

Let $\tilde{\gamma}^0, (\tilde{\gamma}')^0$ have characteristics (A, B), (A', B') uniformly bounded on $[-\delta, 0]$ by a deterministic constant. Let $\tilde{x}(A, B)$ and $\tilde{x}(A', B')$ be corresponding solutions of (5). By Burkholder's inequality and (H)(ii):

$$\|\tilde{x}^t(A, B) - \tilde{x}^t(A', B')\|_2^2$$

$$\leq \|\tilde{\gamma}^0 - (\tilde{\gamma}')^0\|_2^2 + K \int_0^t \|\tilde{x}^s(A, B) - \tilde{x}^s(A', B')\|_2^2 ds \tag{18}$$

By Gronwall's lemma:

$$\|\tilde{x}(A, B) - \tilde{x}(A', B')\|_2^2 \leq C\|\tilde{\gamma}^0 - (\tilde{\gamma}')^0\|_2^2 \tag{19}$$

$\square$

Example-Markov Behavior.

Consider the SDDE:

$$dx(t) = \tau_{t,t-\delta}(x)X\left(x(t-\delta)\right)dW(t) \left.\vphantom{\begin{matrix}dx\\x\end{matrix}}\right\} \qquad (20)$$
$$x^0 = \gamma^0,$$

with $\gamma^0(-\delta) = x \in M$.

Replace x by a random variable $Z \in L^0(\Omega, M)$ independent of of $W(t), t \geq -\delta$.

Fix $t_0 > 0$. The process $x(t), t \geq t_0$ solves the SDDE:

$$dx'(t) = \tau_{t,t-\delta}(x')X\left(x'(t-\delta)\right)dW(t), t \geq t_0 \left.\vphantom{\begin{matrix}dx\\x\end{matrix}}\right\} \qquad (21)$$
$$x'(s) = x(s), \;\; s \in [t_0 - \delta, t_0]$$

$x(t_0 - \delta)$ is independent of $dW(t), t \geq t_0 - \delta$, and parallel transport in (20) depends only on the path between $t - \delta$ and t.

Uniqueness implies

$$x'(t) = x(t), \ t \geq t_0.$$

For any semi-martingale $\gamma(t), \ t \geq -\delta$ in M, let $\gamma_t := \gamma|[t - \delta, t]$.
$x(\cdot)(\gamma^0)(W) :=$ solution of (20) with initial condition γ^0.

Then

$$x(t)(\gamma^0)(W) = x(t - t')(x_{t'}(\gamma^0))(W(t' + \cdot)), t \geq t' \quad (22)$$

$W(t' + \cdot) :=$ Brownian shift

$$s \mapsto W(t' + s) - W(t').$$

Differentiability in Chen-Souriau Sense:

Consider family of SDDE's:

$$\left. \begin{aligned} dx(t)(u) &= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta)(u)) \circ dW(t), t \geq 0 \\ x^0(u) &= \gamma^0(u) \end{aligned} \right\}$$

$$(23)$$

parametrized by $u \in U$, open subset of $\mathbf{R}^n$.

Embed M into $R^{d'}$.

Seek differentiability of $x(t)(u)$ in u. Can use Kolmogorov's lemma, Sobolev's imbedding theorem because u is finite-dimensional.

Flat version of (23) given by SDDE (8′) with an added parameter u.

For a parametrized semimartingale $\gamma(u)$ on M, the couple

$$(\gamma(u), \tau_{t,-\delta}(\gamma(u))) = \hat{x}_t$$

satisfies an Itô SDE depending on the parameter u:

$$d\hat{x}(t) = \hat{Z}(\hat{x}(t))A(u)(t)\,dW(t) + \hat{Y}(\hat{x}(t))A(u)(t)^2\,dt$$

$$+\hat{Z}(\hat{x}(t))B(u)(t)\,dt \tag{24}$$

$\hat{Z}$ and $\hat{Y}$ are smooth.

Introduce family of norms:

$$\|\tilde{\gamma}\|_p^p := E[\int_{-\delta}^T |A(s)|^p\,ds + \int_{-\delta}^T |B(s)|^p\,ds] \tag{25}$$

on the space $\tilde{\mathcal{S}}_\infty^T$ of all semimartingales

$$\tilde{\gamma} \in \mathcal{S}([-\delta, T], T_x(M); -\delta, 0)$$

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where $\tilde{\gamma}(t) = \int_{-\delta}^t A(s)\,dW(s) + \int_{-\delta}^t B(s)\,ds,\ 0 \le t \le T$ and $\|\tilde{\gamma}\|_p$ is finite for every $p \ge 1$.

Suppose $A(u)(\cdot)$ and $B(u)(\cdot)$ are bounded by a deterministic constant C independent of u, and

$$u \mapsto (A(u)(\cdot), B(u)(\cdot))$$

is Fréchet smooth in the Fréchet space $\tilde{\mathcal{S}}_\infty^T$ defined by the family of norms $\|\cdot\|_p$.

Theorem 3.

Consider the parametrized SDDE's:

$$\left. \begin{array}{l} dx(t)(u) = \tau_{t,t-\delta}(x^t(u))X(x(t-\delta)(u)) \circ dW(t), t \geq 0, \\ \\ x^0(u) = \gamma^0(u) \end{array} \right\}$$

$$(26)$$

where X is smooth and $\gamma^0(u)$ is smooth in u as above.

Then $x(t)(u)$ has a version which is a.s. smooth in u.

Theorem also holds if noise has a smooth parameter u:

$dx(t)(u)$

$$= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta))(\circ A(u)(t)\, dW(t) + B(u)(t)\, dt)$$

$$(27)$$

with initial conditions $x^0(u) = \gamma^0(u)$.

Proof of Theorem 3-Outline.

$\alpha := (\alpha_1, \cdots \alpha_k)$ multi-index.

$D^\alpha :=$ partial derivatives of order

$$|\alpha| := \sum_{i=1}^{k} \alpha_i.$$

- For a parametrized semimartingale $\gamma(u)$ on M, the couple

$$(\gamma(u), \tau_{t,-\delta}^{-1}(\gamma(u))) := \hat{x}(t)(u)$$

satisfies an Itô SDE depending on the parameter u:

$d\hat{x}(t)(u) = \hat{Z}(\hat{x}(t)(u))A(u)(t)\,dW(t)$

$\qquad + \hat{Y}(\hat{x}(t)(u))A(u)(t)^2\,dt + \hat{Z}(\hat{x}(t)(u))B(u)(t)\,dt$

Since the inverse of the parallel transport is bounded, then $\hat{Z}$ and $\hat{Y}$ have

bounded derivatives of all orders. If $\gamma(u) \in \mathcal{S}_\infty^T$ has a.s. bounded character-istics $(A(u), B(u))$ which are smooth in u into the Fréchet space $\mathcal{S}_\infty^T$, then the pair $\hat{x}(t)(u) := (\gamma(u), \tau_{t,-\delta}^{-1}(\gamma(u)))$ has characetris-tics Fréchet smooth in u. Follows by differentiating above SDE and apply-ing Burkholder's inequality and Gron-wall's lemma.

- Write the SDDE

$$\left.\begin{aligned}dx(t)(u) &= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta)(u)) \circ dW(t), t \geq 0,\\ x^0(u) &= \gamma^0(u)\end{aligned}\right\}$$
$$(26)$$

in the form:

$$\left.\begin{aligned}d\tilde{x}(t)(u) &= g(\hat{x}((t-\delta))(u))dW(t)\\ \tilde{x}^0(u) &= \tilde{\gamma}^0(u)\end{aligned}\right\} \quad (*)$$

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where $\hat{x}(t) := (x(t), \tau_{t,-\delta}^{-1}(x))$,

$g(y,z) := zX(y)$, and z represents parallel transport (orthogonal matrix), $y \in M$. Then g is bounded and has bounded derivatives of all orders.

$\tilde{\gamma}(t)^0(u) := \int_{-\delta}^t A_s^0(u) dw_s + \int_{-\delta}^t B_s^0 ds$ for $t < 0$ where $A^0(u)(\cdot)$ and $B^0(u)(\cdot)$ are bounded independently of u and differentiable in u in all the L^p semi-martingale norms $\|\cdot\|_p$.

Hence $\tilde{\gamma}(t)^0(u)$ has u-derivatives of all orders in all L^p semi-martingale norms. Follows from Kolmogorov's lemma and Burkholder's inequality.

- For $t \in [0, \delta]$, $\tilde{x}(t)(u)$ is a.s. differentiable in u and

$$dD^\alpha \tilde{x}(t)(u)$$

$$= Dg(\hat{x}(t - \delta)(u))D^\alpha \hat{x}(t - \delta)(u) \, dW(t) + l.o.$$

where $l.o.$ are terms containing lower-order derivatives of $\tilde{x}(t)(u)$.

- Get estimate:

$$\sup_{u \in U} \|D^\alpha \tilde{x}(\cdot)(u)\|_p \leq C(p, \alpha)$$

- Use forward steps of length δ to prove that $\tilde{x}(t)(u)$ has a smooth version in u for all $t \in [0, T]$.

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